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Impulsive quantum difference systems with boundary conditions

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Abstract
This article studies the existence and uniqueness of solutions for coupled systems of nonlinear impulsive quantum difference equations with coupled and uncoupled boundary conditions. The existence and uniqueness of solutions is established by Banach's contraction principle, while the existence of solutions is derived by using Leray-Schauder's alternative. Examples illustrating our results are also presented.

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1 Introduction and preliminaries

In this paper, we concentrate on the study of the existence and uniqueness of solutions for a coupled system of nonlinear impulsive quantum difference equations,

$$\begin{cases} D_{q_k}x(t) = f(t,x(t),y(t)), & t \in J := [0,T], t \neq t_k, \\ D_{p_k}y(t) = g(t,x(t),y(t)), & t \in J, t \neq t_k, \\ \Delta x(t_k) = I_k(x(t_k)), \quad \Delta y(t_k) = I_k^*(y(t_k)), & k = 1,2,\ldots,m, \\ a_1x(0) + b_1y(T) = \lambda_1, \quad a_2y(0) + b_2x(T) = \lambda_2, \end{cases} \tag{1.1}$$

where $0 = t_0 < t_1 < t_2 < \cdots < t_k < \cdots < t_m < t_{m+1} = T$, $f,g : J \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions, $I_k, I_k^* \in C(\mathbb{R},\mathbb{R})$, $\Delta u(t_k) = u(t_k^+) - u(t_k)$, $u(t_k^+) = \lim_{h\to 0^+} u(t_k + h)$, $u \in \{x,y\}$, for $k = 1,2,\ldots,m$, and $0 < p_k, q_k < 1$ for $k = 0,1,2,\ldots,m$ are given quantum numbers, a_i, b_i, λ_i, $i = 1,2$ are real constants with $a_1a_2 \neq b_1b_2$.

The notions of quantum calculus on finite intervals, q_k-derivatives, and q_k-integrals were introduced in [1]. For a fixed $k \in \mathbb{N} \cup \{0\}$ let $J_k := [t_k, t_{k+1}] \subset \mathbb{R}$ be an interval and $0 < q_k < 1$, $k = 1,2,\ldots,m$ be a constant. We define the q_k-derivative of a function $f : J_k \to \mathbb{R}$ at a point $t \in J_k$ as follows.

Definition 1.1 Assume $f : J_k \to \mathbb{R}$ is a continuous function and let $t \in J_k$. Then the expression

$$D_{q_k}f(t) = \frac{f(t) - f(q_kt + (1-q_k)t_k)}{(1-q_k)(t-t_k)}, \quad t \neq t_k, \qquad D_{q_k}f(t_k) = \lim_{t\to t_k} D_{q_k}f(t), \tag{1.2}$$

is called the q_k-derivative of function f at t.

We say that f is q_k-differentiable on J_k provided $D_{q_k}f(t)$ exists for all $t \in J_k$. Note that if $t_k = 0$ and $q_k = q$ in (1.2), then $D_{q_k}f = D_q f$, where D_q is the well-known q-derivative of the function $f(t)$, defined by

$$D_q f(t) = \frac{f(t) - f(qt)}{(1-q)t}. \tag{1.3}$$

The q_k-integral is defined as follows.

Definition 1.2 Assume $f : J_k \to \mathbb{R}$ is a continuous function. Then the q_k-integral is defined by

$$\int_{t_k}^{t} f(s)\, d_{q_k}s = (1-q_k)(t-t_k)\sum_{n=0}^{\infty} q_k^n f\big(q_k^n t + \big(1-q_k^n\big)t_k\big), \tag{1.4}$$

for $t \in J_k$. Moreover, if $a \in (t_k, t)$, then the definite q_k-integral is defined by

$$\begin{aligned}\int_a^t f(s)\, d_{q_k}s &= \int_{t_k}^t f(s)\, d_{q_k}s - \int_{t_k}^a f(s)\, d_{q_k}s = (1-q_k)(t-t_k)\sum_{n=0}^{\infty} q_k^n f\big(q_k^n t + \big(1-q_k^n\big)t_k\big) \\ &\quad - (1-q_k)(a-t_k)\sum_{n=0}^{\infty} q_k^n f\big(q_k^n a + \big(1-q_k^n\big)t_k\big).\end{aligned}$$

Note that if $t_k = 0$ and $q_k = q$, then (1.4) reduces to q-integral of a function $f(t)$, defined by $\int_0^t f(s)\, d_q s = (1-q)t\sum_{n=0}^{\infty} q^n f(q^n t)$ for $t \in [0,\infty)$.

For the basic properties of the q_k-derivative and the q_k-integral we refer to [1].

The book by Kac and Cheung [2] covers many of the fundamental aspects of the quantum calculus. In recent years, the topic of q-calculus has attracted the attention of several researchers and a variety of new results can be found in [3–15] and the references cited therein.

Impulsive differential equations serve as basic models to study the dynamics of processes that are subject to sudden changes in their states. The recent development in this field has been motivated by many applied problems, such as control theory, population dynamics, and medicine. For some recent works on the theory of impulsive differential equations, we refer the interested reader to the monographs [16–18]. Moreover, the interested reader is referred to [19–24] for some recent results on impulsive q_k-difference equations.

In this paper we prove existence and uniqueness results for the impulsive boundary value problem (1.1) by using Banach's contraction mapping principle and Leray-Schauder's nonlinear alternative. The rest of this paper is organized as follows: In Section 2 we present an auxiliary lemma which is used to convert the impulsive boundary value problem (1.1) into an equivalent integral equation. In Section 3, we establish an existence and uniqueness result via Banach's contraction principle, and an existence result by applying Leray-Schauder's alternative. Results on uncoupled integral boundary conditions case are in Section 4. Examples illustrating our results are also presented.

2 An auxiliary lemma

Let $J = [0,T]$, $J_0 = [t_0, t_1]$, $J_k = (t_k, t_{k+1}]$ for $k = 1, 2, \ldots, m$. To define the solutions of problem (1.1) we need the following lemma, which deals with a linear variant of problem (1.1) and gives a representation of the solutions.

Lemma 2.1 *Given $\phi, \psi \in C(J, \mathbb{R})$, the unique solution of the problem*

$$\begin{cases} D_{q_k} x(t) = \phi(t), & t \in J, t \neq t_k, \\ D_{p_k} y(t) = \psi(t), & t \in J, t \neq t_k, \\ \Delta x(t_k) = I_k(x(t_k)), \quad \Delta y(t_k) = I_k^*(y(t_k)), & k = 1, 2, \ldots, m, \\ a_1 x(0) + b_1 y(T) = \lambda_1, \quad a_2 y(0) + b_2 x(T) = \lambda_2, \end{cases} \tag{2.1}$$

is

$$\begin{aligned} x(t) = {} & \frac{1}{\Omega} \Bigg[a_2\lambda_1 - a_2 b_1 \sum_{k=1}^{m} \bigg(\int_{t_{k-1}}^{t_k} \psi(s)\, d_{p_{k-1}} s + I_k^*\big(y(t_k)\big) \bigg) - a_2 b_1 \int_{t_m}^{T} \psi(s)\, d_{p_m} s \\ & - b_1\lambda_2 + b_1 b_2 \sum_{k=1}^{m} \bigg(\int_{t_{k-1}}^{t_k} \phi(s)\, d_{q_{k-1}} s + I_k\big(x(t_k)\big) \bigg) + b_1 b_2 \int_{t_m}^{T} \phi(s)\, d_{q_m} s \Bigg] \\ & + \sum_{0<t_k<t} \bigg(\int_{t_{k-1}}^{t_k} \phi(s)\, d_{q_{k-1}} s + I_k\big(x(t_k)\big) \bigg) + \int_{t_k}^{t} \phi(s)\, d_{q_k} s \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} y(t) = {} & \frac{1}{\Omega} \Bigg[a_1\lambda_2 - a_1 b_2 \sum_{k=1}^{m} \bigg(\int_{t_{k-1}}^{t_k} \phi(s)\, d_{q_{k-1}} s + I_k\big(x(t_k)\big) \bigg) - a_1 b_2 \int_{t_m}^{T} \phi(s)\, d_{q_m} s \\ & - b_2\lambda_1 + b_1 b_2 \sum_{k=1}^{m} \bigg(\int_{t_{k-1}}^{t_k} \psi(s)\, d_{p_{k-1}} s + I_k^*\big(y(t_k)\big) \bigg) + b_1 b_2 \int_{t_m}^{T} \psi(s)\, d_{p_m} s \Bigg] \\ & + \sum_{0<t_k<t} \bigg(\int_{t_{k-1}}^{t_k} \psi(s)\, d_{p_{k-1}} s + I_k^*\big(y(t_k)\big) \bigg) + \int_{t_k}^{t} \psi(s)\, d_{p_k} s, \end{aligned} \tag{2.3}$$

where

$$\Omega = a_1 a_2 - b_1 b_2 \neq 0. \tag{2.4}$$

Proof For $t \in J_0$, q_0-integrating (2.1), it follows that

$$x(t) = x(0) + \int_0^t \phi(s)\, d_{q_0} s,$$

which leads to

$$x(t_1) = x(0) + \int_0^{t_1} \phi(s)\, d_{q_0} s.$$

For $t \in J_1$, taking the q_1-integral for (2.1), we get

$$x(t) = x\big(t_1^+\big) + \int_{t_1}^{t} \phi(s)\, d_{q_1} s.$$

Since $x(t_1^+) = x(t_1) + I_1(x(t_1))$, we have

$$x(t) = x(0) + \int_0^{t_1} \phi(s)\, d_{q_0} s + \int_{t_1}^{t} \phi(s)\, d_{q_1} s + I_1\big(x(t_1)\big).$$

