Liu et al. *Advances in Difference Equations* (2015) 2015:178
DOI 10.1186/s13662-015-0513-8

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Positive solutions and convergence of Mann iterative schemes for a fourth order neutral delay difference equation

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Abstract

The existence of uncountably many positive solutions and convergence of Mann iterative schemes for a fourth order neutral delay difference equation are proved. Seven examples are included.

MSC: 39A10

Keywords: fourth order neutral delay difference equation; positive solutions; Mann iterative methods; Banach fixed point theorem

1 Introduction and preliminaries

This paper is concerned with the following fourth order neutral delay difference equation

$$\begin{aligned}&\Delta\bigl(a_n\Delta^3(x_n+b_nx_{n-\tau})\bigr)+\Delta h(n,x_{h_{1n}},x_{h_{2n}},\dots,x_{h_{kn}})\\&\quad+f(n,x_{f_{1n}},x_{f_{2n}},\dots,x_{f_{kn}})=c_n,\quad\forall n\geq n_0,\end{aligned}\tag{1.1}$$

where $\tau,k,n_0\in\mathbb{N}$, $\{a_n\}_{n\in\mathbb{N}_{n_0}}\subset\mathbb{R}\setminus\{0\}$, $\{b_n\}_{n\in\mathbb{N}_{n_0}},\{c_n\}_{n\in\mathbb{N}_{n_0}}\subset\mathbb{R}$, $h,f\in C(\mathbb{N}_{n_0}\times\mathbb{R}^k,\mathbb{R})$, $\{h_{ln}\}_{n\in\mathbb{N}_{n_0}}$, $\{f_{ln}\}_{n\in\mathbb{N}_{n_0}}\subseteq\mathbb{N}$ and

$$\lim_{n\to\infty}h_{ln}=\lim_{n\to\infty}f_{ln}=+\infty,\quad l\in\{1,2,\dots,k\}.$$

Over the past several decades, a lot of researchers paid much attention to the problems of oscillation, nonoscillation, asymptotic behavior and existence of solutions for some second and third order difference equations, see, for example, [1–14] and the references cited therein. In particular, the researchers [5–8, 12] used fixed point theorems to study the existence of bounded nonoscillatory solutions and positive solutions for the following second and third order nonlinear neutral delay difference equations

$$\begin{aligned}&\Delta^3x_n+f(n,x_n,x_{n-\tau})=0,\quad\forall n\geq n_0,\\&\Delta^2(x_n+b_nx_{n-\tau})+\Delta h(n,x_{h_{1n}},x_{h_{2n}},\dots,x_{h_{kn}})+f(n,x_{f_{1n}},x_{f_{2n}},\dots,x_{f_{kn}})=c_n,\quad\forall n\geq n_0,\\&\Delta\bigl(a_n\Delta(x_n+b_nx_{n-\tau})\bigr)+\Delta h(n,x_{h_{1n}},x_{h_{2n}},\dots,x_{h_{kn}})\\&\quad+f(n,x_{f_{1n}},x_{f_{2n}},\dots,x_{f_{kn}})=c_n,\quad\forall n\geq n_0,\end{aligned}$$

$$\Delta\big(a_n\Delta^2(x_n + p_n x_{n-\tau})\big) + f(n, x_{n-d_{1n}}, x_{n-d_{2n}}, \dots, x_{n-d_{ln}}) = g_n, \quad \forall n \geq n_0$$

and

$$\Delta^3(x_n + b_n x_{n-\tau}) + \Delta h(n, x_{h_{1n}}, x_{h_{2n}}, \dots, x_{h_{kn}}) + f(n, x_{f_{1n}}, x_{f_{2n}}, \dots, x_{f_{kn}}) = c_n, \quad \forall n \geq n_0.$$

The main purpose of this paper is to utilize the Banach fixed point theorem and some new techniques to establish the existence of uncountably many positive solutions of Eq. (1.1). Not only do we construct a few Mann iterative algorithms for approximating these positive solutions, but we also prove convergence and the error estimates of the Mann iterative algorithms relative to these positive solutions. Moreover, seven nontrivial examples are given to illustrate our results.

Throughout this paper, we assume that Δ is the forward difference operator defined by $\Delta x_n = x_{n+1} - x_n$, $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}^+ = [0, +\infty)$, $\mathbb{N}_0$ and $\mathbb{N}$ denote the sets of all nonnegative integers and positive integers, respectively,

$$\mathbb{N}_t = \{n : n \in \mathbb{N} \text{ with } n \geq t\}, \quad \forall t \in \mathbb{N},$$
$$\beta = \min\big\{n_0 - \tau, \inf\{h_{ln}, f_{ln} : 1 \leq l \leq k, n \in \mathbb{N}_{n_0}\}\big\} \in \mathbb{N},$$
$$H_n = \max\big\{h_{ln}^2 : l \in \{1, 2, \dots, k\}\big\}, \qquad F_n = \max\big\{f_{ln}^2 : l \in \{1, 2, \dots, k\}\big\}, \quad \forall n \in \mathbb{N}_{n_0},$$

l_β^∞ represents the Banach space of all real sequences $x = \{x_n\}_{n\in\mathbb{N}_\beta}$ in $\mathbb{N}_\beta$ with norm

$$\|x\| = \sup_{n\in\mathbb{N}_\beta}\left|\frac{x_n}{n^2}\right| < +\infty \quad \text{for each } x = \{x_n\}_{n\in\mathbb{N}_\beta} \in l_\beta^\infty$$

and

$$A(N, M) = \left\{x = \{x_n\}_{n\in\mathbb{N}_\beta} \in l_\beta^\infty : N \leq \frac{x_n}{n^2} \leq M, n \in \mathbb{N}_\beta\right\} \quad \text{for any } M > N > 0.$$

It is clear that $A(N,M)$ is a closed and convex subset of l_β^∞. By a solution of Eq. (1.1), we mean a sequence $\{x_n\}_{n\in\mathbb{N}_\beta}$ with a positive integer $T \geq n_0 + \tau + \beta$ such that Eq. (1.1) holds for all $n \geq T$.

Lemma 1.1 *Let $\{p_t\}_{t\in\mathbb{N}}$ be a nonnegative sequence and $n, \tau \in \mathbb{N}$. Then*

$$\sum_{v=n}^{\infty}\sum_{u=v}^{\infty}\sum_{t=u}^{\infty} p_t \leq \sum_{t=n}^{\infty} t^2 p_t; \tag{1.2}$$

$$\sum_{v=n}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\sum_{t=s}^{\infty} p_t \leq \sum_{t=n}^{\infty} t^3 p_t; \tag{1.3}$$

$$\sum_{i=1}^{\infty}\sum_{u=n+i\tau}^{\infty}\sum_{s=u}^{\infty}\sum_{t=s}^{\infty} p_t \leq \frac{1}{\tau}\sum_{t=n+\tau}^{\infty} t^3 p_t; \tag{1.4}$$

$$\sum_{i=1}^{\infty}\sum_{v=n+i\tau}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\sum_{t=s}^{\infty} p_t \leq \frac{1}{\tau}\sum_{t=n+\tau}^{\infty} t^4 p_t. \tag{1.5}$$

