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Hyers-Ulam stability of the first-order matrix difference equations

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Abstract

In this paper, we prove the Hyers-Ulam stability of the first-order linear homogeneous matrix difference equations $\vec{x}_i = \mathbf{A}\vec{x}_{i-1}$ and $\vec{x}_{i-1} = \mathbf{A}\vec{x}_i$ for all integers $i \in \mathbf{Z}$.

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1 Introduction

Throughout this paper, let n be a fixed positive integer. The nth order linear homogeneous difference equation with constant coefficients is of the form

$$a_i = \alpha_1 a_{i-1} + \alpha_2 a_{i-2} + \cdots + \alpha_n a_{i-n}, \tag{1}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are constants. For example, the second-order difference equation with constant coefficients has the form

$$a_i = \alpha a_{i-1} + \beta a_{i-2}. \tag{2}$$

The solution of (2) is called the Fibonacci numbers when $\alpha = \beta = 1$, $a_0 = 0$, and $a_1 = 1$, Lucas numbers when $\alpha = \beta = 1$, $a_0 = 2$, and $a_1 = 1$, Pell numbers when $\alpha = 2$, $\beta = 1$, $a_0 = 0$, and $a_1 = 1$, Pell-Lucas numbers when $\alpha = 2$, $\beta = 1$, and $a_0 = a_1 = 2$, and Jacobsthal numbers if $\alpha = 1$, $\beta = 2$, $a_0 = 0$, and $a_1 = 1$.

The polynomial

$$p(x) = x^n - \alpha_1 x^{n-1} - \alpha_2 x^{n-2} - \cdots - \alpha_{n-1} x - \alpha_n$$

is called the characteristic polynomial of the difference equation (1).

If the roots $r_1, r_2, \ldots, r_n$ of the characteristic polynomial are distinct, then the solution of the difference equation (1) is given by

$$a_i = k_1 r_1^i + k_2 r_2^i + \cdots + k_n r_n^i,$$

where the coefficients $k_1, k_2, \ldots, k_n$ are uniquely determined under the initial conditions of the difference equation.

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If the characteristic polynomial has roots $r_1, r_2, \ldots, r_d$ with multiplicity $m_1, m_2, \ldots, m_d$, respectively, then the solution of the difference equation (1) is given by

$$a_i = \sum_{j=1}^{d} \sum_{k=1}^{m_j} c_{jk} i^{k-1} r_j^i,$$

where the c_{jk} are constants and $m_1 + m_2 + \cdots + m_d = n$ (see [1, 2]). For the Hyers-Ulam stability of the linear difference equations, we may refer to [3–9].

Let $(\mathbf{C}^n, \|\cdot\|_n)$ be a complex normed space, each of whose elements is a column vector, and let $\mathbf{C}^{n\times n}$ be a vector space consisting of all $(n \times n)$ complex matrices. We choose a norm $\|\cdot\|_{n\times n}$ on $\mathbf{C}^{n\times n}$ which is compatible with $\|\cdot\|_n$, *i.e.*, both norms obey

$$\|\mathbf{AB}\|_{n\times n} \le \|\mathbf{A}\|_{n\times n}\|\mathbf{B}\|_{n\times n} \quad \text{and} \quad \|\mathbf{A}\vec{x}\|_n \le \|\mathbf{A}\|_{n\times n}\|\vec{x}\|_n \tag{3}$$

for all $\mathbf{A}, \mathbf{B} \in \mathbf{C}^{n\times n}$ and $\vec{x} \in \mathbf{C}^n$.

A matrix difference equation is a difference equation with matrix coefficients in which the value of vector of variables at one point is dependent on the values of preceding (succeeding) points.

In this paper, we prove the Hyers-Ulam stability of the first-order linear homogeneous matrix difference equations $\vec{x}_i = \mathbf{A}\vec{x}_{i-1}$ and $\vec{x}_{i-1} = \mathbf{A}\vec{x}_i$ for all integers $i \in \mathbf{Z}$, where the transition matrix $\mathbf{A}$ is nonsingular. More precisely, we prove that if a sequence $\{\vec{y}_i\}_{i\in\mathbf{Z}}$ satisfies the inequality $\|\vec{y}_i - \mathbf{A}\vec{y}_{i-1}\|_n \le \varepsilon$ for all $i \in \mathbf{Z}$ resp. $\|\vec{y}_{i-1} - \mathbf{A}\vec{y}_i\|_n \le \varepsilon$ for all $i \in \mathbf{Z}$, then there exist a solution $\{\vec{x}_i\}_{i\in\mathbf{Z}} \subset \mathbf{C}^n$ of the first-order matrix difference equation (4) resp. (17) and a constant $K > 0$ such that $\|\vec{y}_i - \vec{x}_i\|_n \le K\varepsilon$ for all integers $i \ge 0$. (We refer the reader to [10–14] for the exact definition of Hyers-Ulam stability.)

It should be remarked that many interesting theorems have been proved in [15, 16] concerning the linear (or nonlinear) recurrences. Especially in 2015, the Hyers-Ulam stability of the first-order matrix difference equations has been proved in [17] in a general setting. The substantial difference of this paper from [17] lies in the fact that the stability problems for the 'backward' difference equations have been treated in Section 3 of this paper.

2 Hyers-Ulam stability of $\vec{x}_i = \mathbf{A}\vec{x}_{i-1}$

In this section, we investigate the Hyers-Ulam stability of the first-order linear homogeneous matrix difference equation

$$\vec{x}_i = \mathbf{A}\vec{x}_{i-1} \tag{4}$$

for all integers $i \in \mathbf{Z}$, where

$$\vec{x}_i = \begin{pmatrix} x_{i1} \\ x_{i2} \\ \vdots \\ x_{in} \end{pmatrix} \in \mathbf{C}^n \quad \text{and} \quad \mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \in \mathbf{C}^{n\times n}.$$

Theorem 2.1 *Given a fixed positive integer n, let $(\mathbf{C}^n, \|\cdot\|_n)$ and $(\mathbf{C}^{n\times n}, \|\cdot\|_{n\times n})$ be complex normed spaces, whose elements are column vectors resp. $(n \times n)$ complex matrices, with the* *property* (3). *Assume that the transition matrix* $\mathbf{A} \in \mathbf{C}^{n\times n}$ *is nonsingular and* $\{\varepsilon_i\}_{i\in\mathbf{Z}}$ *is a sequence of nonnegative real numbers. If a sequence* $\{\vec{y}_i\}_{i\in\mathbf{Z}} \subset \mathbf{C}^n$ *satisfies the inequality*

$$\|\vec{y}_i - \mathbf{A}\vec{y}_{i-1}\|_n \le \varepsilon_i \tag{5}$$

for all $i \in \mathbf{Z}$, *then there exists a solution* $\{\vec{x}_i\}_{i\in\mathbf{Z}} \subset \mathbf{C}^n$ *of the first-order matrix difference equation* (4) *such that*

$$\|\vec{y}_i - \vec{x}_i\|_n \le \begin{cases} \sum_{k=1}^{i} \varepsilon_k \|\mathbf{A}\|_{n\times n}^{i-k} + \|\mathbf{A}\|_{n\times n}^{i} \|\vec{y}_0 - \vec{x}_0\|_n & (\textit{for } i \ge 0), \\ \sum_{k=1}^{-i} \varepsilon_{k+i} \|\mathbf{A}^{-1}\|_{n\times n}^{k} + \|\mathbf{A}^{-1}\|_{n\times n}^{-i} \|\vec{y}_0 - \vec{x}_0\|_n & (\textit{for } i < 0). \end{cases}$$