Again q_2-integrating (2.1) from t_2 to t, where $t \in J_2$, then

$$x(t) = x(t_2^+) + \int_{t_2}^{t} \phi(s)\, d_{q_2}s$$
$$= x(0) + \int_0^{t_1} \phi(s)\, d_{q_0}s + \int_{t_1}^{t_2} \phi(s)\, d_{q_1}s + \int_{t_2}^{t} \phi(s)\, d_{q_2}s + I_1(x(t_1)) + I_2(x(t_2)).$$

Repeating the above process, for $t \in J$, we obtain

$$x(t) = x(0) + \sum_{0<t_k<t} \left(\int_{t_{k-1}}^{t_k} \phi(s)\, d_{q_{k-1}}s + I_k(x(t_k)) \right) + \int_{t_k}^{t} \phi(s)\, d_{q_k}s. \tag{2.5}$$

In the same way, we can obtain

$$y(t) = y(0) + \sum_{0<t_k<t} \left(\int_{t_{k-1}}^{t_k} \psi(s)\, d_{p_{k-1}}s + I_k^*(y(t_k)) \right) + \int_{t_k}^{t} \psi(s)\, d_{p_k}s. \tag{2.6}$$

In particular, for $t = T$, we have

$$x(T) = x(0) + \sum_{k=1}^{m} \left(\int_{t_{k-1}}^{t_k} \phi(s)\, d_{q_{k-1}}s + I_k(x(t_k)) \right) + \int_{t_m}^{T} \phi(s)\, d_{q_m}s,$$
$$y(T) = y(0) + \sum_{k=1}^{m} \left(\int_{t_{k-1}}^{t_k} \psi(s)\, d_{p_{k-1}}s + I_k^*(y(t_k)) \right) + \int_{t_m}^{T} \psi(s)\, d_{p_m}s.$$

Applying the boundary conditions of (2.1), we get the system

$$a_1x(0) + b_1y(0) + b_1 \sum_{k=1}^{m} \left(\int_{t_{k-1}}^{t_k} \psi(s)\, d_{p_{k-1}}s + I_k^*(y(t_k)) \right) + b_1 \int_{t_m}^{T} \psi(s)\, d_{p_m}s = \lambda_1,$$
$$a_2y(0) + b_2x(0) + b_2 \sum_{k=1}^{m} \left(\int_{t_{k-1}}^{t_k} \phi(s)\, d_{q_{k-1}}s + I_k(x(t_k)) \right) + b_2 \int_{t_m}^{T} \phi(s)\, d_{q_m}s = \lambda_2,$$

from which we have

$$x(0) = \frac{1}{\Omega} \Bigg[a_2\lambda_1 - a_2b_1 \sum_{k=1}^{m} \left(\int_{t_{k-1}}^{t_k} \psi(s)\, d_{p_{k-1}}s + I_k^*(y(t_k)) \right) - a_2b_1 \int_{t_m}^{T} \psi(s)\, d_{p_m}s$$
$$- b_1\lambda_2 + b_1b_2 \sum_{k=1}^{m} \left(\int_{t_{k-1}}^{t_k} \phi(s)\, d_{q_{k-1}}s + I_k(x(t_k)) \right) + b_1b_2 \int_{t_m}^{T} \phi(s)\, d_{q_m}s \Bigg]$$

and

$$y(0) = \frac{1}{\Omega} \Bigg[a_1\lambda_2 - a_1b_2 \sum_{k=1}^{m} \left(\int_{t_{k-1}}^{t_k} \phi(s)\, d_{q_{k-1}}s + I_k(x(t_k)) \right) - a_1b_2 \int_{t_m}^{T} \phi(s)\, d_{q_m}s$$
$$- b_2\lambda_1 + b_1b_2 \sum_{k=1}^{m} \left(\int_{t_{k-1}}^{t_k} \psi(s)\, d_{p_{k-1}}s + I_k^*(y(t_k)) \right) + b_1b_2 \int_{t_m}^{T} \psi(s)\, d_{p_m}s \Bigg].$$

Substituting the values of $x(0)$ and $y(0)$ in (2.5) and (2.6), we obtain the solutions (2.2) and (2.3). □

3 Main results

Let $PC(J,\mathbb{R}) = \{x : J \to \mathbb{R}; x(t)$ is continuous everywhere except for some t_k at which $x(t_k^+)$ and $x(t_k^-)$ exist and $x(t_k^-) = x(t_k)$, $k = 1, 2, \ldots, m\}$. $PC(J,\mathbb{R})$ is a Banach space with the norm $\|x\|_{PC} = \sup\{|x(t)|, t \in J\}$. Let us introduce the space $X = \{x(t); x(t) \in PC([0,T])\}$ endowed with the norm $\|x\| = \sup\{|x(t)|, t \in [0,T]\}$. Obviously $(X, \|\cdot\|)$ is a Banach space. Also let $Y = \{y(t); y(t) \in PC([0,T])\}$ be endowed with the norm $\|y\| = \sup\{|y(t)|, t \in [0,T]\}$. Obviously the product space $(X \times Y, \|(x,y)\|)$ is a Banach space with norm $\|(x,y)\| = \|x\| + \|y\|$.

In view of Lemma 2.1, we define an operator $\mathcal{T} : X \times Y \to X \times Y$ by

$$\mathcal{T}(x,y)(t) = \begin{pmatrix} \mathcal{T}_1(x,y)(t) \\ \mathcal{T}_2(x,y)(t) \end{pmatrix},$$

where

$$\begin{aligned} \mathcal{T}_1(x,y)(t) = \frac{1}{\Omega}\Bigg[& a_2\lambda_1 - a_2 b_1 \sum_{k=1}^{m} \left(\int_{t_{k-1}}^{t_k} g\big(s, x(s), y(s)\big)\, d_{p_{k-1}}s + I_k^*\big(y(t_k)\big) \right) \\ & - a_2 b_1 \int_{t_m}^{T} g\big(s, x(s), y(s)\big)\, d_{p_m}s - b_1\lambda_2 \\ & + b_1 b_2 \sum_{k=1}^{m} \left(\int_{t_{k-1}}^{t_k} f\big(s, x(s), y(s)\big)\, d_{q_{k-1}}s + I_k\big(x(t_k)\big) \right) \\ & + b_1 b_2 \int_{t_m}^{T} f\big(s, x(s), y(s)\big)\, d_{q_m}s \Bigg] \\ & + \sum_{0<t_k<t} \left(\int_{t_{k-1}}^{t_k} f\big(s, x(s), y(s)\big)\, d_{q_{k-1}}s + I_k\big(x(t_k)\big) \right) + \int_{t_k}^{t} f\big(s, x(s), y(s)\big)\, d_{q_k}s \end{aligned}$$

and

$$\begin{aligned} \mathcal{T}_2(x,y)(t) = \frac{1}{\Omega}\Bigg[& a_1\lambda_2 - a_1 b_2 \sum_{k=1}^{m} \left(\int_{t_{k-1}}^{t_k} f\big(s, x(s), y(s)\big)\, d_{q_{k-1}}s + I_k\big(x(t_k)\big) \right) \\ & - a_1 b_2 \int_{t_m}^{T} f\big(s, x(s), y(s)\big)\, d_{q_m}s - b_2\lambda_1 \\ & + b_1 b_2 \sum_{k=1}^{m} \left(\int_{t_{k-1}}^{t_k} g\big(s, x(s), y(s)\big)\, d_{p_{k-1}}s + I_k^*\big(y(t_k)\big) \right) \\ & + b_1 b_2 \int_{t_m}^{T} g\big(s, x(s), y(s)\big)\, d_{p_m}s \Bigg] \\ & + \sum_{0<t_k<t} \left(\int_{t_{k-1}}^{t_k} g\big(s, x(s), y(s)\big)\, d_{p_{k-1}}s + I_k^*\big(y(t_k)\big) \right) + \int_{t_k}^{t} g\big(s, x(s), y(s)\big)\, d_{p_k}s. \end{aligned}$$