Proof Note that

$$\begin{aligned}
\sum_{v=n}^{\infty}\sum_{u=v}^{\infty}\sum_{t=u}^{\infty}p_t &= \sum_{v=n}^{\infty}\left(\sum_{u=v}^{\infty}\sum_{t=u}^{\infty}p_t\right)\\
&= \sum_{v=n}^{\infty}\left(\sum_{t=v}^{\infty}p_t + \sum_{t=v+1}^{\infty}p_t + \sum_{t=v+2}^{\infty}p_t + \cdots\right)\\
&= \sum_{v=n}^{\infty}\sum_{t=v}^{\infty}(t-v+1)p_t \le \sum_{v=n}^{\infty}\sum_{t=v}^{\infty}tp_t = \sum_{t=n}^{\infty}(t-n+1)tp_t\\
&\le \sum_{t=n}^{\infty}t^2p_t
\end{aligned}$$

and

$$\begin{aligned}
\sum_{i=1}^{\infty}\sum_{u=n+i\tau}^{\infty}\sum_{s=u}^{\infty}\sum_{t=s}^{\infty}p_t &= \sum_{i=1}^{\infty}\sum_{u=n+i\tau}^{\infty}\left(\sum_{s=u}^{\infty}\sum_{t=s}^{\infty}p_t\right)\\
&= \sum_{i=1}^{\infty}\sum_{u=n+i\tau}^{\infty}\sum_{t=u}^{\infty}(t-u+1)p_t \le \sum_{i=1}^{\infty}\sum_{u=n+i\tau}^{\infty}\sum_{t=u}^{\infty}tp_t\\
&= \sum_{i=1}^{\infty}\left(\sum_{t=n+i\tau}^{\infty}tp_t + \sum_{t=n+1+i\tau}^{\infty}tp_t + \sum_{t=n+2+i\tau}^{\infty}tp_t + \cdots\right)\\
&= \sum_{i=1}^{\infty}\sum_{t=n+i\tau}^{\infty}(t-n-i\tau+1)tp_t \le \sum_{i=1}^{\infty}\sum_{t=n+i\tau}^{\infty}t^2p_t\\
&= \sum_{t=n+\tau}^{\infty}t^2p_t + \sum_{t=n+2\tau}^{\infty}t^2p_t + \sum_{t=n+3\tau}^{\infty}t^2p_t + \cdots\\
&\le \sum_{t=n+\tau}^{\infty}\left(\frac{t-n-\tau}{\tau}+1\right)t^2p_t = \sum_{t=n+\tau}^{\infty}\frac{t-n}{\tau}t^2p_t\\
&\le \frac{1}{\tau}\sum_{t=n+\tau}^{\infty}t^3p_t,
\end{aligned}$$

which imply (1.2) and (1.4), respectively. It follows from (1.2) and (1.4) that

$$\begin{aligned}
\sum_{v=n}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\sum_{t=s}^{\infty}p_t &= \sum_{v=n}^{\infty}\left(\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\sum_{t=s}^{\infty}p_t\right) \le \sum_{v=n}^{\infty}\sum_{t=v}^{\infty}t^2p_t = \sum_{t=n}^{\infty}(t-n+1)t^2p_t\\
&\le \sum_{t=n}^{\infty}t^3p_t
\end{aligned}$$

and

$$\begin{aligned}
\sum_{i=1}^{\infty}\sum_{v=n+i\tau}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\sum_{t=s}^{\infty}p_t &= \sum_{i=1}^{\infty}\sum_{v=n+i\tau}^{\infty}\sum_{u=v}^{\infty}\left(\sum_{s=u}^{\infty}\sum_{t=s}^{\infty}p_t\right)\\
&= \sum_{i=1}^{\infty}\sum_{v=n+i\tau}^{\infty}\sum_{u=v}^{\infty}\sum_{t=u}^{\infty}(t-u-1)p_t
\end{aligned}$$

$$\leq \sum_{i=1}^{\infty} \sum_{v=n+i\tau}^{\infty} \sum_{u=v}^{\infty} \sum_{t=u}^{\infty} tp_t$$
$$\leq \frac{1}{\tau} \sum_{t=n+\tau}^{\infty} t^4 p_t,$$

which yields (1.3) and (1.5), respectively. This completes the proof. □

2 Uncountably many positive solutions and Mann iterative sequences

In this section, we discuss the existence of uncountably many positive solutions of Eq. (1.1) and prove convergence and the error estimates of the Mann iterative algorithms with respect to these positive solutions by using the Banach fixed point theorem.

Theorem 2.1 *Assume that there exist two constants M and N with $M > N > 0$ and four nonnegative sequences $\{P_n\}_{n\in\mathbb{N}_{n_0}}$, $\{Q_n\}_{n\in\mathbb{N}_{n_0}}$, $\{R_n\}_{n\in\mathbb{N}_{n_0}}$ and $\{W_n\}_{n\in\mathbb{N}_{n_0}}$ satisfying*

$$\begin{aligned}
&\bigl|f(n,u_1,u_2,\dots,u_k) - f(n,\bar{u}_1,\bar{u}_2,\dots,\bar{u}_k)\bigr| \leq P_n \max\bigl\{|u_l - \bar{u}_l| : 1 \leq l \leq k\bigr\},\\
&\bigl|h(n,u_1,u_2,\dots,u_k) - h(n,\bar{u}_1,\bar{u}_2,\dots,\bar{u}_k)\bigr| \leq R_n \max\bigl\{|u_l - \bar{u}_l| : 1 \leq l \leq k\bigr\},\\
&\quad \forall (n,u_l,\bar{u}_l) \in \mathbb{N}_{n_0} \times \bigl(\mathbb{R}^+ \setminus \{0\}\bigr)^2, 1 \leq l \leq k; \qquad (2.1)
\end{aligned}$$

$$\begin{aligned}
&\bigl|f(n,u_1,u_2,\dots,u_k)\bigr| \leq Q_n \quad \textit{and} \quad \bigl|h(n,u_1,u_2,\dots,u_k)\bigr| \leq W_n,\\
&\quad \forall (n,u_l) \in \mathbb{N}_{n_0} \times \bigl(\mathbb{R}^+ \setminus \{0\}\bigr), 1 \leq l \leq k; \qquad (2.2)
\end{aligned}$$

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{i=1}^{\infty} \sum_{v=n+i\tau}^{\infty} \sum_{u=v}^{\infty} \sum_{s=u}^{\infty} \frac{1}{|a_s|} \max\{R_s H_s, W_s\} = 0; \qquad (2.3)$$