Proof Assume that a sequence $\{\vec{y}_i\}_{i\in\mathbf{Z}} \subset \mathbf{C}^n$ satisfies the inequality (5) for all $i \in \mathbf{Z}$. First, we assume that i is a nonnegative integer. It then follows from (3) and (5) that

$$\begin{aligned} \|\vec{y}_i - \mathbf{A}^i\vec{y}_0\|_n &\le \|\vec{y}_i - \mathbf{A}\vec{y}_{i-1}\|_n + \|\mathbf{A}\vec{y}_{i-1} - \mathbf{A}^2\vec{y}_{i-2}\|_n \\ &\quad + \|\mathbf{A}^2\vec{y}_{i-2} - \mathbf{A}^3\vec{y}_{i-3}\|_n + \cdots + \|\mathbf{A}^{i-1}\vec{y}_1 - \mathbf{A}^i\vec{y}_0\|_n \\ &\le \|\vec{y}_i - \mathbf{A}\vec{y}_{i-1}\|_n + \|\mathbf{A}\|_{n\times n}\|\vec{y}_{i-1} - \mathbf{A}\vec{y}_{i-2}\|_n \\ &\quad + \|\mathbf{A}\|_{n\times n}^2\|\vec{y}_{i-2} - \mathbf{A}\vec{y}_{i-3}\|_n + \cdots + \|\mathbf{A}\|_{n\times n}^{i-1}\|\vec{y}_1 - \mathbf{A}\vec{y}_0\|_n \\ &\le \varepsilon_i + \|\mathbf{A}\|_{n\times n}\varepsilon_{i-1} + \|\mathbf{A}\|_{n\times n}^2\varepsilon_{i-2} + \cdots + \|\mathbf{A}\|_{n\times n}^{i-1}\varepsilon_1 \\ &= \|\mathbf{A}\|_{n\times n}^i \sum_{k=1}^{i} \varepsilon_k \|\mathbf{A}\|_{n\times n}^{-k}. \end{aligned} \tag{6}$$

It is obvious that a sequence $\{\vec{x}_i\}_{i\in\mathbf{Z}} \subset \mathbf{C}^n$ satisfies the first-order matrix difference equation (4) if and only if

$$\vec{x}_i = \mathbf{A}^i\vec{x}_0 \tag{7}$$

for each $i \in \mathbf{Z}$, where we set $\mathbf{A}^i = (\mathbf{A}^{-1})^{-i}$ for all negative integers i. Hence, by (6) and (7), we have

$$\begin{aligned} \|\vec{y}_i - \vec{x}_i\|_n &\le \|\vec{y}_i - \mathbf{A}^i\vec{y}_0\|_n + \|\mathbf{A}^i\vec{y}_0 - \mathbf{A}^i\vec{x}_0\|_n + \|\mathbf{A}^i\vec{x}_0 - \vec{x}_i\|_n \\ &\le \sum_{k=1}^{i} \varepsilon_k \|\mathbf{A}\|_{n\times n}^{i-k} + \|\mathbf{A}\|_{n\times n}^i \|\vec{y}_0 - \vec{x}_0\|_n \end{aligned}$$

for any integer $i \ge 0$.

On the other hand, we suppose i is a negative integer. For this case, it follows from (3) and (5) that

$$\begin{aligned} &\|\vec{y}_i - \mathbf{A}^i\vec{y}_0\|_n \\ &\quad = \|\vec{y}_i - (\mathbf{A}^{-1})^{-i}\vec{y}_0\|_n \\ &\quad \le \|\vec{y}_i - \mathbf{A}^{-1}\vec{y}_{i+1}\|_n + \|\mathbf{A}^{-1}\vec{y}_{i+1} - (\mathbf{A}^{-1})^2\vec{y}_{i+2}\|_n \\ &\qquad + \|(\mathbf{A}^{-1})^2\vec{y}_{i+2} - (\mathbf{A}^{-1})^3\vec{y}_{i+3}\|_n + \cdots + \|(\mathbf{A}^{-1})^{-i-1}\vec{y}_{-1} - (\mathbf{A}^{-1})^{-i}\vec{y}_0\|_n \end{aligned}$$

$$
\begin{aligned}
&\leq \left\|\mathbf{A}^{-1}\right\|_{n\times n}\|\mathbf{A}\vec{y}_i - \vec{y}_{i+1}\|_n + \left\|\mathbf{A}^{-1}\right\|_{n\times n}^{2}\|\mathbf{A}\vec{y}_{i+1} - \vec{y}_{i+2}\|_n \\
&\quad + \left\|\mathbf{A}^{-1}\right\|_{n\times n}^{3}\|\mathbf{A}\vec{y}_{i+2} - \vec{y}_{i+3}\|_n + \cdots + \left\|\mathbf{A}^{-1}\right\|_{n\times n}^{-i}\|\mathbf{A}\vec{y}_{-1} - \vec{y}_0\|_n \\
&\leq \left\|\mathbf{A}^{-1}\right\|_{n\times n}\varepsilon_{i+1} + \left\|\mathbf{A}^{-1}\right\|_{n\times n}^{2}\varepsilon_{i+2} + \left\|\mathbf{A}^{-1}\right\|_{n\times n}^{3}\varepsilon_{i+3} + \cdots + \left\|\mathbf{A}^{-1}\right\|_{n\times n}^{-i}\varepsilon_0 \\
&= \sum_{k=1}^{-i}\varepsilon_{k+i}\left\|\mathbf{A}^{-1}\right\|_{n\times n}^{k}. \qquad (8)
\end{aligned}
$$

Moreover, by (7) and (8), we have

$$
\begin{aligned}
\|\vec{y}_i - \vec{x}_i\|_n &\leq \left\|\vec{y}_i - \left(\mathbf{A}^{-1}\right)^{-i}\vec{y}_0\right\|_n + \left\|\left(\mathbf{A}^{-1}\right)^{-i}\vec{y}_0 - \left(\mathbf{A}^{-1}\right)^{-i}\vec{x}_0\right\|_n \\
&\quad + \left\|\left(\mathbf{A}^{-1}\right)^{-i}\vec{x}_0 - \vec{x}_i\right\|_n \\
&\leq \sum_{k=1}^{-i}\varepsilon_{k+i}\left\|\mathbf{A}^{-1}\right\|_{n\times n}^{k} + \left\|\mathbf{A}^{-1}\right\|_{n\times n}^{-i}\|\vec{y}_0 - \vec{x}_0\|_n
\end{aligned}
$$

for all integers $i < 0$. □

In view of (7), if we assume the initial condition in the previous theorem, we can easily prove the uniqueness of the sequence $\{\vec{x}_i\}_{i\in\mathbf{Z}}$ as we see in the following corollary.