For the sake of convenience, we set

$$M_1 = \frac{1}{|\Omega|}\big[T\big(L_1|a_2||b_1| + K_1|b_1||b_2| + K_1|\Omega|\big) + mK_3\big(|b_1||b_2| + |\Omega|\big)\big], \tag{3.1}$$

$$M_2 = \frac{1}{|\Omega|}\big[T\big(L_2|a_2||b_1| + K_2|b_1||b_2| + K_2|\Omega|\big) + mL_3|a_2||b_1|\big], \tag{3.2}$$

$$M_3 = \frac{1}{|\Omega|}\big[T\big(N_2|a_2||b_1| + N_1|b_1||b_2| + N_1|\Omega|\big) + m\big(N_4|a_2||b_1| + N_3|b_1||b_2| + N_3|\Omega|\big) + |a_2||\lambda_1| + |b_1||\lambda_2|\big], \tag{3.3}$$

$$M_4 = \frac{1}{|\Omega|}\big[T\big(K_1|a_1||b_2| + L_1|b_1||b_2| + L_1|\Omega|\big) + mK_3|a_1||b_2|\big], \tag{3.4}$$

$$M_5 = \frac{1}{|\Omega|}\big[T\big(K_2|a_1||b_2| + L_2|b_1||b_2| + L_2|\Omega|\big) + mL_3\big(|b_1||b_2| + |\Omega|\big)\big], \tag{3.5}$$

$$M_6 = \frac{1}{|\Omega|}\big[T\big(N_1|a_1||b_2| + N_2|b_1||b_2| + N_2|\Omega|\big) + m\big(N_3|a_1||b_2| + N_4|b_1||b_2| + N_4|\Omega|\big) + |a_1||\lambda_2| + |b_2||\lambda_1|\big]. \tag{3.6}$$

The first result is concerned with the existence and uniqueness of solutions for the problem (1.1) and is based on Banach's contraction mapping principle.

Theorem 3.1 *Assume that*:

(H_1) *The functions* $f, g : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$ *are continuous and there exist constants* $K_i, L_i > 0$, $i = 1, 2$ *such that for all* $t \in [0, T]$ *and* $u_i, v_i \in \mathbb{R}$, $i = 1, 2$,

$$\big|f(t, u_1, u_2) - f(t, v_1, v_2)\big| \le K_1|u_1 - v_1| + K_2|u_2 - v_2|$$

and

$$\big|g(t, u_1, u_2) - g(t, v_1, v_2)\big| \le L_1|u_1 - v_1| + L_2|u_2 - v_2|.$$

(H_2) *The functions* $I_k, I_k^* : \mathbb{R} \to \mathbb{R}$ *are continuous and there exist constants* $K_3, L_3 > 0$ *such that for all* $t \in [0, T]$ *and* $u_3, v_3 \in \mathbb{R}$, $k = 1, 2, \ldots, m$,

$$\big|I_k(u_3) - I_k(v_3)\big| \le K_3|u_3 - v_3|$$

and

$$\big|I_k^*(u_3) - I_k^*(v_3)\big| \le L_3|u_3 - v_3|.$$

In addition, assume that

$$M_1 + M_2 + M_4 + M_5 < 1,$$

where M_i, $i = 1, 2, 4, 5$, *are given by* (3.1)-(3.2) *and* (3.4)-(3.5). *Then the boundary value problem* (1.1) *has a unique solution*.

Proof Define $\sup_{t\in[0,T]} f(t, 0, 0) = N_1 < \infty$, $\sup_{t\in[0,T]} g(t, 0, 0) = N_2 < \infty$, $\sup\{|I_k(0)| : k = 1, 2, \ldots, m\} = N_3 < \infty$ and $\sup\{|I_k^*(0)| : k = 1, 2, \ldots, m\} = N_4 < \infty$ such that

$$r \ge \max\left\{\frac{M_3}{1 - (M_1 + M_2)}, \frac{M_6}{1 - (M_4 + M_5)}\right\},$$

where M_3 and M_6 are defined by (3.3) and (3.6), respectively.

We show that $\mathcal{T}B_r \subset B_r$, where $B_r = \{(x,y) \in X \times Y : \|(x,y)\| \leq r\}$.

For $(x,y) \in B_r$, we have

$$\begin{aligned}
&\bigl|\mathcal{T}_1(x,y)(t)\bigr| \\
&\quad= \sup_{t\in[0,T]}\Biggl\{\frac{1}{\Omega}\Biggl[a_2\lambda_1 - a_2b_1\sum_{k=1}^{m}\biggl(\int_{t_{k-1}}^{t_k} g\bigl(s,x(s),y(s)\bigr)\,d_{p_{k-1}}s + I_k^*\bigl(y(t_k)\bigr)\biggr) \\
&\qquad - a_2b_1\int_{t_m}^{T} g\bigl(s,x(s),y(s)\bigr)\,d_{p_m}s - b_1\lambda_2 \\
&\qquad + b_1b_2\sum_{k=1}^{m}\biggl(\int_{t_{k-1}}^{t_k} f\bigl(s,x(s),y(s)\bigr)\,d_{q_{k-1}}s + I_k\bigl(x(t_k)\bigr)\biggr) + b_1b_2\int_{t_m}^{T} f\bigl(s,x(s),y(s)\bigr)\,d_{q_m}s\Biggr] \\
&\qquad + \sum_{0<t_k<t}\biggl(\int_{t_{k-1}}^{t_k} f\bigl(s,x(s),y(s)\bigr)\,d_{q_{k-1}}s + I_k\bigl(x(t_k)\bigr)\biggr) + \int_{t_k}^{t} f\bigl(s,x(s),y(s)\bigr)\,d_{q_k}s\Biggr\} \\
&\quad\leq \frac{1}{|\Omega|}\Biggl[|a_2||\lambda_1| + |a_2||b_1|\sum_{k=1}^{m}\biggl(\int_{t_{k-1}}^{t_k}\bigl|g\bigl(s,x(s),y(s)\bigr) - g(s,0,0)\bigr| + \bigl|g(s,0,0)\bigr|\,d_{p_{k-1}}s \\
&\qquad + \bigl|I_k^*\bigl(y(t_k)\bigr) - I_k^*(0)\bigr| + \bigl|I_k^*(0)\bigr|\biggr) \\
&\qquad + |a_2||b_1|\int_{t_m}^{T}\bigl|g\bigl(s,x(s),y(s)\bigr) - g(s,0,0)\bigr| + \bigl|g(s,0,0)\bigr|\,d_{p_m}s + |b_1||\lambda_2| \\
&\qquad + |b_1||b_2|\sum_{k=1}^{m}\biggl(\int_{t_{k-1}}^{t_k}\bigl|f\bigl(s,x(s),y(s)\bigr) - f(s,0,0)\bigr| + \bigl|f(s,0,0)\bigr|\,d_{q_{k-1}}s \\
&\qquad + \bigl|I_k\bigl(x(t_k)\bigr) - I_k(0)\bigr| + \bigl|I_k(0)\bigr|\biggr) \\
&\qquad + |b_1||b_2|\int_{t_m}^{T}\bigl|f\bigl(s,x(s),y(s)\bigr) - f(s,0,0)\bigr| + \bigl|f(s,0,0)\bigr|\,d_{q_m}s\Biggr] \\
&\qquad + \sum_{k=1}^{m}\biggl(\int_{t_{k-1}}^{t_k}\bigl|f\bigl(s,x(s),y(s)\bigr) - f(s,0,0)\bigr| + \bigl|f(s,0,0)\bigr|\,d_{q_{k-1}}s \\
&\qquad + \bigl|I_k\bigl(x(t_k)\bigr) - I_k(0)\bigr| + \bigl|I_k(0)\bigr|\biggr) + \int_{t_m}^{t}\bigl|f\bigl(s,x(s),y(s)\bigr) - f(s,0,0)\bigr| + \bigl|f(s,0,0)\bigr|\,d_{q_m}s \\
&\quad\leq \frac{1}{|\Omega|}\Biggl[|a_2||\lambda_1| + |a_2||b_1|\sum_{k=1}^{m}\bigl(\bigl(L_1\|x\| + L_2\|y\| + N_2\bigr)(t_k - t_{k-1}) + L_3\|y\| + N_4\bigr) \\
&\qquad + |a_2||b_1|\bigl(L_1\|x\| + L_2\|y\| + N_2\bigr)(T - t_m) + |b_1||\lambda_2| \\
&\qquad + |b_1||b_2|\sum_{k=1}^{m}\bigl(\bigl(K_1\|x\| + K_2\|y\| + N_1\bigr)(t_k - t_{k-1}) + K_3\|x\| + N_3\bigr) \\
&\qquad + |b_1||b_2|\bigl(K_1\|x\| + K_2\|y\| + N_1\bigr)(T - t_m)\Biggr] \\
&\qquad + \sum_{k=1}^{m}\bigl(\bigl(K_1\|x\| + K_2\|y\| + N_1\bigr)(t_k - t_{k-1}) + K_3\|x\| + N_3\bigr)
\end{aligned}$$