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{i=1}^{\infty} \sum_{v=n+i\tau}^{\infty} \sum_{u=v}^{\infty} \sum_{s=u}^{\infty} \sum_{t=s}^{\infty} \frac{1}{|a_s|} \max\bigl\{P_t F_t, Q_t, |c_t|\bigr\} = 0; \qquad (2.4)$$

$$b_n = -1 \quad \textit{eventually}. \qquad (2.5)$$

Then

(a) *for any $L \in (N,M)$, there exist $\theta \in (0,1)$ and $T \geq n_0 + \tau + \beta$ such that for each $x_0 = \{x_{0n}\}_{n\in\mathbb{N}_\beta} \in A(N,M)$, the Mann iterative sequence $\{x_m\}_{m\in\mathbb{N}_0} = \{\{x_{mn}\}_{n\in\mathbb{N}_\beta}\}_{m\in\mathbb{N}_0}$ generated by the scheme:*

$$x_{m+1n} = \begin{cases} (1-\alpha_m)x_{mn} + \alpha_m\{n^2 L \\ \quad - \sum_{i=1}^{\infty} \sum_{v=n+i\tau}^{\infty} \sum_{u=v}^{\infty} \sum_{s=u}^{\infty} \frac{1}{a_s}[h(s,x_{mh_{1s}},x_{mh_{2s}},\dots,x_{mh_{ks}}) \\ \quad - \sum_{t=s}^{\infty}(f(t,x_{mf_{1t}},x_{mf_{2t}},\dots,x_{mf_{kt}}) - c_t)]\}, \quad m \geq 0, n \geq T, \\ (1-\alpha_m)\frac{n^2}{T^2}x_{mT} + \alpha_m \frac{n^2}{T^2}\{T^2 L \\ \quad - \sum_{i=1}^{\infty} \sum_{v=T+i\tau}^{\infty} \sum_{u=v}^{\infty} \sum_{s=u}^{\infty} \frac{1}{a_s}[h(s,x_{mh_{1s}},x_{mh_{2s}},\dots,x_{mh_{ks}}) \\ \quad - \sum_{t=s}^{\infty}(f(t,x_{mf_{1t}},x_{mf_{2t}},\dots,x_{mf_{kt}}) - c_t)]\}, \quad m \geq 0, \beta \leq n < T \end{cases} \qquad (2.6)$$

converges to a positive solution $w = \{w_n\}_{n\in\mathbb{N}_\beta} \in A(N,M)$ of Eq. (1.1) with

$$\lim_{n\to\infty} \frac{w_n}{n^2} = L \in (N,M) \qquad (2.7)$$

and has the following error estimate:

$$\|x_{m+1} - w\| \le e^{-(1-\theta)\sum_{i=0}^{m}\alpha_i}\|x_0 - w\|, \quad \forall m \in \mathbb{N}_0, \tag{2.8}$$

where $\{\alpha_m\}_{m\in\mathbb{N}_0}$ *is an arbitrary sequence in* $[0,1]$ *with*

$$\sum_{m=0}^{\infty}\alpha_m = +\infty; \tag{2.9}$$

(b) *Equation* (1.1) *possesses uncountably many positive solutions in* $A(N,M)$.

Proof In the first place we show that (a) holds. Set $L \in (N,M)$. It follows from (2.3)-(2.5) that there exist $\theta \in (0,1)$ and $T \ge n_0 + \tau + \beta$ satisfying

$$\theta = \frac{1}{T^2}\sum_{i=1}^{\infty}\sum_{v=T+i\tau}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\frac{1}{|a_s|}\left(R_sH_s + \sum_{t=s}^{\infty}P_tF_t\right); \tag{2.10}$$

$$\frac{1}{T^2}\sum_{i=1}^{\infty}\sum_{v=T+i\tau}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\frac{1}{|a_s|}\left(W_s + \sum_{t=s}^{\infty}\bigl(Q_t + |c_t|\bigr)\right) < \min\{M-L, L-N\}; \tag{2.11}$$

$$b_n = -1, \quad \forall n \ge T. \tag{2.12}$$

Define a mapping $S_L : A(N,M) \to l_\beta^\infty$ by

$$S_Lx_n = \begin{cases} n^2L - \sum_{i=1}^{\infty}\sum_{v=n+i\tau}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\frac{1}{a_s}\{h(s,x_{h_{1s}},x_{h_{2s}},\ldots,x_{h_{ks}}) \\ \quad - \sum_{t=s}^{\infty}[f(t,x_{f_{1t}},x_{f_{2t}},\ldots,x_{f_{kt}}) - c_t]\}, \quad n \ge T, \\ \frac{n^2}{T^2}S_Lx_T, \quad \beta \le n < T \end{cases} \tag{2.13}$$

for each $x = \{x_n\}_{n\in\mathbb{N}_\beta} \in A(N,M)$. By virtue of (2.1), (2.2), (2.10), (2.11) and (2.13), we gain that for each $x = \{x_n\}_{n\in\mathbb{N}_\beta}, y = \{y_n\}_{n\in\mathbb{N}_\beta} \in A(N,M)$

$$\begin{aligned}
&\left|\frac{S_Lx_n}{n^2} - \frac{S_Ly_n}{n^2}\right| \\
&\quad \le \frac{1}{n^2}\sum_{i=1}^{\infty}\sum_{v=n+i\tau}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\frac{1}{|a_s|}\Biggl(\bigl|h(s,x_{h_{1s}},x_{h_{2s}},\ldots,x_{h_{ks}}) - h(s,y_{h_{1s}},y_{h_{2s}},\ldots,y_{h_{ks}})\bigr| \\
&\qquad + \sum_{t=s}^{\infty}\bigl|f(t,x_{f_{1t}},x_{f_{2t}},\ldots,x_{f_{kt}}) - f(t,y_{f_{1t}},y_{f_{2t}},\ldots,y_{f_{kt}})\bigr|\Biggr) \\
&\quad \le \frac{1}{n^2}\sum_{i=1}^{\infty}\sum_{v=n+i\tau}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\frac{1}{|a_s|}\Biggl(R_s\max\bigl\{|x_{h_{ls}} - y_{h_{ls}}| : 1 \le l \le k\bigr\} \\
&\qquad + \sum_{t=s}^{\infty}P_t\max\bigl\{|x_{f_{lt}} - y_{f_{lt}}| : 1 \le l \le k\bigr\}\Biggr) \\
&\quad \le \frac{\|x-y\|}{n^2}\sum_{i=1}^{\infty}\sum_{v=n+i\tau}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\frac{1}{|a_s|}\Biggl(R_s\max\bigl\{h_{ls}^2 : 1 \le l \le k\bigr\}
\end{aligned}$$

$$
\begin{aligned}
&\quad + \sum_{t=s}^{\infty} P_t \max\{f_{lt}^2 : 1 \le l \le k\}\Bigg) \\
&\le \frac{\|x-y\|}{T^2} \sum_{i=1}^{\infty} \sum_{v=T+i\tau}^{\infty} \sum_{u=v}^{\infty} \sum_{s=u}^{\infty} \frac{1}{|a_s|} \left(R_s H_s + \sum_{t=s}^{\infty} P_t F_t \right) \\
&= \theta \|x-y\|, \quad \forall n \ge T,
\end{aligned}
$$

$$
\left| \frac{S_L x_n}{n^2} - \frac{S_L y_n}{n^2} \right| = \left| \frac{S_L x_T}{T^2} - \frac{S_L y_T}{T^2} \right| \le \theta \|x-y\|, \quad \beta \le n < T,
$$