Corollary 2.2 *Given a fixed positive integer n, let $(\mathbf{C}^n, \|\cdot\|_n)$ and $(\mathbf{C}^{n\times n}, \|\cdot\|_{n\times n})$ be complex normed spaces, whose elements are column vectors resp. $(n \times n)$ complex matrices, with the property* (3). *Assume that the transition matrix $\mathbf{A} \in \mathbf{C}^{n\times n}$ is nonsingular and $\{\varepsilon_i\}_{i\in\mathbf{Z}}$ is a sequence of nonnegative real numbers. If a sequence $\{\vec{y}_i\}_{i\in\mathbf{Z}} \subset \mathbf{C}^n$ satisfies the inequality* (5) *for all $i \in \mathbf{Z}$, then there exists a unique solution $\{\vec{x}_i\}_{i\in\mathbf{Z}} \subset \mathbf{C}^n$ of the first-order matrix difference equation* (4) *with the initial condition $\vec{x}_0 = \vec{y}_0$ such that*

$$
\|\vec{y}_i - \vec{x}_i\|_n \leq \begin{cases} \sum_{k=1}^{i}\varepsilon_k\|\mathbf{A}\|_{n\times n}^{i-k} & (\textit{for } i \geq 0), \\ \sum_{k=1}^{-i}\varepsilon_{k+i}\|\mathbf{A}^{-1}\|_{n\times n}^{k} & (\textit{for } i < 0). \end{cases}
$$

Some of the most important matrix norms are induced by p-norms. For $1 \leq p \leq \infty$, the matrix norm induced by the p-norm,

$$
\|\mathbf{A}\|_p := \sup_{\vec{x}\neq\vec{0}}\frac{\|\mathbf{A}\vec{x}\|_p}{\|\vec{x}\|_p},
$$

is called the matrix p-norm. For example, we get

$$
\|\mathbf{A}\|_1 = \max_{1\leq j\leq n}\sum_{i=1}^{n}|a_{ij}| \quad \text{and} \quad \|\mathbf{A}\|_\infty = \max_{1\leq i\leq n}\sum_{j=1}^{n}|a_{ij}|.
$$

It is well known that the matrix p-norm, together with the p-norm, satisfies the conditions in (3), where

$$
\|\vec{x}\|_1 = \sum_{j=1}^{n}|x_j| \quad \text{and} \quad \|\vec{x}\|_\infty = \max_{1\leq j\leq n}|x_j|
$$

for any $\vec{x} \in \mathbf{C}^n$.

In the following corollary, we prove the Hyers-Ulam stability of the second-order linear homogeneous difference equation with constant coefficients.

Corollary 2.3 *Let* $(\mathbf{C}^2, \|\cdot\|_\infty)$ *and* $(\mathbf{C}^{2\times 2}, \|\cdot\|_\infty)$ *be complex normed spaces and let* α, β, γ *be complex numbers satisfying the conditions*

$$\alpha^2 + 4\beta \neq 0, \qquad \beta \neq 0, \qquad \gamma \neq 0. \tag{9}$$

Assume that $\varepsilon > 0$ *is an arbitrary constant. If a sequence* $\{a_i\}_{i\in\mathbf{Z}}$ *of complex numbers satisfies the inequality*

$$|a_i - \alpha a_{i-1} - \beta a_{i-2}| \leq \varepsilon \tag{10}$$

for all $i \in \mathbf{Z}$, *then there exists a sequence* $\{c_i\}_{i\in\mathbf{Z}}$ *of complex numbers such that* $c_{-1} = a_{-1}$, $c_0 = a_0$, $c_i = \alpha c_{i-1} + \beta c_{i-2}$, *and*

$$|a_i - c_i| \leq \begin{cases} \sum_{k=1}^{i} \varepsilon \|\mathbf{A}\|_\infty^{i-k} & (\text{for } i \geq 0), \\ \sum_{k=1}^{-i} \varepsilon \|\mathbf{A}^{-1}\|_\infty^{k} & (\text{for } i < 0), \end{cases}$$

where $\|\mathbf{A}\|_\infty = \max\{|\alpha| + |\beta/\gamma|, |\gamma|\}$ *and* $\|\mathbf{A}^{-1}\|_\infty = \max\{|1/\gamma|, |\alpha/\beta| + |\gamma/\beta|\}$.

Proof If we define a sequence $\{b_i\}_{i\in\mathbf{Z}}$ of complex numbers by $b_i = \gamma a_{i-1}$, it then follows from (10) that

$$\begin{cases} |a_i - \alpha a_{i-1} - \frac{\beta}{\gamma} b_{i-1}| \leq \varepsilon, \\ |b_i - \gamma a_{i-1}| = 0 \end{cases}$$

for any $i \in \mathbf{Z}$. If we set

$$\vec{y}_i := \begin{pmatrix} a_i \\ b_i \end{pmatrix} \quad \text{and} \quad \mathbf{A} := \begin{pmatrix} \alpha & \frac{\beta}{\gamma} \\ \gamma & 0 \end{pmatrix},$$

then we get

$$\|\vec{y}_i - \mathbf{A}\vec{y}_{i-1}\|_\infty \leq \varepsilon$$

for each $i \in \mathbf{Z}$.

According to Corollary 2.2, there exists a unique solution $\{\vec{x}_i\}_{i\in\mathbf{Z}} \subset \mathbf{C}^2$ of the first-order matrix difference equation (4) with the initial condition $\vec{x}_0 = \binom{a_0}{\gamma a_{-1}}$ such that

$$\|\vec{y}_i - \vec{x}_i\|_\infty \leq \begin{cases} \sum_{k=1}^{i} \varepsilon \|\mathbf{A}\|_\infty^{i-k} & (\text{for } i \geq 0), \\ \sum_{k=1}^{-i} \varepsilon \|\mathbf{A}^{-1}\|_\infty^{k} & (\text{for } i < 0). \end{cases}$$

In view of (7), this last inequality implies that

$$\left\| \begin{pmatrix} a_i \\ \gamma a_{i-1} \end{pmatrix} - \mathbf{A}^i \begin{pmatrix} a_0 \\ \gamma a_{-1} \end{pmatrix} \right\|_\infty \leq \begin{cases} \sum_{k=1}^{i} \varepsilon \|\mathbf{A}\|_\infty^{i-k} & (\text{for } i \geq 0), \\ \sum_{k=1}^{-i} \varepsilon \|\mathbf{A}^{-1}\|_\infty^{k} & (\text{for } i < 0). \end{cases} \tag{11}$$