$$
\begin{aligned}
&+ \bigl(K_1\|x\| + K_2\|y\| + N_1\bigr)(T - t_m) \\
&= \frac{1}{|\Omega|}\Biggl[|a_2||\lambda_1| + |a_2||b_1|\sum_{k=1}^{m+1}\bigl(\bigl(L_1\|x\| + L_2\|y\| + N_2\bigr)(t_k - t_{k-1})\bigr) + |a_2||b_1|mL_3\|y\| \\
&+ |a_2||b_1|mN_4 + |b_1||\lambda_2| + |b_1||b_2|\sum_{k=1}^{m+1}\bigl(\bigl(K_1\|x\| + K_2\|y\| + N_1\bigr)(t_k - t_{k-1})\bigr) \\
&+ |b_1||b_2|mK_3\|x\| + |b_1||b_2|mN_3\Biggr] + \sum_{k=1}^{m+1}\bigl(\bigl(K_1\|x\| + K_2\|y\| + N_1\bigr)(t_k - t_{k-1})\bigr) \\
&+ mK_3\|x\| + mN_3 \\
&= \|x\|\Biggl\{\frac{1}{|\Omega|}\Biggl[\sum_{k=1}^{m+1}(t_k - t_{k-1})\bigl(L_1|a_2||b_1| + K_1|b_1||b_2| + K_1|\Omega|\bigr) + mK_3\bigl(|b_1||b_2| + |\Omega|\bigr)\Biggr]\Biggr\} \\
&+ \|y\|\Biggl\{\frac{1}{|\Omega|}\Biggl[\sum_{k=1}^{m+1}(t_k - t_{k-1})\bigl(L_2|a_2||b_1| + K_2|b_1||b_2| + K_2|\Omega|\bigr) + mL_3|a_2||b_1|\Biggr]\Biggr\} \\
&+ \frac{1}{|\Omega|}\Biggl[\sum_{k=1}^{m+1}(t_k - t_{k-1})\bigl(N_2|a_2||b_1| + N_1|b_1||b_2| + N_1|\Omega|\bigr) \\
&+ m\bigl(N_4|a_2||b_1| + N_3|b_1||b_2| + N_3|\Omega|\bigr) + |a_2||\lambda_1| + |b_1||\lambda_2|\Biggr] \\
&= M_1\|x\| + M_2\|y\| + M_3 \\
&\le (M_1 + M_2)r + M_3 \le r.
\end{aligned}
$$

In the same way, we can obtain

$$
\begin{aligned}
&\bigl|\mathcal{T}_2(x,y)(t)\bigr| \\
&\le \|x\|\Biggl\{\frac{1}{|\Omega|}\Biggl[\sum_{k=1}^{m+1}(t_k - t_{k-1})\bigl(K_1|a_1||b_2| + L_1|b_1||b_2| + L_1|\Omega|\bigr) + mK_3|a_1||b_2|\Biggr]\Biggr\} \\
&+ \|y\|\Biggl\{\frac{1}{|\Omega|}\Biggl[\sum_{k=1}^{m+1}(t_k - t_{k-1})\bigl(K_2|a_1||b_2| + L_2|b_1||b_2| + L_2|\Omega|\bigr) \\
&+ mL_3\bigl(|b_1||b_2| + |\Omega|\bigr)\Biggr]\Biggr\} \\
&+ \frac{1}{|\Omega|}\Biggl[\sum_{k=1}^{m+1}(t_k - t_{k-1})\bigl(N_1|a_1||b_2| + N_2|b_1||b_2| + N_2|\Omega|\bigr) \\
&+ m\bigl(N_3|a_1||b_2| + N_4|b_1||b_2| + N_4|\Omega|\bigr) + |a_1||\lambda_2| + |b_2||\lambda_1|\Biggr] \\
&= M_4\|x\| + M_5\|y\| + M_6 \\
&\le (M_4 + M_5)r + M_6 \le r.
\end{aligned}
$$

Consequently, $\|\mathcal{T}(x,y)(t)\| \le r$.

Now for $(x_2, y_2), (x_1, y_1) \in X \times Y$ and for any $t \in [0, T]$, we get

$$
\begin{aligned}
&\bigl|\mathcal{T}_1(x_2, y_2)(t) - \mathcal{T}_1(x_1, y_1)(t)\bigr| \\
&\quad\le \frac{1}{|\Omega|}\Biggl[|a_2||b_1|\Biggl(\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k}\bigl|g\bigl(s, x_2(s), y_2(s)\bigr) - g\bigl(s, x_1(s), y_1(s)\bigr)\bigr|\,d_{p_{k-1}}s \\
&\qquad + \sum_{k=1}^{m}\bigl|I_k^*\bigl(y_2(t_k)\bigr) - I_k^*\bigl(y_1(t_k)\bigr)\bigr|\Biggr) \\
&\qquad + |b_1||b_2|\Biggl(\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k}\bigl|f\bigl(s, x_2(s), y_2(s)\bigr) - f\bigl(s, x_1(s), y_1(s)\bigr)\bigr|\,d_{q_{k-1}}s \\
&\qquad + \sum_{k=1}^{m}\bigl|I_k\bigl(x_2(t_k)\bigr) - I_k\bigl(x_1(t_k)\bigr)\bigr|\Biggr)\Biggr] \\
&\qquad + \sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k}\bigl|f\bigl(s, x_2(s), y_2(s)\bigr) - f\bigl(s, x_1(s), y_1(s)\bigr)\bigr|\,d_{q_{k-1}}s + \sum_{k=1}^{m}\bigl|I_k\bigl(x_2(t_k)\bigr) - I_k\bigl(x_1(t_k)\bigr)\bigr| \\
&\quad\le \frac{1}{|\Omega|}\Biggl[|a_2||b_1|\Biggl(\sum_{k=1}^{m+1}(t_k - t_{k-1})\bigl(L_1\|x_2 - x_1\| + L_2\|y_2 - y_1\|\bigr) + mL_3\|y_2 - y_1\|\Biggr) \\
&\qquad + |b_1||b_2|\Biggl(\sum_{k=1}^{m+1}(t_k - t_{k-1})\bigl(K_1\|x_2 - x_1\| + K_2\|y_2 - y_1\|\bigr) + mK_3\|x_2 - x_1\|\Biggr)\Biggr] \\
&\qquad + \sum_{k=1}^{m+1}(t_k - t_{k-1})\bigl(K_1\|x_2 - x_1\| + K_2\|y_2 - y_1\|\bigr) + mK_3\|x_2 - x_1\| \\
&\quad= \|x_2 - x_1\|\Biggl\{\frac{1}{|\Omega|}\Biggl[\sum_{k=1}^{m+1}(t_k - t_{k-1})\bigl(L_1|a_2||b_1| + K_1|b_1||b_2| + K_1|\Omega|\bigr) \\
&\qquad + mK_3\bigl(|b_1||b_2| + |\Omega|\bigr)\Biggr]\Biggr\} \\
&\qquad + \|y_2 - y_1\|\Biggl\{\frac{1}{|\Omega|}\Biggl[\sum_{k=1}^{m+1}(t_k - t_{k-1})\bigl(L_2|a_2||b_1| + K_2|b_1||b_2| + K_2|\Omega|\bigr) + mL_3|a_2||b_1|\Biggr]\Biggr\} \\
&\quad= M_1\|x_2 - x_1\| + M_2\|y_2 - y_1\|,
\end{aligned}
$$

and consequently we obtain

$$
\bigl\|\mathcal{T}_1(x_2, y_2)(t) - \mathcal{T}_1(x_1, y_1)\bigr\| \le (M_1 + M_2)\bigl[\|x_2 - x_1\| + \|y_2 - y_1\|\bigr]. \tag{3.7}
$$

Similarly,

$$
\bigl\|\mathcal{T}_2(x_2, y_2)(t) - \mathcal{T}_2(x_1, y_1)\bigr\| \le (M_4 + M_5)\bigl[\|x_2 - x_1\| + \|y_2 - y_1\|\bigr]. \tag{3.8}
$$

It follows from (3.7) and (3.8) that

$$
\bigl\|\mathcal{T}(x_2, y_2)(t) - \mathcal{T}(x_1, y_1)(t)\bigr\| \le (M_1 + M_2 + M_4 + M_5)\bigl[\|x_2 - x_1\| + \|y_2 - y_1\|\bigr].
$$

Since $M_1 + M_2 + M_4 + M_5 < 1$, therefore, $\mathcal{T}$ is a contraction operator. So, by Banach's fixed point theorem, the operator $\mathcal{T}$ has a unique fixed point, which is the unique solution of problem (1.1). This completes the proof. □

In the next result, we prove the existence of solutions for problem (1.1) by applying the Leray-Schauder alternative.