$$
\begin{aligned}
&\left| \frac{S_L x_n}{n^2} - L \right| \\
&= \Bigg| \frac{1}{n^2} \sum_{i=1}^{\infty} \sum_{v=n+i\tau}^{\infty} \sum_{u=v}^{\infty} \sum_{s=u}^{\infty} \frac{1}{a_s} \Bigg(h(s, x_{h_{1s}}, x_{h_{2s}}, \ldots, x_{h_{ks}}) \\
&\quad - \sum_{t=s}^{\infty} \big[f(t, x_{f_{1t}}, x_{f_{2t}}, \ldots, x_{f_{kt}}) - c_t \big] \Bigg) \Bigg| \\
&\le \frac{1}{n^2} \sum_{i=1}^{\infty} \sum_{v=n+i\tau}^{\infty} \sum_{u=v}^{\infty} \sum_{s=u}^{\infty} \frac{1}{|a_s|} \Bigg(\big| h(s, x_{h_{1s}}, x_{h_{2s}}, \ldots, x_{h_{ks}}) \big| \\
&\quad + \sum_{t=s}^{\infty} \big[\big| f(t, x_{f_{1t}}, x_{f_{2t}}, \ldots, x_{f_{kt}}) \big| + |c_t| \big] \Bigg) \\
&\le \frac{1}{T^2} \sum_{i=1}^{\infty} \sum_{v=T+i\tau}^{\infty} \sum_{u=v}^{\infty} \sum_{s=u}^{\infty} \frac{1}{|a_s|} \left(W_s + \sum_{t=s}^{\infty} \big(Q_t + |c_t| \big) \right) \\
&< \min\{M-L, L-N\}, \quad \forall n \ge T,
\end{aligned}
$$

$$
\left| \frac{S_L x_n}{n^2} - L \right| = \left| \frac{S_L x_T}{T^2} - L \right| < \min\{M-L, L-N\}, \quad \beta \le n < T,
$$

which yield that

$$
S_L\big(A(N,M)\big) \subseteq A(N,M), \qquad \|S_L x - S_L y\| \le \theta \|x-y\|, \quad \forall x, y \in A(N,M), \tag{2.14}
$$

which means that S_L is a contraction in $A(N,M)$. Utilizing the Banach fixed point theorem, we conclude that S_L has a unique fixed point $w = \{w_n\}_{n\in\mathbb{N}_\beta} \in A(N,M)$, that is,

$$
\begin{aligned}
w_n = S_L w_n = n^2 L - \sum_{i=1}^{\infty} \sum_{v=n+i\tau}^{\infty} \sum_{u=v}^{\infty} \sum_{s=u}^{\infty} \frac{1}{a_s} \Bigg\{ & h(s, w_{h_{1s}}, w_{h_{2s}}, \ldots, w_{h_{ks}}) \\
& - \sum_{t=s}^{\infty} \big[f(t, w_{f_{1t}}, w_{f_{2t}}, \ldots, w_{f_{kt}}) - c_t \big] \Bigg\}, \quad \forall n \ge T
\end{aligned} \tag{2.15}
$$

and

$$
w_n = S_L w_n = \frac{n^2}{T^2} S_L w_T = \frac{n^2}{T^2} w_T, \quad \beta \le n < T. \tag{2.16}
$$

It is obvious that (2.15) yields that

$$\begin{aligned} w_{n-\tau} &= (n-\tau)^2 L - \sum_{i=1}^{\infty} \sum_{v=n+(i-1)\tau}^{\infty} \sum_{u=v}^{\infty} \sum_{s=u}^{\infty} \frac{1}{a_s} \Biggl\{ h(s, w_{h_{1s}}, w_{h_{2s}}, \ldots, w_{h_{ks}}) \\ &\quad - \sum_{t=s}^{\infty} \bigl[f(t, w_{f_{1t}}, w_{f_{2t}}, \ldots, w_{f_{kt}}) - c_t \bigr] \Biggr\}, \quad \forall n \geq T+\tau, \\ w_n - w_{n-\tau} &= \bigl(2n\tau - \tau^2\bigr) L + \sum_{v=n}^{\infty} \sum_{u=v}^{\infty} \sum_{s=u}^{\infty} \frac{1}{a_s} \Biggl\{ h(s, w_{h_{1s}}, w_{h_{2s}}, \ldots, w_{h_{ks}}) \\ &\quad - \sum_{t=s}^{\infty} \bigl[f(t, w_{f_{1t}}, w_{f_{2t}}, \ldots, w_{f_{kt}}) - c_t \bigr] \Biggr\}, \quad \forall n \geq T+\tau, \end{aligned}$$

which implies that

$$\begin{aligned} \Delta(w_n - w_{n-\tau}) &= 2\tau L - \sum_{u=n}^{\infty} \sum_{s=u}^{\infty} \frac{1}{a_s} \Biggl\{ h(s, w_{h_{1s}}, w_{h_{2s}}, \ldots, w_{h_{ks}}) \\ &\quad - \sum_{t=s}^{\infty} \bigl[f(t, w_{f_{1t}}, w_{f_{2t}}, \ldots, w_{f_{kt}}) - c_t \bigr] \Biggr\}, \quad \forall n \geq T+\tau, \\ \Delta^2(w_n - w_{n-\tau}) &= \sum_{s=n}^{\infty} \frac{1}{a_s} \Biggl\{ h(s, w_{h_{1s}}, w_{h_{2s}}, \ldots, w_{h_{ks}}) \\ &\quad - \sum_{t=s}^{\infty} \bigl[f(t, w_{f_{1t}}, w_{f_{2t}}, \ldots, w_{f_{kt}}) - c_t \bigr] \Biggr\}, \quad \forall n \geq T+\tau, \\ a_n \Delta^3(w_n - w_{n-\tau}) &= -h(n, w_{h_{1n}}, w_{h_{2n}}, \ldots, w_{h_{kn}}) \\ &\quad + \sum_{t=n}^{\infty} \bigl[f(t, w_{f_{1t}}, w_{f_{2t}}, \ldots, w_{f_{kt}}) - c_t \bigr], \quad \forall n \geq T+\tau \end{aligned}$$

and

$$\begin{aligned} \Delta\bigl(a_n \Delta^3(w_n - w_{n-\tau})\bigr) &= -\Delta h(n, w_{h_{1n}}, w_{h_{2n}}, \ldots, w_{h_{kn}}) \\ &\quad - f(n, w_{f_{1n}}, w_{f_{2n}}, \ldots, w_{f_{kn}}) + c_n, \quad \forall n \geq T+\tau, \end{aligned}$$

which together with (2.12) means that $w = \{w_n\}_{n\in\mathbb{N}_\beta}$ is a positive solution of Eq. (1.1) in $A(N,M)$. It follows from (2.2)-(2.4) and (2.15) that

$$\begin{aligned} &\left| \frac{w_n}{n^2} - L \right| \\ &\quad = \left| \frac{1}{n^2} \sum_{i=1}^{\infty} \sum_{v=n+i\tau}^{\infty} \sum_{u=v}^{\infty} \sum_{s=u}^{\infty} \frac{1}{a_s} \Biggl(h(s, w_{h_{1s}}, w_{h_{2s}}, \ldots, w_{h_{ks}}) \right. \\ &\qquad \left. - \sum_{t=s}^{\infty} \bigl[f(t, w_{f_{1t}}, w_{f_{2t}}, \ldots, w_{f_{kt}}) - c_t \bigr] \Biggr) \right| \\ &\quad \leq \frac{1}{n^2} \sum_{i=1}^{\infty} \sum_{v=n+i\tau}^{\infty} \sum_{u=v}^{\infty} \sum_{s=u}^{\infty} \frac{1}{|a_s|} \Biggl(\bigl| h(s, w_{h_{1s}}, w_{h_{2s}}, \ldots, w_{h_{ks}}) \bigr| \end{aligned}$$