Since the transition matrix $\mathbf{A}$ has two distinct eigenvalues $\lambda_1 = \frac{\alpha-\sqrt{\alpha^2+4\beta}}{2}$ and $\lambda_2 = \frac{\alpha+\sqrt{\alpha^2+4\beta}}{2}$, which are the roots of the characteristic equation $\lambda^2 - \alpha\lambda - \beta = 0$, the matrix $\mathbf{A}$ can be expressed as

$$\mathbf{A} = \mathbf{C}\mathbf{D}\mathbf{C}^{-1} \tag{12}$$

with

$$\mathbf{C} = \begin{pmatrix} \lambda_1 & \lambda_2 \\ \gamma & \gamma \end{pmatrix}, \qquad \mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \qquad \mathbf{C}^{-1} = \frac{1}{\gamma(\lambda_1 - \lambda_2)} \begin{pmatrix} \gamma & -\lambda_2 \\ -\gamma & \lambda_1 \end{pmatrix}.$$

By (12), we obtain

$$\begin{aligned} \mathbf{A}^i &= \mathbf{C}\mathbf{D}^i\mathbf{C}^{-1} \\ &= \frac{1}{\gamma(\lambda_1 - \lambda_2)} \begin{pmatrix} \lambda_1 & \lambda_2 \\ \gamma & \gamma \end{pmatrix} \begin{pmatrix} \lambda_1^i & 0 \\ 0 & \lambda_2^i \end{pmatrix} \begin{pmatrix} \gamma & -\lambda_2 \\ -\gamma & \lambda_1 \end{pmatrix} \\ &= \frac{1}{\gamma(\lambda_1 - \lambda_2)} \begin{pmatrix} \gamma(\lambda_1^{i+1} - \lambda_2^{i+1}) & -\lambda_1\lambda_2(\lambda_1^i - \lambda_2^i) \\ \gamma^2(\lambda_1^i - \lambda_2^i) & -\gamma\lambda_1\lambda_2(\lambda_1^{i-1} - \lambda_2^{i-1}) \end{pmatrix} \end{aligned}$$

for every integer $i \geq 0$. Using this equality, it follows from (11) that

$$\left\| \begin{pmatrix} a_i - \frac{a_0 - a_{-1}\lambda_2}{\lambda_1 - \lambda_2}\lambda_1^{i+1} + \frac{a_0 - a_{-1}\lambda_1}{\lambda_1 - \lambda_2}\lambda_2^{i+1} \\ \gamma a_{i-1} - \gamma\frac{a_0 - a_{-1}\lambda_2}{\lambda_1 - \lambda_2}\lambda_1^i + \gamma\frac{a_0 - a_{-1}\lambda_1}{\lambda_1 - \lambda_2}\lambda_2^i \end{pmatrix} \right\|_\infty \leq \sum_{k=1}^{i} \varepsilon \|\mathbf{A}\|_\infty^{i-k} \tag{13}$$

for all integers $i \geq 0$.

On the other hand, the inverse matrix $\mathbf{A}^{-1}$ has two distinct eigenvalues $\omega_1 = \frac{-\alpha-\sqrt{\alpha^2+4\beta}}{2\beta} = -\frac{1}{\beta}\lambda_2$ and $\omega_2 = \frac{-\alpha+\sqrt{\alpha^2+4\beta}}{2\beta} = -\frac{1}{\beta}\lambda_1$, which are roots of the characteristic equation $\omega^2 + \frac{\alpha}{\beta}\omega - \frac{1}{\beta} = 0$. Hence, the matrix $\mathbf{A}^{-1}$ may be expressed as

$$\mathbf{A}^{-1} = \begin{pmatrix} 0 & \frac{1}{\gamma} \\ \frac{\gamma}{\beta} & -\frac{\alpha}{\beta} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \gamma\omega_1 & \gamma\omega_2 \end{pmatrix} \begin{pmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \gamma\omega_1 & \gamma\omega_2 \end{pmatrix}^{-1}. \tag{14}$$

Using (14), we have

$$\begin{aligned} \mathbf{A}^i &= (\mathbf{A}^{-1})^{-i} \\ &= \begin{pmatrix} 1 & 1 \\ \gamma\omega_1 & \gamma\omega_2 \end{pmatrix} \begin{pmatrix} \omega_1^{-i} & 0 \\ 0 & \omega_2^{-i} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \gamma\omega_1 & \gamma\omega_2 \end{pmatrix}^{-1} \\ &= \frac{-1}{\gamma(\omega_1 - \omega_2)} \begin{pmatrix} \gamma\omega_1\omega_2(\omega_1^{-i-1} - \omega_2^{-i-1}) & \omega_2^{-i} - \omega_1^{-i} \\ \gamma^2\omega_1\omega_2(\omega_1^{-i} - \omega_2^{-i}) & \gamma(\omega_2^{1-i} - \omega_1^{1-i}) \end{pmatrix} \end{aligned}$$

for all integers $i < 0$. Thus, the inequality (11) yields

$$\left\| \begin{pmatrix} a_i - \frac{a_{-1} - a_0\omega_2}{\omega_1 - \omega_2}\omega_1^{-i} + \frac{a_{-1} - a_0\omega_1}{\omega_1 - \omega_2}\omega_2^{-i} \\ \gamma a_{i-1} - \gamma\frac{a_{-1} - a_0\omega_2}{\omega_1 - \omega_2}\omega_1^{1-i} + \gamma\frac{a_{-1} - a_0\omega_1}{\omega_1 - \omega_2}\omega_2^{1-i} \end{pmatrix} \right\|_\infty \leq \sum_{k=1}^{-i} \varepsilon \left\|\mathbf{A}^{-1}\right\|_\infty^k \tag{15}$$

for any integer $i < 0$.