For the sake of convenience, we set

$$M_7 = \frac{1}{|\Omega|}\Big[T\big(B_1|a_2||b_1| + A_1|b_1||b_2| + A_1|\Omega|\big) + mA_4\big(|b_1||b_2| + |\Omega|\big)\Big], \tag{3.9}$$

$$M_8 = \frac{1}{|\Omega|}\Big[T\big(B_2|a_2||b_1| + A_2|b_1||b_2| + A_2|\Omega|\big) + mB_4|a_2||b_1|\Big], \tag{3.10}$$

$$\begin{aligned} M_9 = {}& \frac{1}{|\Omega|}\Big[T\big(B_0|a_2||b_1| + A_0|b_1||b_2| + A_0|\Omega|\big) \\ & + m\big(B_3|a_2||b_1| + A_3|b_1||b_2| + A_3|\Omega|\big) + |a_2||\lambda_1| + |b_1||\lambda_2|\Big], \end{aligned} \tag{3.11}$$

$$M_{10} = \frac{1}{|\Omega|}\Big[T\big(A_1|a_1||b_2| + B_1|b_1||b_2| + B_1|\Omega|\big) + mA_4|a_1||b_2|\Big], \tag{3.12}$$

$$M_{11} = \frac{1}{|\Omega|}\Big[T\big(A_2|a_1||b_2| + B_2|b_1||b_2| + B_2|\Omega|\big) + mB_4\big(|b_1||b_2| + |\Omega|\big)\Big], \tag{3.13}$$

$$\begin{aligned} M_{12} = {}& \frac{1}{|\Omega|}\Big[T\big(A_0|a_1||b_2| + B_0|b_1||b_2| + B_0|\Omega|\big) \\ & + m\big(A_3|a_1||b_2| + B_3|b_1||b_2| + B_3|\Omega|\big) + |a_1||\lambda_2| + |b_2||\lambda_1|\Big], \end{aligned} \tag{3.14}$$

and

$$M_0 = \min\big\{1 - (M_7 + M_{10}), 1 - (M_8 + M_{11})\big\}. \tag{3.15}$$

Lemma 3.1 (Leray-Schauder alternative) ([25], p.4) *Let $F : E \to E$ be a completely continuous operator* (*i.e., a map that is restricted to any bounded set in E is compact*). *Let*

$$\mathcal{E}(F) = \big\{x \in E : x = \lambda F(x) \text{ for some } 0 < \lambda < 1\big\}.$$

Then either the set $\mathcal{E}(F)$ is unbounded, or F has at least one fixed point.

Theorem 3.2 *Assume that*:

(H_3) *The functions $f, g : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous and there exist constants $A_i, B_i \geq 0$* ($i = 1,2$) *and $A_0, B_0 > 0$ such that $\forall x_i \in \mathbb{R}$* ($i = 1,2$)

$$\big|f(t,x_1,x_2)\big| \leq A_0 + A_1|x_1| + A_2|x_2|$$

and

$$\big|g(t,x_1,x_2)\big| \leq B_0 + B_1|x_1| + B_2|x_2|.$$

(H$_4$) *The functions* $I_k, I_k^* : \mathbb{R} \to \mathbb{R}$ *are continuous and there exist constants* $A_4, B_4 \geq 0$ *and* $A_3, B_3 > 0$ *such that* $\forall x_3 \in \mathbb{R}$, $k = 1, 2, \ldots, m$

$$\left|I_k(x_3)\right| \leq A_3 + A_4|x_3|$$

and

$$\left|I_k^*(x_3)\right| \leq B_3 + B_4|x_3|.$$

In addition it is assumed that

$$M_7 + M_{10} < 1 \quad \text{and} \quad M_8 + M_{11} < 1,$$

where M_7, M_8, M_{10}, M_{11} *are given by* (3.9)-(3.10) *and* (3.12)-(3.13). *Then there exists at least one solution for the boundary value problem* (1.1).

To prove the theorem we use the following lemma.

Lemma 3.2 *Assume that* (H$_3$) *and* (H$_4$) *hold. Then the operator* $\mathcal{T} : X \times Y \to X \times Y$ *is completely continuous.*

Proof By continuity of functions f and g, the operator $\mathcal{T}$ is continuous.

Let $\Theta \subset X \times Y$ be bounded. Then there exist positive constants P_1, P_2, P_3, and P_4 such that

$$\begin{aligned} &\left|f\bigl(t, x(t), y(t)\bigr)\right| \leq P_1, \qquad \left|g\bigl(t, x(t), y(t)\bigr)\right| \leq P_2, \quad \forall (x,y) \in \Theta, \\ &\left|I_k\bigl(x(t)\bigr)\right| \leq P_3, \qquad \left|I_k^*\bigl(y(t)\bigr)\right| \leq P_4, \quad k = 1, 2, \ldots, m. \end{aligned}$$

Then for any $(x, y) \in \Theta$, we have

$$\begin{aligned} &\left\|\mathcal{T}_1(x,y)\right\| \\ &\quad \leq \frac{1}{|\Omega|}\Biggl[|a_2||\lambda_1| + |a_2||b_1|\Biggl(\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k}\left|g\bigl(s, x(s), y(s)\bigr)\right| d_{p_{k-1}}s + \sum_{k=1}^{m}\left|I_k^*\bigl(y(t_k)\bigr)\right|\Biggr) \\ &\qquad + |b_1||\lambda_2| + |b_1||b_2|\Biggl(\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k}\left|f\bigl(s, x(s), y(s)\bigr)\right| d_{q_{k-1}}s + \sum_{k=1}^{m}\left|I_k\bigl(x(t_k)\bigr)\right|\Biggr)\Biggr] \\ &\qquad + \sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k}\left|f\bigl(s, x(s), y(s)\bigr)\right| d_{q_{k-1}}s + \sum_{k=1}^{m}\left|I_k\bigl(x(t_k)\bigr)\right| \\ &\quad \leq \frac{1}{|\Omega|}\bigl[|a_2||\lambda_1| + |a_2||b_1|(P_2T + mP_4) + |b_1||\lambda_2| + |b_1||b_2|(P_1T + mP_3)\bigr] \\ &\qquad + P_1T + mP_3 \\ &\quad := D_1. \end{aligned}$$

Similarly, we get

$$\begin{aligned}\|\mathcal{T}_2(x,y)\| &\le \frac{1}{|\Omega|}\big[|a_1||\lambda_2| + |a_1||b_2|(P_1T + mP_3) + |b_2||\lambda_1| + |b_1||b_2|(P_2T + mP_4)\big] \\ &\quad + P_2T + mP_4 \\ &:= D_2.\end{aligned}$$

Thus, it follows from the above inequalities that the operator $\mathcal{T}$ is uniformly bounded.

Next, we show that $\mathcal{T}$ is equicontinuous. Let $\nu_1, \nu_2 \in (t_l, t_{l+1})$ for some $l = 0, 1, \dots, m$ with $\nu_1 < \nu_2$. Then we have

$$\begin{aligned}&\big|\mathcal{T}_1\big(x(\nu_2), y(\nu_2)\big) - \mathcal{T}_1\big(x(\nu_1), y(\nu_1)\big)\big| \\ &\quad = \left|\int_{t_l}^{\nu_2} f\big(s, x(s), y(s)\big)\, d_{q_l}s - \int_{t_l}^{\nu_1} f\big(s, x(s), y(s)\big)\, d_{q_l}s\right| \\ &\quad \le P_1|\nu_2 - \nu_1|.\end{aligned}$$

Analogously, we can obtain

$$\begin{aligned}&\big|\mathcal{T}_2\big(x(\nu_2), y(\nu_2)\big) - \mathcal{T}_2\big(x(\nu_1), y(\nu_1)\big)\big| \\ &\quad = \left|\int_{t_l}^{\nu_2} g\big(s, x(s), y(s)\big)\, d_{p_l}s - \int_{t_l}^{\nu_1} g\big(s, x(s), y(s)\big)\, d_{p_l}s\right| \\ &\quad \le P_2|\nu_2 - \nu_1|.\end{aligned}$$

Therefore, the operator $\mathcal{T}(x,y)$ is equicontinuous, and thus the operator $\mathcal{T}(x,y)$ is completely continuous. □

Proof of Theorem 3.2 By Lemma 3.2 the operator $\mathcal{T}(x,y)$ is completely continuous.