$$
\begin{aligned}
&+\sum_{t=s}^{\infty}\left[\left|f(t,w_{f_{1t}},w_{f_{2t}},\dots,w_{f_{kt}})\right|+|c_t|\right]\Bigg)\\
&\le \frac{1}{n^2}\sum_{i=1}^{\infty}\sum_{v=n+i\tau}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\frac{1}{|a_s|}\left(W_s+\sum_{t=s}^{\infty}(Q_t+|c_t|)\right)\\
&\to 0 \quad \text{as } n\to\infty,
\end{aligned}
$$

that is, (2.7) holds. It follows from (2.6), (2.10), (2.12) and (2.14)-(2.16) that

$$
\begin{aligned}
&\frac{|x_{m+1n}-w_n|}{n^2}\\
&=\frac{1}{n^2}\Bigg|(1-\alpha_m)x_{mn}+\alpha_m\Bigg\{n^2L-\sum_{i=1}^{\infty}\sum_{v=n+i\tau}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\frac{1}{a_s}\Bigg[h(s,x_{mh_{1s}},x_{mh_{2s}},\dots,x_{mh_{ks}})\\
&\quad-\sum_{t=s}^{\infty}\left[f(t,x_{mf_{1t}},x_{mf_{2t}},\dots,x_{mf_{kt}})-c_t\right]\Bigg]\Bigg\}-w_n\Bigg|\\
&\le(1-\alpha_m)\frac{|x_{mn}-w_n|}{n^2}+\alpha_m\frac{|S_Lx_{mn}-S_Lw_n|}{n^2}\\
&\le(1-\alpha_m)\|x_m-w\|+\theta\alpha_m\|x_m-w\|\\
&=\left[1-(1-\theta)\alpha_m\right]\|x_m-w\|\\
&\le e^{-(1-\theta)\alpha_m}\|x_m-w\|,\quad \forall m\in\mathbb{N}_0, n\ge T
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{|x_{m+1n}-w_n|}{n^2}\\
&=\frac{1}{n^2}\Bigg|(1-\alpha_m)\frac{n^2}{T^2}x_{mT}\\
&\quad+\alpha_m\frac{n^2}{T^2}\Bigg\{T^2L-\sum_{i=1}^{\infty}\sum_{v=T+i\tau}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\frac{1}{a_s}\Bigg[h(s,x_{mh_{1s}},x_{mh_{2s}},\dots,x_{mh_{ks}})\\
&\quad-\sum_{t=s}^{\infty}\left[f(t,x_{mf_{1t}},x_{mf_{2t}},\dots,x_{mf_{kt}})-c_t\right]\Bigg]\Bigg\}-w_n\Bigg|\\
&\le(1-\alpha_m)\frac{|x_{mT}-w_T|}{T^2}+\alpha_m\frac{|S_Lx_{mT}-S_Lw_T|}{T^2}\\
&\le\left[1-(1-\theta)\alpha_m\right]\|x_m-w\|\\
&\le e^{-(1-\theta)\alpha_m}\|x_m-w\|,\quad \forall m\in\mathbb{N}_0, \beta\le n<T,
\end{aligned}
$$

which imply that

$$
\|x_{m+1}-w\|\le e^{-(1-\theta)\alpha_m}\|x_m-w\|\le e^{-(1-\theta)\sum_{i=0}^{m}\alpha_i}\|x_0-w\|,\quad \forall m\in\mathbb{N}_0,
$$

that is, (2.8) holds. Thus (2.8) and (2.9) guarantee that $\lim_{m\to\infty}x_m=w$.

In the next place we show that (b) holds. Let $L_1,L_2\in(N,M)$ and $L_1\neq L_2$. As in the proof of (a), we deduce similarly that for each $c\in\{1,2\}$, there exist constants $\theta_c\in(0,1)$, $T_c\ge$

$n_0 + \tau + \beta$ and a mapping S_{L_c} satisfying (2.10)-(2.14), where θ, L and T are replaced by θ_c, L_c and T_c, respectively, and the mapping S_{L_c} has a fixed point $z^c = \{z_n^c\}_{n\in\mathbb{N}_\beta} \in A(N,M)$, which is a positive solution of Eq. (1.1) in $A(N,M)$, that is,

$$\begin{aligned} z_n^c = n^2 L_c - \sum_{i=1}^{\infty} \sum_{v=n+i\tau}^{\infty} \sum_{u=v}^{\infty} \sum_{s=u}^{\infty} \frac{1}{a_s} \Bigg\{ h\big(s, z^c_{h_{1s}}, z^c_{h_{2s}}, \ldots, z^c_{h_{ks}}\big) \\ - \sum_{t=s}^{\infty} \Big[f\big(t, z^c_{f_{1t}}, z^c_{f_{2t}}, \ldots, z^c_{f_{kt}}\big) - c_t \Big] \Bigg\}, \quad \forall n \geq T_c, \end{aligned}$$

which together with (2.1), (2.10) and (2.12) implies that

$$\begin{aligned} &\left| \frac{z_n^1}{n^2} - \frac{z_n^2}{n^2} \right| \\ &\geq |L_1 - L_2| - \frac{1}{n^2} \sum_{i=1}^{\infty} \sum_{v=n+i\tau}^{\infty} \sum_{u=v}^{\infty} \sum_{s=u}^{\infty} \frac{1}{|a_s|} \Bigg(\big| h\big(s, z^1_{h_{1s}}, z^1_{h_{2s}}, \ldots, z^1_{h_{ks}}\big) \\ &\quad - h\big(s, z^2_{h_{1s}}, z^2_{h_{2s}}, \ldots, z^2_{h_{ks}}\big)\big| + \sum_{t=s}^{\infty} \big| f\big(t, z^1_{f_{1t}}, z^1_{f_{2t}}, \ldots, z^1_{f_{kt}}\big) - f\big(t, z^2_{f_{1t}}, z^2_{f_{2t}}, \ldots, z^2_{f_{kt}}\big)\big| \Bigg) \\ &\geq |L_1 - L_2| - \frac{1}{n^2} \sum_{i=1}^{\infty} \sum_{v=n+i\tau}^{\infty} \sum_{u=v}^{\infty} \sum_{s=u}^{\infty} \frac{1}{|a_s|} \Bigg(R_s \max\big\{ \big| z^1_{h_{ls}} - z^2_{h_{ls}} \big| : 1 \leq l \leq k \big\} \\ &\quad + \sum_{t=s}^{\infty} P_t \max\big\{ \big| z^1_{f_{lt}} - z^2_{f_{lt}} \big| : 1 \leq l \leq k \big\} \Bigg) \\ &\geq |L_1 - L_2| - \frac{\|z^1 - z^2\|}{n^2} \sum_{i=1}^{\infty} \sum_{v=n+i\tau}^{\infty} \sum_{u=v}^{\infty} \sum_{s=u}^{\infty} \frac{1}{|a_s|} \left(R_s H_s + \sum_{t=s}^{\infty} P_t F_t \right) \\ &\geq |L_1 - L_2| - \frac{\|z^1 - z^2\|}{\max\{T_1^2, T_2^2\}} \sum_{i=1}^{\infty} \sum_{v=\max\{T_1,T_2\}+i\tau}^{\infty} \sum_{u=v}^{\infty} \sum_{s=u}^{\infty} \frac{1}{|a_s|} \left(R_s H_s + \sum_{t=s}^{\infty} P_t F_t \right) \\ &\geq |L_1 - L_2| - \max\{\theta_1, \theta_2\} \big\| z^1 - z^2 \big\|, \quad \forall n \geq \max\{T_1, T_2\}, \end{aligned}$$