Finally, considering (13), (15), and [18], Theorem 10.1, if we set

$$c_i := \begin{cases} \frac{a_0-a_{-1}\lambda_2}{\lambda_1-\lambda_2}\lambda_1^{i+1} - \frac{a_0-a_{-1}\lambda_1}{\lambda_1-\lambda_2}\lambda_2^{i+1} & (\text{for } i \geq 0), \\ \frac{a_{-1}-a_0\omega_2}{\omega_1-\omega_2}\omega_1^{-i} - \frac{a_{-1}-a_0\omega_1}{\omega_1-\omega_2}\omega_2^{-i} & (\text{for } i < 0), \end{cases}$$

then we get $c_{-1} = a_{-1}$, $c_0 = a_0$, and it follows from (13) and (15) that

$$|a_i - c_i| \leq \begin{cases} \sum_{k=1}^{i} \varepsilon \|\mathbf{A}\|_\infty^{i-k} & (\text{for } i \geq 0), \\ \sum_{k=1}^{-i} \varepsilon \|\mathbf{A}^{-1}\|_\infty^{k} & (\text{for } i < 0). \end{cases}$$

Furthermore, it is not difficult to show that the sequence $\{c_i\}_{i\in\mathbf{Z}}$ satisfies the second-order linear difference equation

$$c_i = \alpha c_{i-1} + \beta c_{i-2}$$

for any integer i. □

If we set $\gamma = \frac{\pm\alpha\pm\sqrt{\alpha^2+4\beta}}{2}$ in Corollary 2.3, then we get

$$\lim_{\beta\to\infty} \|\mathbf{A}\|_\infty \cdot \left\|\mathbf{A}^{-1}\right\|_\infty = 1.$$

For example, if we set $\gamma = \frac{\alpha+\sqrt{\alpha^2+4\beta}}{2}$ and $\beta > 0$, then we have

$$\begin{cases} \|\mathbf{A}\|_\infty = \max\left\{\frac{\alpha+\sqrt{\alpha^2+4\beta}}{2}, \frac{-3\alpha+\sqrt{\alpha^2+4\beta}}{2}\right\}, \\ \|\mathbf{A}^{-1}\|_\infty = \max\left\{\frac{3\alpha+\sqrt{\alpha^2+4\beta}}{2\beta}, \frac{-\alpha+\sqrt{\alpha^2+4\beta}}{2\beta}\right\}, \end{cases} \tag{16}$$

and hence

$$\lim_{\beta\to\infty} \|\mathbf{A}\|_\infty \cdot \left\|\mathbf{A}^{-1}\right\|_\infty = \lim_{\beta\to\infty} \sqrt{\beta} \cdot \frac{1}{\sqrt{\beta}} = 1.$$

For the case when $\gamma = \frac{\alpha-\sqrt{\alpha^2+4\beta}}{2}$, $\gamma = \frac{-\alpha+\sqrt{\alpha^2+4\beta}}{2}$, or $\gamma = \frac{-\alpha-\sqrt{\alpha^2+4\beta}}{2}$, we analogously obtain $\lim_{\beta\to\infty} \|\mathbf{A}\|_\infty \cdot \|\mathbf{A}^{-1}\|_\infty = 1$.

If α and β are simultaneously small in absolute value, then the second-order difference equation (2) has the Hyers-Ulam stability as we see in the following example.

Example 2.4 Given an $\varepsilon > 0$, assume that a sequence $\{a_i\}_{i\in\mathbf{Z}}$ of complex numbers satisfies the inequality

$$\left| a_i - \frac{1}{3}a_{i-1} - \frac{1}{4}a_{i-2} \right| \leq \varepsilon$$

for all $i \in \mathbf{Z}$. With $\alpha = \frac{1}{3}$ and $\beta = \frac{1}{4}$, it follows from (16) that $\|\mathbf{A}\|_\infty = \frac{1+\sqrt{10}}{6}$ and $\|\mathbf{A}^{-1}\|_\infty = \frac{6+2\sqrt{10}}{3}$. Using these values, Corollary 2.3 implies that there exists a sequence $\{c_i\}_{i\in\mathbf{Z}}$ of complex numbers such that $c_{-1} = a_{-1}$, $c_0 = a_0$, $c_i = \frac{1}{3}c_{i-1} + \frac{1}{4}c_{i-2}$, and

$$|a_i - c_i| \leq \begin{cases} \frac{2\sqrt{10}+10}{5}\left(1 - \left(\frac{\sqrt{10}+1}{6}\right)^i\right)\varepsilon & (\text{for } i \geq 0), \\ \frac{6\sqrt{10}+22}{31}\left(\left(\frac{2\sqrt{10}+6}{3}\right)^{-i} - 1\right)\varepsilon & (\text{for } i < 0). \end{cases}$$

3 Hyers-Ulam stability of $\vec{x}_{i-1} = \mathbf{A}\vec{x}_i$

In practical applications, we sometimes consider the first-order linear homogeneous matrix difference equation

$$\vec{x}_{i-1} = \mathbf{A}\vec{x}_i \tag{17}$$

instead of (4), where the transition matrix $\mathbf{A}$ is a nonsingular matrix of $\mathbf{C}^{n\times n}$.

We now investigate the Hyers-Ulam stability of the matrix difference equation (17).

Theorem 3.1 *Given a fixed positive integer n, let $(\mathbf{C}^n, \|\cdot\|_n)$ and $(\mathbf{C}^{n\times n}, \|\cdot\|_{n\times n})$ be complex normed spaces, whose elements are column vectors resp. $(n\times n)$ complex matrices, with the property* (3). *Assume that the transition matrix $\mathbf{A}\in\mathbf{C}^{n\times n}$ is nonsingular and $\{\varepsilon_i\}_{i\in\mathbf{Z}}$ is a sequence of nonnegative real numbers. If a sequence $\{\vec{y}_i\}_{i\in\mathbf{Z}}\subset\mathbf{C}^n$ satisfies the inequality*

$$\|\vec{y}_{i-1} - \mathbf{A}\vec{y}_i\|_n \le \varepsilon_i \tag{18}$$

for all $i\in\mathbf{Z}$, then there exists a solution $\{\vec{x}_i\}_{i\in\mathbf{Z}}\subset\mathbf{C}^n$ of the first-order matrix difference equation (17) *such that*

$$\|\vec{y}_i - \vec{x}_i\|_n \le \begin{cases} \sum_{k=1}^{i} \varepsilon_k \|\mathbf{A}^{-1}\|_{n\times n}^{i+1-k} + \|\mathbf{A}^{-1}\|_{n\times n}^{i}\|\vec{y}_0 - \vec{x}_0\|_n & (\textit{for } i\ge 0), \\ \sum_{k=1}^{-i} \varepsilon_{k+i} \|\mathbf{A}\|_{n\times n}^{k-1} + \|\mathbf{A}\|_{n\times n}^{-i}\|\vec{y}_0 - \vec{x}_0\|_n & (\textit{for } i< 0). \end{cases} \tag{19}$$