Now, it will be verified that the set $\mathcal{E} = \{(x,y) \in X \times Y | (x,y) = \lambda\mathcal{T}(x,y), 0 \le \lambda \le 1\}$ is bounded. Let $(x,y) \in \mathcal{E}$, then $(x,y) = \lambda\mathcal{T}(x,y)$. For any $t \in [0,T]$, we have

$$x(t) = \lambda\mathcal{T}_1(x,y)(t), \qquad y(t) = \lambda\mathcal{T}_2(x,y)(t).$$

Then

$$\begin{aligned}\big|x(t)\big| &\le \|x\|\left\{\frac{1}{|\Omega|}\left[\sum_{k=1}^{m+1}(t_k - t_{k-1})\big(B_1|a_2||b_1| + A_1|b_1||b_2| + A_1|\Omega|\big)\right.\right. \\ &\quad \left.\left. + mA_4\big(|b_1||b_2| + |\Omega|\big)\right]\right\} \\ &\quad + \|y\|\left\{\frac{1}{|\Omega|}\left[\sum_{k=1}^{m+1}(t_k - t_{k-1})\big(B_2|a_2||b_1| + A_2|b_1||b_2| + A_2|\Omega|\big) + mB_4|a_2||b_1|\right]\right\} \\ &\quad + \frac{1}{|\Omega|}\left[\sum_{k=1}^{m+1}(t_k - t_{k-1})\big(B_0|a_2||b_1| + A_0|b_1||b_2| + A_0|\Omega|\big)\right. \\ &\quad \left. + m\big(B_3|a_2||b_1| + A_3|b_1||b_2| + A_3|\Omega|\big) + |a_2||\lambda_1| + |b_1||\lambda_2|\right]\end{aligned}$$

and

$$\begin{aligned}|y(t)| \le{}& \|x\|\left\{\frac{1}{|\Omega|}\left[\sum_{k=1}^{m+1}(t_k-t_{k-1})\big(A_1|a_1||b_2|+B_1|b_1||b_2|+B_1|\Omega|\big)+mA_4|a_1||b_2|\right]\right\}\\ &+\|y\|\left\{\frac{1}{|\Omega|}\left[\sum_{k=1}^{m+1}(t_k-t_{k-1})\big(A_2|a_1||b_2|+B_2|b_1||b_2|+B_2|\Omega|\big)\right.\right.\\ &\left.\left.+mB_4\big(|b_1||b_2|+|\Omega|\big)\right]\right\}\\ &+\frac{1}{|\Omega|}\left[\sum_{k=1}^{m+1}(t_k-t_{k-1})\big(A_0|a_1||b_2|+B_0|b_1||b_2|+B_0|\Omega|\big)\right.\\ &\left.+m\big(A_3|a_1||b_2|+B_3|b_1||b_2|+B_3|\Omega|\big)+|a_1||\lambda_2|+|b_2||\lambda_1|\right].\end{aligned}$$

Hence we have

$$\|x\| \le M_7\|x\|+M_8\|y\|+M_9$$

and

$$\|y\| \le M_{10}\|x\|+M_{11}\|y\|+M_{12},$$

which imply that

$$\|x\|+\|y\| \le (M_7+M_{10})\|x\|+(M_8+M_{11})\|y\|+M_9+M_{12}.$$

Consequently,

$$\big\|(x,y)\big\| \le \frac{M_9+M_{12}}{M_0},$$

for any $t\in[0,T]$, where M_0 is defined by (3.15), which proves that $\mathcal{E}$ is bounded. Thus, by Lemma 3.1, the operator $\mathcal{T}$ has at least one fixed point. Hence the boundary value problem (1.1) has at least one solution. The proof is complete. □

3.1 Examples

Example 3.1 Consider the following coupled system of impulsive quantum difference equations with coupled boundary conditions

$$\begin{cases} D_{\frac{2k+1}{k^2+k+2}}x(t)=\frac{t\cos^2(\pi t)}{(3e^t+4)^2}\frac{|x(t)|}{|x(t)|+1}+\frac{t+1}{(2t+4)^3}\frac{|y(t)|}{|y(t)|+1}+\frac{3}{2}, & t\in[0,2], t\neq t_k,\\ D_{\frac{\sqrt{k+1}}{e^k+1}}y(t)=\frac{1}{(2^{t+1}+5)^2}\sin x(t)+\frac{e^{-2(t+1)}}{7}\cos y(t)+\frac{t^2+1}{3}, & t\in[0,2], t\neq t_k,\\ \Delta x(t_k)=\frac{|x(t_k)|}{6(k+5)+|x(t_k)|},\qquad \Delta y(t_k)=\frac{|y(t_k)|}{7(k+4)+|y(t_k)|}, & t_k=\frac{k}{2}, k=1,2,3,\\ 2x(0)+4y(2)=5,\qquad 3y(0)-2x(2)=-6. \end{cases} \tag{3.16}$$

Here $q_k = (2k+1)/(k^2+k+2)$, $p_k = (\sqrt{k+1})/(e^k+1)$, $k = 0,1,2,3$, $m = 3$, $T = 2$, $a_1 = 2$, $a_2 = 3$, $b_1 = 4$, $b_2 = -2$, $\lambda_1 = 5$, $\lambda_2 = -6$, $f(t,x,y) = (t\cos^2(\pi t)|x|)/(((3e^t+4)^2)(|x|+1)) + ((t+1)|y|)/(((2t+4)^3)(|y|+1)) + 3/2$, $g(t,x,y) = (\sin x)/(2^{t+1}+5)^2 + (e^{-2(t+1)}\cos y)/7 + (t^2+1)/3$, $I_k(x) = |x|/(6(k+5)+|x|)$, and $I_k^*(y) = |y|/(7(k+4)+|y|)$. We have $|f(t,x_1,y_1) - f(t,x_2,y_2)| \leq ((2/49)|x_1-x_2| + (3/64)|y_1-y_2|)$, $|g(t,x_1,y_1) - g(t,x_2,y_2)| \leq ((1/49)|x_1-x_2| + (1/(7e^2))|y_1-y_2|)$, $|I_k(x) - I_k(y)| \leq (1/36)|x-y|$, and $|I_k^*(x) - I_k^*(y)| \leq (1/35)|x-y|$. We can find

$$\Omega = a_1a_2 - b_1b_2 = 14 \neq 0.$$

With the given values, it is found that $K_1 = 2/49$, $K_2 = 3/64$, $K_3 = 1/36$, $L_1 = 1/49$, $L_2 = 1/(7e^2)$, $L_3 = 1/35$, $M_1 \simeq 0.29422$, $M_2 \simeq 0.25393$, $M_4 \simeq 0.11127$, $M_5 \simeq 0.22224$,, and

$$M_1 + M_2 + M_4 + M_5 \simeq 0.88167 < 1.$$

Thus all the conditions of Theorem 3.1 are satisfied. Therefore, by the conclusion of Theorem 3.1, problem (3.16) has a unique solution on $[0,2]$.

Example 3.2 Consider the following coupled system of impulsive quantum difference equations with coupled boundary conditions:

$$\begin{cases} D_{\frac{k+1}{\sqrt{k^2+e^k+1}}} x(t) = \frac{1}{4} + \frac{1}{2(t+5)^2}\sin x(t) + \frac{1}{7\pi^2}\tan^{-1} y(t), & t \in [0,3], t \neq t_k, \\ D_{\frac{1}{3}\sin(\frac{k+1}{10}\pi)} y(t) = \frac{t+2}{e} + \frac{1}{40}x(t)\cos y(t) + \frac{1}{2^t+45}y(t), & t \in [0,3], t \neq t_k, \\ \Delta x(t_k) = \frac{1}{4}\tan^{-1}(\frac{x(t_k)}{8}) + 2, \quad t_k = \frac{k}{3}, k = 1,2,\ldots,8, \\ \Delta y(t_k) = \frac{1}{5}\sin(\frac{y(t_k)}{6}) + 3, \quad t_k = \frac{k}{3}, k = 1,2,\ldots,8, \\ -x(0) + 5y(3) = -2, \qquad 2y(0) + 3x(3) = 5. \end{cases} \tag{3.17}$$

Here $q_k = (k+1)/(\sqrt{k^2+e^k+1})$, $p_k = (\sin(((k+1)\pi)/10))/3$, $k = 0,1,2,\ldots,8$, $m = 8$, $T = 3$, $a_1 = -1$, $a_2 = 2$, $b_1 = 5$, $b_2 = 3$, $\lambda_1 = -2$, $\lambda_2 = 5$, $f(t,x,y) = (1/4) + (\sin x)/(2(t+5)^2) + (\tan^{-1} y)/(7\pi^2)$, $g(t,x,y) = ((t+2)/e) + (x\cos y)/40 + (y)/(2^t+45)$, $I_k(x) = (\tan^{-1}(x/8))/4 + 2$, and $I_k^*(y) = (\sin(y/6))/5 + 3$. We get

$$\Omega = a_1a_2 - b_1b_2 = -17 \neq 0.$$

Since $|f(t,x,y)| \leq A_0 + A_1|x| + A_2|y|$, $|g(t,x,y)| \leq B_0 + B_1|x| + B_2|y|$, where $A_0 = 1/4$, $A_1 = 1/50$, $A_2 = 1/(7\pi^2)$, $B_0 = 5/e$, $B_1 = 1/40$, $B_2 = 1/46$, it is found that $M_7 \simeq 0.62765$, $M_8 \simeq 0.27696$, $M_{10} \simeq 0.19588$, $M_{11} \simeq 0.63239$. Furthermore,

$$M_7 + M_{10} \approx 0.82353 < 1$$

and

$$M_8 + M_{11} \approx 0.90935 < 1.$$

Thus all the conditions of Theorem 3.2 holds true and consequently the conclusion of Theorem 3.2; problem (3.17) has at least one solution on $[0,3]$.