which yields that

$$\big\| z^1 - z^2 \big\| \geq \frac{|L_1 - L_2|}{1 + \max\{\theta_1, \theta_2\}} > 0,$$

that is, $z^1 \neq z^2$. This completes the proof. □

Theorem 2.2 *Assume that there exist two constants M and N with $M > N > 0$ and four nonnegative sequences $\{P_n\}_{n\in\mathbb{N}_{n_0}}$, $\{Q_n\}_{n\in\mathbb{N}_{n_0}}$, $\{R_n\}_{n\in\mathbb{N}_{n_0}}$ and $\{W_n\}_{n\in\mathbb{N}_{n_0}}$ satisfying* (2.1), (2.2),

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{v=n}^{\infty} \sum_{u=v}^{\infty} \sum_{s=u}^{\infty} \frac{1}{|a_s|} \max\{R_s H_s, W_s\} = 0; \tag{2.17}$$

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{v=n}^{\infty} \sum_{u=v}^{\infty} \sum_{s=u}^{\infty} \sum_{t=s}^{\infty} \frac{1}{|a_s|} \max\big\{P_t F_t, Q_t, |c_t|\big\} = 0; \tag{2.18}$$

$$b_n = 1 \quad \textit{eventually}. \tag{2.19}$$

Then

(a) *for any $L \in (N,M)$, there exist $\theta \in (0,1)$ and $T \ge n_0 + \tau + \beta$ such that for each $x_0 = \{x_{0n}\}_{n\in\mathbb{N}_\beta} \in A(N,M)$, the Mann iterative sequence $\{x_m\}_{m\in\mathbb{N}_0} = \{\{x_{mn}\}_{n\in\mathbb{N}_\beta}\}_{m\in\mathbb{N}_0}$ generated by the scheme*:

$$x_{m+1n} = \begin{cases} (1-\alpha_m)x_{mn} + \alpha_m\{n^2 L \\ \quad + \sum_{i=1}^{\infty}\sum_{v=n+(2i-1)\tau}^{n+2i\tau-1}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\frac{1}{a_s}[h(s,x_{mh_{1s}},x_{mh_{2s}},\ldots,x_{mh_{ks}}) \\ \quad - \sum_{t=s}^{\infty}(f(t,x_{mf_{1t}},x_{mf_{2t}},\ldots,x_{mf_{kt}}) - c_t)]\}, \quad m \ge 0, n \ge T, \\ (1-\alpha_m)\frac{n^2}{T^2}x_{mT} + \alpha_m\frac{n^2}{T^2}\{T^2 L \\ \quad + \sum_{i=1}^{\infty}\sum_{v=T+(2i-1)\tau}^{T+2i\tau-1}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\frac{1}{a_s}[h(s,x_{mh_{1s}},x_{mh_{2s}},\ldots,x_{mh_{ks}}) \\ \quad - \sum_{t=s}^{\infty}(f(t,x_{mf_{1t}},x_{mf_{2t}},\ldots,x_{mf_{kt}}) - c_t)]\}, \quad m \ge 0, \beta \le n < T \end{cases} \tag{2.20}$$

converges to a positive solution $w = \{w_n\}_{n\in\mathbb{N}_\beta} \in A(N,M)$ of Eq. (1.1) *with* (2.7) *and has the error estimate* (2.8), *where $\{\alpha_m\}_{m\in\mathbb{N}_0}$ is an arbitrary sequence in* $[0,1]$ *satisfying* (2.9);

(b) *Equation* (1.1) *possesses uncountably many positive solutions in $A(N,M)$.*

Proof Set $L \in (N,M)$. It follows from (2.17)-(2.19) that there exist $\theta \in (0,1)$ and $T \ge n_0 + \tau + \beta$ satisfying

$$\theta = \frac{1}{T^2}\sum_{v=T}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\frac{1}{|a_s|}\left(R_sH_s + \sum_{t=s}^{\infty}P_tF_t\right); \tag{2.21}$$

$$\frac{1}{T^2}\sum_{v=T}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\frac{1}{|a_s|}\left(W_s + \sum_{t=s}^{\infty}\bigl(Q_t + |c_t|\bigr)\right) < \min\{M-L, L-N\}; \tag{2.22}$$

$$b_n = 1, \quad \forall n \ge T. \tag{2.23}$$

Define a mapping $S_L : A(N,M) \to l_\beta^\infty$ by

$$S_L x_n = \begin{cases} n^2 L + \sum_{i=1}^{\infty}\sum_{v=n+(2i-1)\tau}^{n+2i\tau-1}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\frac{1}{a_s}\{h(s,x_{h_{1s}},x_{h_{2s}},\ldots,x_{h_{ks}}) \\ \quad - \sum_{t=s}^{\infty}[f(t,x_{f_{1t}},x_{f_{2t}},\ldots,x_{f_{kt}}) - c_t]\}, \quad n \ge T, \\ \frac{n^2}{T^2}S_L x_T, \quad \beta \le n < T \end{cases} \tag{2.24}$$

for each $x = \{x_n\}_{n\in\mathbb{N}_\beta} \in A(N,M)$. Using (2.1), (2.2), (2.21)-(2.24), we obtain that for each $x = \{x_n\}_{n\in\mathbb{N}_\beta}, y = \{y_n\}_{n\in\mathbb{N}_\beta} \in A(N,M)$

$$\begin{aligned} &\left|\frac{S_L x_n}{n^2} - \frac{S_L y_n}{n^2}\right| \\ &\quad \le \frac{1}{n^2}\sum_{i=1}^{\infty}\sum_{v=n+(2i-1)\tau}^{n+2i\tau-1}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\frac{1}{|a_s|}\Biggl(\bigl|h(s,x_{h_{1s}},x_{h_{2s}},\ldots,x_{h_{ks}}) - h(s,y_{h_{1s}},y_{h_{2s}},\ldots,y_{h_{ks}})\bigr| \\ &\qquad + \sum_{t=s}^{\infty}\bigl|f(t,x_{f_{1t}},x_{f_{2t}},\ldots,x_{f_{kt}}) - f(t,y_{f_{1t}},y_{f_{2t}},\ldots,y_{f_{kt}})\bigr|\Biggr) \\ &\quad \le \frac{\|x-y\|}{n^2}\sum_{i=1}^{\infty}\sum_{v=n+(2i-1)\tau}^{n+2i\tau-1}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\frac{1}{|a_s|}\left(R_sH_s + \sum_{t=s}^{\infty}P_tF_t\right) \end{aligned}$$