Proof Assume that a sequence $\{\vec{y}_i\}_{i\in\mathbf{Z}}\subset\mathbf{C}^n$ satisfies the inequality (18) for all $i\in\mathbf{Z}$. First, we assume that i is a nonnegative integer. Then, by (3) and (18), we have

$$\begin{aligned} \left\|\vec{y}_i - \mathbf{A}^{-i}\vec{y}_0\right\|_n &\le \left\|\vec{y}_i - \mathbf{A}^{-1}\vec{y}_{i-1}\right\|_n + \left\|\mathbf{A}^{-1}\vec{y}_{i-1} - \mathbf{A}^{-2}\vec{y}_{i-2}\right\|_n \\ &\quad + \left\|\mathbf{A}^{-2}\vec{y}_{i-2} - \mathbf{A}^{-3}\vec{y}_{i-3}\right\|_n + \cdots + \left\|\mathbf{A}^{-i+1}\vec{y}_1 - \mathbf{A}^{-i}\vec{y}_0\right\|_n \\ &\le \varepsilon_i\left\|\mathbf{A}^{-1}\right\|_{n\times n} + \varepsilon_{i-1}\left\|\mathbf{A}^{-1}\right\|_{n\times n}^2 + \varepsilon_{i-2}\left\|\mathbf{A}^{-1}\right\|_{n\times n}^3 + \cdots + \varepsilon_1\left\|\mathbf{A}^{-1}\right\|_{n\times n}^i \\ &= \sum_{k=1}^{i} \varepsilon_k \left\|\mathbf{A}^{-1}\right\|_{n\times n}^{i+1-k}. \end{aligned}$$

Obviously, a sequence $\{\vec{x}_i\}_{i\in\mathbf{Z}}\subset\mathbf{C}^n$ satisfies the first-order matrix difference equation (17) if and only if

$$\vec{x}_i = \mathbf{A}^{-i}\vec{x}_0 \tag{20}$$

for all $i\in\mathbf{Z}$, where we set $\mathbf{A}^{-i} = (\mathbf{A}^{-1})^i$ for each integer $i\ge 0$. Hence, we get

$$\begin{aligned} \|\vec{y}_i - \vec{x}_i\|_n &\le \left\|\vec{y}_i - \mathbf{A}^{-i}\vec{y}_0\right\|_n + \left\|\mathbf{A}^{-i}\vec{y}_0 - \mathbf{A}^{-i}\vec{x}_0\right\|_n + \left\|\mathbf{A}^{-i}\vec{x}_0 - \vec{x}_i\right\|_n \\ &\le \sum_{k=1}^{i} \varepsilon_k \left\|\mathbf{A}^{-1}\right\|_{n\times n}^{i+1-k} + \left\|\mathbf{A}^{-1}\right\|_{n\times n}^{i}\|\vec{y}_0 - \vec{x}_0\|_n \end{aligned}$$

for all integers $i\ge 0$.

On the other hand, if i is a negative integer, then it follows from (3) and (18) that

$$\begin{aligned}\left\|\vec{y}_i - \mathbf{A}^{-i}\vec{y}_0\right\|_n &= \|\vec{y}_i - \mathbf{A}\vec{y}_{i+1}\|_n + \left\|\mathbf{A}\vec{y}_{i+1} - \mathbf{A}^2\vec{y}_{i+2}\right\|_n \\ &\quad + \left\|\mathbf{A}^2\vec{y}_{i+2} - \mathbf{A}^3\vec{y}_{i+3}\right\|_n + \cdots + \left\|\mathbf{A}^{-i-1}\vec{y}_{-1} - \mathbf{A}^{-i}\vec{y}_0\right\|_n \\ &\le \varepsilon_{i+1} + \varepsilon_{i+2}\|\mathbf{A}\|_{n\times n} + \varepsilon_{i+3}\|\mathbf{A}\|_{n\times n}^2 + \cdots + \varepsilon_0\|\mathbf{A}\|_{n\times n}^{-i-1} \\ &= \sum_{k=1}^{-i} \varepsilon_{k+i}\|\mathbf{A}\|_{n\times n}^{k-1}.\end{aligned}$$

Thus, by (20) and the last inequality, we obtain

$$\begin{aligned}\|\vec{y}_i - \vec{x}_i\|_n &\le \left\|\vec{y}_i - \mathbf{A}^{-i}\vec{y}_0\right\|_n + \left\|\mathbf{A}^{-i}\vec{y}_0 - \mathbf{A}^{-i}\vec{x}_0\right\|_n + \left\|\mathbf{A}^{-i}\vec{x}_0 - \vec{x}_i\right\|_n \\ &\le \sum_{k=1}^{-i} \varepsilon_{k+i}\|\mathbf{A}\|_{n\times n}^{k-1} + \|\mathbf{A}\|_{n\times n}^{-i}\|\vec{y}_0 - \vec{x}_0\|_n\end{aligned}$$

for any integer $i < 0$. □

We now remark that if we apply Theorem 2.1 in place of the proof of Theorem 3.1, then we would obtain an inequality (21) below, which seems not to be better than the inequality (19) given in Theorem 3.1, as we see in the following remark, whose proof we omit.

Remark 3.2 Given a fixed positive integer n, let $(\mathbf{C}^n, \|\cdot\|_n)$ and $(\mathbf{C}^{n\times n}, \|\cdot\|_{n\times n})$ be complex normed spaces, whose elements are column vectors resp. $(n \times n)$ complex matrices, with the property (3). Assume that the transition matrix $\mathbf{A} \in \mathbf{C}^{n\times n}$ is nonsingular and $\{\varepsilon_i\}_{i\in\mathbf{Z}}$ is a sequence of nonnegative real numbers. If a sequence $\{\vec{y}_i\}_{i\in\mathbf{Z}} \subset \mathbf{C}^n$ satisfies the inequality (18) for all $i \in \mathbf{Z}$, then there exists a solution $\{\vec{x}_i\}_{i\in\mathbf{Z}} \subset \mathbf{C}^n$ of the first-order matrix difference equation (17) such that

$$\begin{aligned}&\|\vec{y}_i - \vec{x}_i\|_n \\ &\quad\le \begin{cases}\varepsilon_{i+1} + \sum_{k=1}^{i+1}\varepsilon_k\|\mathbf{A}^{-1}\|_{n\times n}^{i+1-k} + \|\mathbf{A}^{-1}\|_{n\times n}^{i+1}\|\mathbf{A}\vec{y}_0 - \vec{x}_{-1}\|_n & (\text{for } i \ge 0), \\ \varepsilon_{i+1} + \sum_{k=1}^{-i-1}\varepsilon_{k+i+1}\|\mathbf{A}\|_{n\times n}^{k} + \|\mathbf{A}\|_{n\times n}^{-i-1}\|\mathbf{A}\vec{y}_0 - \vec{x}_{-1}\|_n & (\text{for } i < 0).\end{cases}\end{aligned} \tag{21}$$

In view of (20), assuming the initial condition in the previous theorem leads to the uniqueness of the sequence $\{\vec{x}_i\}_{i\in\mathbf{Z}}$, as we see in the following corollary.