4 Uncoupled boundary conditions case

In this section, we consider again the system

$$\begin{cases} D_{q_k}x(t) = f(t, x(t), y(t)), \quad t \in J, t \neq t_k, \\ D_{p_k}y(t) = g(t, x(t), y(t)), \quad t \in J, t \neq t_k, \\ \Delta x(t_k) = I_k(x(t_k)), \qquad \Delta y(t_k) = I_k^*(y(t_k)), \quad k = 1, 2, \ldots, m, \\ a_1 x(0) + b_1 x(T) = \lambda_1, \qquad a_2 y(0) + b_2 y(T) = \lambda_2. \end{cases} \tag{4.1}$$

Lemma 4.1 (Auxiliary lemma) *For $h \in C([0, T], \mathbb{R})$, the unique solution of the problem*

$$\begin{cases} D_{q_k}x(t) = h(t), \quad t \in J, t \neq t_k, \\ \Delta x(t_k) = I_k(x(t_k)), \quad k = 1, 2, \ldots, m, \\ a_1 x(0) + b_1 x(T) = \lambda_1, \end{cases} \tag{4.2}$$

is given by

$$\begin{aligned} x(t) = {} & \frac{\lambda_1}{\Lambda} - \frac{b_1}{\Lambda}\left(\sum_{k=1}^{m+1} \int_{t_{k-1}}^{t_k} h(s)\, d_{q_{k-1}}s + \sum_{k=1}^{m} I_k\bigl(x(t_k)\bigr) \right) \\ & + \sum_{0<t_k<t} \left(\int_{t_{k-1}}^{t_k} h(s)\, d_{q_{k-1}}s + I_k\bigl(x(t_k)\bigr) \right) + \int_{t_k}^{t} h(s)\, d_{q_k}s, \end{aligned} \tag{4.3}$$

where

$$\Lambda := a_1 + b_1 \neq 0. \tag{4.4}$$

In view of Lemma 4.1, we define an operator $\mathfrak{T} : X \times Y \to X \times Y$ by

$$\mathfrak{T}(u, v)(t) = \begin{pmatrix} \mathfrak{T}_1(u, v)(t) \\ \mathfrak{T}_2(u, v)(t) \end{pmatrix},$$

where

$$\begin{aligned} \mathfrak{T}_1(u, v)(t) = {} & \frac{\lambda_1}{\Lambda} - \frac{b_1}{\Lambda}\left(\sum_{k=1}^{m+1} \int_{t_{k-1}}^{t_k} f\bigl(s, u(s), v(s)\bigr)\, d_{q_{k-1}}s + \sum_{k=1}^{m} I_k\bigl(u(t_k)\bigr) \right) \\ & + \sum_{0<t_k<t} \left(\int_{t_{k-1}}^{t_k} f\bigl(s, u(s), v(s)\bigr)\, d_{q_{k-1}}s + I_k\bigl(u(t_k)\bigr) \right) + \int_{t_k}^{t} f\bigl(s, u(s), v(s)\bigr)\, d_{q_k}s \end{aligned}$$

and

$$\begin{aligned} \mathfrak{T}_2(u, v)(t) = {} & \frac{\lambda_2}{\Phi} - \frac{b_2}{\Phi}\left(\sum_{k=1}^{m+1} \int_{t_{k-1}}^{t_k} g\bigl(s, u(s), v(s)\bigr)\, d_{p_{k-1}}s + \sum_{k=1}^{m} I_k^*\bigl(v(t_k)\bigr) \right) \\ & + \sum_{0<t_k<t} \left(\int_{t_{k-1}}^{t_k} g\bigl(s, u(s), v(s)\bigr)\, d_{p_{k-1}}s + I_k^*\bigl(v(t_k)\bigr) \right) \\ & + \int_{t_k}^{t} g\bigl(s, u(s), v(s)\bigr)\, d_{p_k}s, \end{aligned}$$

where

$$\Phi := a_2 + b_2 \neq 0.$$

We remark that $\mathfrak{T}_1$ depends only on f and $\mathfrak{T}_2$ only on g. We call the above system, for convenience, a 'coupled system with uncoupled boundary conditions'.

In the sequel, we set the constants

$$\overline{M}_1 = \frac{1}{|\Lambda|}(T\overline{K}_1 + m\overline{K}_3)\big(|b_1| + |\Lambda|\big), \tag{4.5}$$

$$\overline{M}_2 = \frac{1}{|\Lambda|}T\overline{K}_2\big(|b_1| + |\Lambda|\big), \tag{4.6}$$

$$\overline{M}_3 = \frac{1}{|\Lambda|}\big[(T\overline{N}_1 + m\overline{N}_3)\big(|b_1| + |\Lambda|\big) + |\lambda_1|\big], \tag{4.7}$$

$$\overline{M}_4 = \frac{1}{|\Phi|}T\overline{L}_1\big(|b_2| + |\Phi|\big), \tag{4.8}$$

$$\overline{M}_5 = \frac{1}{|\Phi|}(T\overline{L}_2 + m\overline{L}_3)\big(|b_2| + |\Phi|\big), \tag{4.9}$$

$$\overline{M}_6 = \frac{1}{|\Phi|}\big[(T\overline{N}_2 + m\overline{N}_4)\big(|b_2| + |\Phi|\big) + |\lambda_2|\big]. \tag{4.10}$$

Now we present the existence and uniqueness result for problem (4.1). We do not provide the proof of this result as it is similar to the one for Theorem 3.1.

Theorem 4.1 *Assume that*:

(H_5) *The functions* $f, g : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$ *are continuous and there exist constants* $\overline{K}_i, \overline{L}_i > 0$, $i = 1, 2$ *such that for all* $t \in [0, T]$ *and* $u_i, v_i \in \mathbb{R}$, $i = 1, 2$,

$$\big|f(t, u_1, u_2) - f(t, v_1, v_2)\big| \le \overline{K}_1|u_1 - v_1| + \overline{K}_2|u_2 - v_2|$$

and

$$\big|g(t, u_1, u_2) - g(t, v_1, v_2)\big| \le \overline{L}_1|u_1 - v_1| + \overline{L}_2|u_2 - v_2|.$$

(H_6) *The functions* $I_k, I_k^* : \mathbb{R} \to \mathbb{R}$ *are continuous and there exist constants* $\overline{K}_3, \overline{L}_3 > 0$ *such that for all* $t \in [0, T]$ *and* $u_3, v_3 \in \mathbb{R}$, $k = 1, 2, \dots, m$

$$\big|I_k(u_3) - I_k(v_3)\big| \le \overline{K}_3|u_3 - v_3|$$

and

$$\big|I_k^*(u_3) - I_k^*(v_3)\big| \le \overline{L}_3|u_3 - v_3|.$$

In addition, assume that

$$\overline{M}_1 + \overline{M}_2 + \overline{M}_4 + \overline{M}_5 < 1,$$

where $\overline{M}_1$, $\overline{M}_2$, $\overline{M}_4$, $\overline{M}_5$ *are given by* (4.5)-(4.6) *and* (4.8)-(4.9), *respectively. Then the boundary value problem* (4.1) *has a unique solution*.