$$
\begin{aligned}
&\le \frac{\|x-y\|}{T^2}\sum_{v=T}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\frac{1}{|a_s|}\left(R_sH_s+\sum_{t=s}^{\infty}P_tF_t\right)\\
&= \theta\|x-y\|, \quad \forall n\ge T,
\end{aligned}
$$

$$
\left|\frac{S_Lx_n}{n^2}-\frac{S_Ly_n}{n^2}\right| = \left|\frac{S_Lx_T}{T^2}-\frac{S_Ly_T}{T^2}\right| \le \theta\|x-y\|, \quad \beta\le n<T,
$$

$$
\begin{aligned}
\left|\frac{S_Lx_n}{n^2}-L\right| &\le \frac{1}{n^2}\sum_{i=1}^{\infty}\sum_{v=n+(2i-1)\tau}^{n+2i\tau-1}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\frac{1}{|a_s|}\Biggl(\bigl|h(s,x_{h_{1s}},x_{h_{2s}},\dots,x_{h_{ks}})\bigr|\\
&\quad+\sum_{t=s}^{\infty}\bigl[\bigl|f(t,x_{f_{1t}},x_{f_{2t}},\dots,x_{f_{kt}})\bigr|+|c_t|\bigr]\Biggr)\\
&\le \frac{1}{T^2}\sum_{v=T}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\frac{1}{|a_s|}\left(W_s+\sum_{t=s}^{\infty}\bigl(Q_t+|c_t|\bigr)\right)\\
&< \min\{M-L,L-N\}, \quad \forall n\ge T
\end{aligned}
$$

and

$$
\left|\frac{S_Lx_n}{n^2}-L\right| = \left|\frac{S_Lx_T}{T^2}-L\right| < \min\{M-L,L-N\}, \quad \beta\le n<T,
$$

which mean (2.14). Consequently, (2.14) gives that S_L is a contraction in $A(N,M)$ and has a unique fixed point $w=\{w_n\}_{n\in\mathbb{N}_\beta}\in A(N,M)$, that is,

$$
\begin{aligned}
w_n = S_Lw_n = n^2L+\sum_{i=1}^{\infty}\sum_{v=n+(2i-1)\tau}^{n+2i\tau-1}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\frac{1}{a_s}\Biggl\{&h(s,w_{h_{1s}},w_{h_{2s}},\dots,w_{h_{ks}})\\
&-\sum_{t=s}^{\infty}\bigl[f(t,w_{f_{1t}},w_{f_{2t}},\dots,w_{f_{kt}})-c_t\bigr]\Biggr\}, \quad \forall n\ge T
\end{aligned} \tag{2.25}
$$

and (2.16) holds. It follows from (2.25) that

$$
\begin{aligned}
\Delta(w_n+w_{n-\tau}) &= (4n+2-2\tau)L-\sum_{u=n}^{\infty}\sum_{s=u}^{\infty}\frac{1}{a_s}\Biggl\{h(s,w_{h_{1s}},w_{h_{2s}},\dots,w_{h_{ks}})\\
&\quad-\sum_{t=s}^{\infty}\bigl[f(t,w_{f_{1t}},w_{f_{2t}},\dots,w_{f_{kt}})-c_t\bigr]\Biggr\}, \quad \forall n\ge T+\tau,\\
\Delta^2(w_n+w_{n-\tau}) &= 4L+\sum_{s=n}^{\infty}\frac{1}{a_s}\Biggl\{h(s,w_{h_{1s}},w_{h_{2s}},\dots,w_{h_{ks}})\\
&\quad-\sum_{t=s}^{\infty}\bigl[f(t,w_{f_{1t}},w_{f_{2t}},\dots,w_{f_{kt}})-c_t\bigr]\Biggr\}, \quad \forall n\ge T+\tau,\\
a_n\Delta^3(w_n+w_{n-\tau}) &= -h(n,w_{h_{1n}},w_{h_{2n}},\dots,w_{h_{kn}})\\
&\quad+\sum_{t=n}^{\infty}\bigl[f(t,w_{f_{1t}},w_{f_{2t}},\dots,w_{f_{kt}})-c_t\bigr], \quad \forall n\ge T+\tau
\end{aligned}
$$

and

$$\begin{aligned}\Delta\big(a_n\Delta^3(w_n+w_{n-\tau})\big) &= -\Delta h(n, w_{h_{1n}}, w_{h_{2n}}, \ldots, w_{h_{kn}}) \\ &\quad - \big[f(n, w_{f_{1n}}, w_{f_{2n}}, \ldots, w_{f_{kn}}) - c_n\big], \quad \forall n \geq T+\tau,\end{aligned}$$

that is, $w = \{w_n\}_{n\in\mathbb{N}_\beta}$ is a positive solution of Eq. (1.1) in $A(N,M)$. In terms of (2.2), (2.17), (2.18) and (2.25), we infer that

$$\begin{aligned}\left|\frac{w_n}{n^2} - L\right| &= \frac{1}{n^2}\left|\sum_{i=1}^{\infty}\sum_{v=n+(2i-1)\tau}^{n+2i\tau-1}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\frac{1}{a_s}\left\{h(s, w_{h_{1s}}, w_{h_{2s}}, \ldots, w_{h_{ks}})\right.\right. \\ &\quad \left.\left. - \sum_{t=s}^{\infty}\big[f(t, w_{f_{1t}}, w_{f_{2t}}, \ldots, w_{f_{kt}}) - c_t\big]\right\}\right| \\ &\leq \frac{1}{n^2}\sum_{i=1}^{\infty}\sum_{v=n+(2i-1)\tau}^{n+2i\tau-1}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\frac{1}{|a_s|}\left(\big|h(s, w_{h_{1s}}, w_{h_{2s}}, \ldots, w_{h_{ks}})\big|\right. \\ &\quad \left. + \sum_{t=s}^{\infty}\big[\big|f(t, w_{f_{1t}}, w_{f_{2t}}, \ldots, w_{f_{kt}})\big| + |c_t|\big]\right) \\ &\leq \frac{1}{n^2}\sum_{i=1}^{\infty}\sum_{v=n+(2i-1)\tau}^{n+2i\tau-1}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\frac{1}{|a_s|}\left(W_s + \sum_{t=s}^{\infty}\big(Q_t + |c_t|\big)\right) \\ &\leq \frac{1}{n^2}\sum_{v=n}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\frac{1}{|a_s|}\left(W_s + \sum_{t=s}^{\infty}\big(Q_t + |c_t|\big)\right) \\ &\to 0 \quad \text{as } n\to\infty,\end{aligned}$$

that is, (2.7) holds. Linking (2.14), (2.16), (2.20), (2.21) and (2.25), we infer that