Corollary 3.3 *Given a fixed positive integer n, let $(\mathbf{C}^n, \|\cdot\|_n)$ and $(\mathbf{C}^{n\times n}, \|\cdot\|_{n\times n})$ be complex normed spaces, whose elements are column vectors resp. $(n \times n)$ complex matrices, with the property* (3). *Assume that the transition matrix $\mathbf{A} \in \mathbf{C}^{n\times n}$ is nonsingular and $\{\varepsilon_i\}_{i\in\mathbf{Z}}$ is a sequence of nonnegative real numbers. If a sequence $\{\vec{y}_i\}_{i\in\mathbf{Z}} \subset \mathbf{C}^n$ satisfies the inequality* (18) *for all $i \in \mathbf{Z}$, then there exists a solution $\{\vec{x}_i\}_{i\in\mathbf{Z}} \subset \mathbf{C}^n$ of the first-order matrix difference equation* (17) *with the initial condition $\vec{x}_0 = \vec{y}_0$ such that*

$$\|\vec{y}_i - \vec{x}_i\|_n \le \begin{cases}\sum_{k=1}^{i}\varepsilon_k\|\mathbf{A}^{-1}\|_{n\times n}^{i+1-k} & (\textit{for } i \ge 0), \\ \sum_{k=1}^{-i}\varepsilon_{k+i}\|\mathbf{A}\|_{n\times n}^{k-1} & (\textit{for } i < 0).\end{cases}$$

In the next corollary, we investigate the Hyers-Ulam stability of the second-order linear homogeneous difference equation with constant coefficients

$$a_i = \alpha a_{i+1} + \beta a_{i+2}. \tag{22}$$

Corollary 3.4 *Let $(\mathbf{C}^2, \|\cdot\|_\infty)$ and $(\mathbf{C}^{2\times 2}, \|\cdot\|_\infty)$ be complex normed spaces and let α, β, γ be complex numbers satisfying the conditions*

$$\alpha^2 + 4\beta \neq 0, \qquad \beta \neq 0, \qquad \gamma \neq 0. \tag{23}$$

Assume that $\varepsilon > 0$ is an arbitrary constant. If a sequence $\{a_i\}_{i\in\mathbf{Z}}$ of complex numbers satisfies the inequality

$$|a_i - \alpha a_{i+1} - \beta a_{i+2}| \leq \varepsilon \tag{24}$$

for all $i \in \mathbf{Z}$, then there exists a sequence $\{c_i\}_{i\in\mathbf{Z}}$ of complex numbers such that $c_0 = a_0$, $c_1 = a_1$, $c_i = \alpha c_{i+1} + \beta c_{i+2}$, and

$$|a_i - c_i| \leq \begin{cases} \sum_{k=1}^{i} \varepsilon \|\mathbf{A}^{-1}\|_\infty^{i+1-k} & (\textit{for } i \geq 0), \\ \sum_{k=1}^{-i} \varepsilon \|\mathbf{A}\|_\infty^{k-1} & (\textit{for } i < 0), \end{cases}$$

where $\|\mathbf{A}\|_\infty = \max\{|\alpha| + |\beta/\gamma|, |\gamma|\}$ and $\|\mathbf{A}^{-1}\|_\infty = \max\{|1/\gamma|, |\alpha/\beta| + |\gamma/\beta|\}$.

Proof If we define a sequence $\{b_i\}_{i\in\mathbf{Z}}$ of complex numbers by $b_i = \gamma a_{i+1}$, it then follows from (24) that

$$\begin{cases} |a_i - \alpha a_{i+1} - \frac{\beta}{\gamma} b_{i+1}| \leq \varepsilon, \\ |b_i - \gamma a_{i+1}| = 0 \end{cases}$$

for every $i \in \mathbf{Z}$. Hence, if we set

$$\vec{y}_i := \begin{pmatrix} a_i \\ b_i \end{pmatrix} \quad \text{and} \quad \mathbf{A} := \begin{pmatrix} \alpha & \frac{\beta}{\gamma} \\ \gamma & 0 \end{pmatrix},$$

then we get

$$\|\vec{y}_i - \mathbf{A}\vec{y}_{i+1}\|_\infty \leq \varepsilon$$

for all $i \in \mathbf{Z}$.

According to Corollary 3.3, there exists a unique solution $\{\vec{x}_i\}_{i\in\mathbf{Z}}$ of the first-order matrix difference equation (17) with the initial condition $\vec{x}_0 = \binom{a_0}{\gamma a_1}$ such that

$$\|\vec{y}_i - \vec{x}_i\|_\infty \leq \begin{cases} \sum_{k=1}^{i} \varepsilon \|\mathbf{A}^{-1}\|_\infty^{i+1-k} & (\text{for } i \geq 0), \\ \sum_{k=1}^{-i} \varepsilon \|\mathbf{A}\|_\infty^{k-1} & (\text{for } i < 0). \end{cases}$$

In view of (20) and the last inequality, we have

$$\left\| \begin{pmatrix} a_i \\ \gamma a_{i+1} \end{pmatrix} - \mathbf{A}^{-i} \begin{pmatrix} a_0 \\ \gamma a_1 \end{pmatrix} \right\|_\infty \leq \begin{cases} \sum_{k=1}^{i} \varepsilon \|\mathbf{A}^{-1}\|_\infty^{i+1-k} & (\text{for } i \geq 0), \\ \sum_{k=1}^{-i} \varepsilon \|\mathbf{A}\|_\infty^{k-1} & (\text{for } i < 0). \end{cases} \tag{25}$$

Since the matrix $\mathbf{A}$ has two distinct eigenvalues $\lambda_1 = \frac{\alpha-\sqrt{\alpha^2+4\beta}}{2}$ and $\lambda_2 = \frac{\alpha+\sqrt{\alpha^2+4\beta}}{2}$, as we did in the proof of Corollary 2.3, if i is a negative integer, then we get

$$\mathbf{A}^{-i} = \frac{1}{\gamma(\lambda_1-\lambda_2)}\begin{pmatrix} \gamma(\lambda_1^{1-i}-\lambda_2^{1-i}) & -\lambda_1\lambda_2(\lambda_1^{-i}-\lambda_2^{-i}) \\ \gamma^2(\lambda_1^{-i}-\lambda_2^{-i}) & -\gamma\lambda_1\lambda_2(\lambda_1^{-i-1}-\lambda_2^{-i-1}) \end{pmatrix}$$

for all integers $i < 0$. By using (25) and this equality, we have

$$\left\| \begin{pmatrix} a_i - \frac{a_0-a_1\lambda_2}{\lambda_1-\lambda_2}\lambda_1^{1-i} + \frac{a_0-a_1\lambda_1}{\lambda_1-\lambda_2}\lambda_2^{1-i} \\ \gamma a_{i+1} - \gamma\frac{a_0-a_1\lambda_2}{\lambda_1-\lambda_2}\lambda_1^{-i} + \gamma\frac{a_0-a_1\lambda_1}{\lambda_1-\lambda_2}\lambda_2^{-i} \end{pmatrix} \right\|_\infty \le \sum_{k=1}^{-i} \varepsilon \|\mathbf{A}\|_\infty^{k-1} \tag{26}$$

for any integer $i < 0$.