Example 4.1 Consider the following coupled system of impulsive quantum difference equations with uncoupled boundary conditions

$$\begin{cases} D_{(\frac{2}{7})^k}x(t) = \frac{\sin(\pi t)}{(e^t+5)^2}\frac{|x(t)|}{|x(t)|+1} + \frac{\pi^t}{(t+4)^3}\frac{|y(t)|}{|y(t)|+1} + 3, \quad t \in [0,1], t \neq t_k, \\ D_{(\frac{3+k}{4+2k})^k}y(t) = \frac{1}{10(2^t+4)}\cos x(t) + \frac{1}{6\pi(t+3)}|y(t)| + 2, \quad t \in [0,1], t \neq t_k, \\ \Delta x(t_k) = \frac{|x(t_k)|}{3(k+9)+|x(t_k)|}, \qquad \Delta y(t_k) = \frac{|y(t_k)|}{5(k+6)+|y(t_k)|}, \quad t_k = \frac{k}{5}, k = 1,2,3,4, \\ 3x(0) - 8x(1) = 7, \qquad 4y(0) + 5y(1) = 2. \end{cases} \tag{4.11}$$

Here $q_k = (2/7)^k$, $p_k = ((3+k)/(4+2k))^k$, $k = 0,1,2,3,4$, $m = 4$, $T = 1$, $a_1 = 3$, $a_2 = 4$, $b_1 = -8$, $b_2 = 5$, $\lambda_1 = 7$, $\lambda_2 = 2$, $f(t,x,y) = (\sin(\pi t)|x|)/(((e^t+5)^2)(|x|+1)) + (\pi^t|y|)/(((t+4)^3)(|y|+1)) + 3$, $g(t,x,y) = (\cos x)/(10(2^t+4)) + (|y|)/(6\pi(t+3)) + 2$, $I_k(x) = |x|/(3(k+9)+|x|)$, and $I_k^*(y) = |y|/(5(k+6)+|y|)$. Since $|f(t,x_1,y_1) - f(t,x_2,y_2)| \leq ((1/36)|x_1 - x_2| + (\pi/64)|y_1 - y_2|)$, $|g(t,x_1,y_1) - g(t,x_2,y_2)| \leq ((1/50)|x_1 - x_2| + (1/(18\pi))|y_1 - y_2|)$, $|I_k(x) - I_k(y)| \leq (1/30)|x - y|$, and $|I_k^*(x) - I_k^*(y)| \leq (1/35)|x - y|$. We can find

$$\Lambda = a_1 + b_1 = -5 \neq 0 \quad \text{and} \quad \Phi = a_2 + b_2 = 9 \neq 0.$$

With the given values, it is found that $\overline{K}_1 = 1/36$, $\overline{K}_2 = \pi/64$, $\overline{K}_3 = 1/30$, $\overline{L}_1 = 1/50$, $\overline{L}_2 = 1/(18\pi)$, $\overline{L}_3 = 1/35$, $\overline{M}_1 \simeq 0.41889$, $\overline{M}_2 \simeq 0.12763$, $\overline{M}_4 \simeq 0.03111$, $\overline{M}_5 \simeq 0.20529$, and

$$\overline{M}_1 + \overline{M}_2 + \overline{M}_4 + \overline{M}_5 \simeq 0.78291 < 1.$$

Thus all the conditions of Theorem 4.1 are satisfied. Therefore, by the conclusion of Theorem 4.1, problem (4.11) has a unique solution on $[0,1]$.

The second result dealing with the existence of solutions for the problem (4.1) is analogous to Theorem 3.2 and is given below.

In the sequel, we set constants

$$\overline{M}_7 = \frac{1}{|\Lambda|}(T\overline{A}_1 + m\overline{A}_4)\big(|b_1| + |\Lambda|\big), \tag{4.12}$$

$$\overline{M}_8 = \frac{1}{|\Lambda|}T\overline{A}_2\big(|b_1| + |\Lambda|\big), \tag{4.13}$$

$$\overline{M}_9 = \frac{1}{|\Lambda|}\big[(T\overline{A}_0 + m\overline{A}_3)\big(|b_1| + |\Lambda|\big) + |\lambda_1|\big], \tag{4.14}$$

$$\overline{M}_{10} = \frac{1}{|\Phi|}T\overline{B}_1\big(|b_2| + |\Phi|\big), \tag{4.15}$$

$$\overline{M}_{11} = \frac{1}{|\Phi|}(T\overline{B}_2 + m\overline{B}_4)\big(|b_2| + |\Phi|\big), \tag{4.16}$$

$$\overline{M}_{12} = \frac{1}{|\Phi|}\big[(T\overline{B}_0 + m\overline{B}_3)\big(|b_2| + |\Phi|\big) + |\lambda_2|\big]. \tag{4.17}$$

Theorem 4.2 *Assume that*:

(H_7) *The functions* $f, g : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ *are continuous and there exist constants* $\overline{A}_i, \overline{B}_i \geq 0$ $(i = 1,2)$ *and* $\overline{A}_0, \overline{B}_0 > 0$ *such that* $\forall x_i \in \mathbb{R}$ $(i = 1,2)$

$$\big|f(t,x_1,x_2)\big| \leq \overline{A}_0 + \overline{A}_1|x_1| + \overline{A}_2|x_2|$$

and

$$\big|g(t,x_1,x_2)\big| \le \overline{B}_0 + \overline{B}_1|x_1| + \overline{B}_2|x_2|.$$

(H_8) *The functions* $I_k, I_k^* : \mathbb{R} \to \mathbb{R}$ *are continuous and there exist constants* $\overline{A}_4, \overline{B}_4 \ge 0$ *and* $\overline{A}_3, \overline{B}_3 > 0$ *such that* $\forall x_3 \in \mathbb{R}$, $k = 1,2,\ldots,m$,

$$\big|I_k(x_3)\big| \le \overline{A}_3 + \overline{A}_4|x_3|$$

and

$$\big|I_k^*(x_3)\big| \le \overline{B}_3 + \overline{B}_4|x_3|.$$

In addition it is assumed that

$$\overline{M}_7 + \overline{M}_{10} < 1 \quad \text{and} \quad \overline{M}_8 + \overline{M}_{11} < 1,$$

where $\overline{M}_7, \overline{M}_8, \overline{M}_{10}, \overline{M}_{11}$ *are given by* (4.12)-(4.13) *and* (4.15)-(4.16), *respectively. Then the boundary value problem* (4.1) *has at least one solution.*

Proof Setting

$$\overline{M}_0 = \min\big\{1 - (\overline{M}_7 + \overline{M}_{10}), 1 - (\overline{M}_8 + \overline{M}_{11})\big\},$$

the proof is similar to that of Theorem 3.2. So we omit it. □

Example 4.2 Consider the following coupled system of impulsive quantum difference equations with uncoupled boundary conditions:

$$\begin{cases} D_{\frac{2}{3+k}} x(t) = 3 + \frac{1}{20}\sin(\frac{x(t)}{2}) + \frac{1}{(t+3)^2}\tan^{-1}(\frac{y(t)}{4}), & t \in [0,1], t \neq t_k, \\ D_{\frac{3+k}{5+2k+k^2}} y(t) = 4 + \frac{t}{10(2^t+1)}x(t) + \frac{\sin(\pi t)}{2(2t+5)^2}y(t), & t \in [0,1], t \neq t_k, \\ \Delta x(t_k) = \frac{1}{10\pi^2}\sin(\frac{\pi x(t_k)}{2}) + \frac{1}{2}, & t_k = \frac{k}{10}, k = 1,2,\ldots,9, \\ \Delta y(t_k) = \frac{1}{5}\sin(\frac{y(t_k)}{6}) + 3, & t_k = \frac{k}{10}, k = 1,2,\ldots,9, \\ 2x(0) - 7x(1) = -3, \qquad 3y(0) - 5y(1) = -10. \end{cases} \tag{4.18}$$

Here $q_k = 2/(3+k)$, $p_k = (3+k)/(5+2k+k^2)$, $k = 0,1,2,\ldots,9$, $m = 9$, $T = 1$, $a_1 = 2$, $a_2 = 3$, $b_1 = -7$, $b_2 = -5$, $\lambda_1 = -3$, $\lambda_2 = -10$, $f(t,x,y) = 3 + (\sin(x/2))/20 + (\tan^{-1}(y/4))/(t+3)^2$, $g(t,x,y) = 4 + (tx)/(10(2^t+1)) + (\sin(\pi t)y)/(2(2t+5)^2)$, $I_k(x) = (\sin(\pi x/2))/(10\pi^2) + 1/2$, and $I_k^*(y) = y/(20e + y^2) + \pi/4$. We get

$$\Lambda = a_1 + b_1 = -5 \neq 0 \quad \text{and} \quad \Phi = a_2 + b_2 = -2 \neq 0.$$

Since $|f(t,x,y)| \le \overline{A}_0 + \overline{A}_1|x| + \overline{A}_2|y|$, $|g(t,x,y)| \le \overline{B}_0 + \overline{B}_1|x| + \overline{B}_2|y|$, where $\overline{A}_0 = 3$, $\overline{A}_1 = 1/40$, $\overline{A}_2 = 1/36$, $\overline{B}_0 = 4$, $\overline{B}_1 = 1/20$, $\overline{B}_2 = 1/50$, it is found that $\overline{M}_7 \simeq 0.40377$, $\overline{M}_8 \simeq 0.06667$, $\overline{M}_{10} \simeq 0.175$, $\overline{M}_{11} \simeq 0.64941$. Furthermore,

$$\overline{M}_7 + \overline{M}_{10} \approx 0.57877 < 1$$

and

$$\overline{M}_8 + \overline{M}_{11} \approx 0.71608 < 1.$$

Thus all the conditions of Theorem 4.2 holds true and consequently the conclusion of Theorem 4.2; problem (4.18) has at least one solution on $[0,1]$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally in this article. They read and approved the final manuscript.

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