On the other hand, the inverse matrix $\mathbf{A}^{-1}$ has two distinct eigenvalues $\omega_1 = \frac{-\alpha-\sqrt{\alpha^2+4\beta}}{2\beta} = -\frac{1}{\beta}\lambda_2$ and $\omega_2 = \frac{-\alpha+\sqrt{\alpha^2+4\beta}}{2\beta} = -\frac{1}{\beta}\lambda_1$. In a similar way to the proof of Corollary 2.3, if i is a nonnegative integer, then we obtain

$$\mathbf{A}^{-i} = \left(\mathbf{A}^{-1}\right)^i = \frac{-1}{\gamma(\omega_1-\omega_2)}\begin{pmatrix} \gamma\omega_1\omega_2(\omega_1^{i-1}-\omega_2^{i-1}) & \omega_2^i-\omega_1^i \\ \gamma^2\omega_1\omega_2(\omega_1^i-\omega_2^i) & \gamma(\omega_2^{i+1}-\omega_1^{i+1}) \end{pmatrix}$$

for all integers $i \ge 0$. Thus, it follows from (25) and the last equality that

$$\left\| \begin{pmatrix} a_i - \frac{a_1-a_0\omega_2}{\omega_1-\omega_2}\omega_1^i + \frac{a_1-a_0\omega_1}{\omega_1-\omega_2}\omega_2^i \\ \gamma a_{i+1} - \gamma\frac{a_1-a_0\omega_2}{\omega_1-\omega_2}\omega_1^{i+1} + \gamma\frac{a_1-a_0\omega_1}{\omega_1-\omega_2}\omega_2^{i+1} \end{pmatrix} \right\|_\infty \le \sum_{k=1}^{i} \varepsilon \left\|\mathbf{A}^{-1}\right\|_\infty^{i+1-k} \tag{27}$$

for any integer $i \ge 0$.

Finally, considering (26) and (27), we define

$$c_i := \begin{cases} \frac{a_1-a_0\omega_2}{\omega_1-\omega_2}\omega_1^i - \frac{a_1-a_0\omega_1}{\omega_1-\omega_2}\omega_2^i & (\text{for } i \ge 0), \\ \frac{a_0-a_1\lambda_2}{\lambda_1-\lambda_2}\lambda_1^{1-i} - \frac{a_0-a_1\lambda_1}{\lambda_1-\lambda_2}\lambda_2^{1-i} & (\text{for } i < 0). \end{cases}$$

We then have $c_0 = a_0$, $c_1 = a_1$, and it follows from (26) and (27) that

$$|a_i - c_i| \le \begin{cases} \sum_{k=1}^{i} \varepsilon \|\mathbf{A}^{-1}\|_\infty^{i+1-k} & (\text{for } i \ge 0), \\ \sum_{k=1}^{-i} \varepsilon \|\mathbf{A}\|_\infty^{k-1} & (\text{for } i < 0). \end{cases}$$

Furthermore, it is easy to verify that the sequence $\{c_i\}_{i\in\mathbf{Z}}$ satisfies

$$c_i = \alpha c_{i+1} + \beta c_{i+2}$$

for any integer i. □

If we set $\gamma = \frac{\alpha+\sqrt{\alpha^2+4\beta}}{2}$ and $\beta > 0$ in Corollary 3.4, then we obtain the equalities in (16):

$$\begin{cases} \|\mathbf{A}\|_\infty = \max\left\{\frac{\alpha+\sqrt{\alpha^2+4\beta}}{2}, \frac{-3\alpha+\sqrt{\alpha^2+4\beta}}{2}\right\}, \\ \|\mathbf{A}^{-1}\|_\infty = \max\left\{\frac{3\alpha+\sqrt{\alpha^2+4\beta}}{2\beta}, \frac{-\alpha+\sqrt{\alpha^2+4\beta}}{2\beta}\right\}. \end{cases}$$

Thus, we get

$$\lim_{\beta\to\infty} \|\mathbf{A}\|_\infty \cdot \left\|\mathbf{A}^{-1}\right\|_\infty = \lim_{\beta\to\infty} \sqrt{\beta} \cdot \frac{1}{\sqrt{\beta}} = 1.$$

If β is large in absolute value, then the second-order difference equation (22) has the Hyers-Ulam stability as we see in the next example.

Example 3.5 Given an $\varepsilon > 0$, assume that a sequence $\{a_i\}_{i\in\mathbf{Z}}$ of complex numbers satisfies the inequality

$$|a_i - a_{i+1} - 4a_{i+2}| \le \varepsilon \tag{28}$$

for all $i \in \mathbf{Z}$. With $\alpha = 1$, $\beta = 4$, and $\gamma = \frac{1+\sqrt{17}}{2}$, it follows from (16) that $\|\mathbf{A}\|_\infty = \frac{1+\sqrt{17}}{2}$ and $\|\mathbf{A}^{-1}\|_\infty = \frac{3+\sqrt{17}}{8}$. Using these values, Corollary 3.4 implies that there exists a sequence $\{c_i\}_{i\in\mathbf{Z}}$ of complex numbers such that $c_0 = a_0$, $c_1 = a_1$, $c_i = c_{i+1} + 4c_{i+2}$, and

$$|a_i - c_i| \le \begin{cases} (\sqrt{17}+4)\bigl(1 - \bigl(\frac{\sqrt{17}+3}{8}\bigr)^i\bigr)\varepsilon & \text{(for } i \ge 0\text{)}, \\ \frac{\sqrt{17}+1}{8}\bigl(\bigl(\frac{\sqrt{17}+1}{2}\bigr)^{-i} - 1\bigr)\varepsilon & \text{(for } i < 0\text{)}. \end{cases}$$

If we apply Corollary 2.3 with the inequality

$$\left| a_i + \frac{1}{4}a_{i-1} - \frac{1}{4}a_{i-2} \right| \le \frac{1}{4}\varepsilon, \tag{29}$$

where we set $\alpha = -\frac{1}{4}$, $\beta = \frac{1}{4}$, and $\gamma = \frac{\sqrt{17}-1}{8}$, then there exists a sequence $\{c_i\}_{i\in\mathbf{Z}}$ of complex numbers such that $c_{-1} = a_{-1}$, $c_0 = a_0$, $c_i = c_{i+1} + 4c_{i+2}$, and

$$|a_i - c_i| \le \begin{cases} \frac{\sqrt{17}+5}{4}\bigl(1 - \bigl(\frac{\sqrt{17}+3}{8}\bigr)^i\bigr)\varepsilon & \text{(for } i \ge 0\text{)}, \\ \frac{\sqrt{17}+9}{32}\bigl(\bigl(\frac{\sqrt{17}+1}{2}\bigr)^{-i} - 1\bigr)\varepsilon & \text{(for } i < 0\text{)}. \end{cases}$$

The inequalities (28) and (29) are equivalent. In this case with inequality (28) or (29), there is more efficiency with Corollary 2.3 than Corollary 3.4 for any integer i.

Competing interests
The author declares that he has no competing interests.

Author's contributions
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