$$\begin{aligned}&\frac{|x_{m+1n} - w_n|}{n^2} \\ &\quad = \frac{1}{n^2}\left|(1-\alpha_m)x_{mn} + \alpha_m\left\{n^2L\right.\right. \\ &\qquad + \sum_{i=1}^{\infty}\sum_{v=n+(2i-1)\tau}^{n+2i\tau-1}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\frac{1}{a_s}\left[h(s, x_{mh_{1s}}, x_{mh_{2s}}, \ldots, x_{mh_{ks}})\right. \\ &\qquad \left.\left.\left. - \sum_{t=s}^{\infty}\big[f(t, x_{mf_{1t}}, x_{mf_{2t}}, \ldots, x_{mf_{kt}}) - c_t\big]\right]\right\} - w_n\right| \\ &\quad \leq (1-\alpha_m)\frac{|x_{mn} - w_n|}{n^2} + \alpha_m\frac{|S_Lx_{mn} - S_Lw_n|}{n^2} \\ &\quad \leq (1-\alpha_m)\|x_m - w\| + \theta\alpha_m\|x_m - w\| \\ &\quad = \big[1 - (1-\theta)\alpha_m\big]\|x_m - w\| \\ &\quad \leq e^{-(1-\theta)\alpha_m}\|x_m - w\|, \quad \forall m\in\mathbb{N}_0, n\geq T\end{aligned}$$

and

$$\begin{aligned}
&\frac{|x_{m+1n} - w_n|}{n^2} \\
&= \frac{1}{n^2}\left|(1-\alpha_m)\frac{n^2}{T^2}x_{mT}\right. \\
&\quad + \alpha_m \frac{n^2}{T^2}\left\{T^2L + \sum_{i=1}^{\infty}\sum_{\nu=T+(2i-1)\tau}^{T+2i\tau-1}\sum_{u=\nu}^{\infty}\sum_{s=u}^{\infty}\frac{1}{a_s}\left[h(s, x_{mh_{1s}}, x_{mh_{2s}}, \dots, x_{mh_{ks}})\right.\right. \\
&\quad \left.\left.\left. - \sum_{t=s}^{\infty}\bigl[f(t, x_{mf_{1t}}, x_{mf_{2t}}, \dots, x_{mf_{kt}}) - c_t\bigr]\right]\right\} - w_n\right| \\
&\le (1-\alpha_m)\frac{|x_{mT} - w_T|}{T^2} + \alpha_m \frac{|S_L x_{mT} - S_L w_T|}{T^2} \\
&\le \bigl[1 - (1-\theta)\alpha_m\bigr]\|x_m - w\| \\
&\le e^{-(1-\theta)\alpha_m}\|x_m - w\|, \quad \forall m \in \mathbb{N}_0, \beta \le n < T,
\end{aligned}$$

which imply that

$$\|x_{m+1} - w\| \le e^{-(1-\theta)\alpha_m}\|x_m - w\| \le e^{-(1-\theta)\sum_{i=0}^{m}\alpha_i}\|x_0 - w\|, \quad \forall m \in \mathbb{N}_0,$$

that is, (2.8) holds. It follows from (2.8) and (2.9) that $\lim_{m\to\infty} x_m = w$.

Next we show that (b) holds. Let $L_1, L_2 \in (N, M)$ and $L_1 \neq L_2$. Similar to the proof of (a), we get that for each $c \in \{1, 2\}$, there exist constants $\theta_c \in (0, 1)$, $T_c \ge n_0 + \tau + \beta$ and a mapping S_{L_c} satisfying (2.21)-(2.24), where θ, L and T are replaced by θ_c, L_c and T_c, respectively, and the mapping S_{L_c} has a fixed point $z^c = \{z^c_n\}_{n\in\mathbb{N}_\beta} \in A(N, M)$, which is a positive solution of Eq. (1.1) in $A(N, M)$, that is,

$$\begin{aligned}
z^c_n = n^2 L_c - \sum_{i=1}^{\infty}\sum_{\nu=n+(2i-1)\tau}^{n+2i\tau}\sum_{u=\nu}^{\infty}\sum_{s=u}^{\infty}\frac{1}{a_s}\left\{h\bigl(s, z^c_{h_{1s}}, z^c_{h_{2s}}, \dots, z^c_{h_{ks}}\bigr)\right. \\
\left. - \sum_{t=s}^{\infty}\bigl[f\bigl(t, z^c_{f_{1t}}, z^c_{f_{2t}}, \dots, z^c_{f_{kt}}\bigr) - c_t\bigr]\right\}, \quad \forall n \ge T_c,
\end{aligned}$$

which together with (2.1), (2.10) and (2.23) implies that

$$\begin{aligned}
&\left|\frac{z^1_n}{n^2} - \frac{z^2_n}{n^2}\right| \\
&\ge |L_1 - L_2| - \frac{1}{n^2}\sum_{i=1}^{\infty}\sum_{\nu=n+(2i-1)\tau}^{n+2i\tau}\sum_{u=\nu}^{\infty}\sum_{s=u}^{\infty}\frac{1}{|a_s|}\Bigl(\bigl|h\bigl(s, z^1_{h_{1s}}, z^1_{h_{2s}}, \dots, z^1_{h_{ks}}\bigr) \\
&\quad - h\bigl(s, z^2_{h_{1s}}, z^2_{h_{2s}}, \dots, z^2_{h_{ks}}\bigr)\bigr| + \sum_{t=s}^{\infty}\bigl|f\bigl(t, z^1_{f_{1t}}, z^1_{f_{2t}}, \dots, z^1_{f_{kt}}\bigr) - f\bigl(t, z^2_{f_{1t}}, z^2_{f_{2t}}, \dots, z^2_{f_{kt}}\bigr)\bigr|\Bigr) \\
&\ge |L_1 - L_2| - \frac{1}{n^2}\sum_{i=1}^{\infty}\sum_{\nu=n+(2i-1)\tau}^{n+2i\tau}\sum_{u=\nu}^{\infty}\sum_{s=u}^{\infty}\frac{1}{|a_s|}\Bigl(R_s \max\bigl\{\bigl|z^1_{h_{ls}} - z^2_{h_{ls}}\bigr| : 1 \le l \le k\bigr\}
\end{aligned}$$

$$
\begin{aligned}
&\quad + \sum_{t=s}^{\infty} P_t \max\left\{\left|z^1_{f_{lt}} - z^2_{f_{lt}}\right| : 1 \le l \le k\right\}\Bigg) \\
&\ge |L_1 - L_2| - \frac{\|z^1 - z^2\|}{n^2} \sum_{i=1}^{\infty} \sum_{v=n+(2i-1)\tau}^{n+2i\tau} \sum_{u=v}^{\infty}\sum_{s=u}^{\infty} \frac{1}{|a_s|}\left(R_s H_s + \sum_{t=s}^{\infty} P_t F_t\right) \\
&\ge |L_1 - L_2| - \frac{\|z^1 - z^2\|}{\max\{T_1^2, T_2^2\}} \sum_{v=\max\{T_1,T_2\}}^{\infty} \sum_{u=v}^{\infty}\sum_{s=u}^{\infty} \frac{1}{|a_s|}\left(R_s H_s + \sum_{t=s}^{\infty} P_t F_t\right) \\
&\ge |L_1 - L_2| - \max\{\theta_1, \theta_2\}\left\|z^1 - z^2\right\|, \quad \forall n \ge \max\{T_1, T_2\},
\end{aligned}
$$

which yields that

$$
\left\|z^1 - z^2\right\| \ge \frac{|L_1 - L_2|}{1 + \max\{\theta_1, \theta_2\}} > 0,
$$

that is, $z^1 \neq z^2$. This completes the proof. □

Theorem 2.3 *Assume that there exist three constants b, M and N with* $(1 - b)M > N > 0$ *and four nonnegative sequences* $\{P_n\}_{n\in\mathbb{N}_{n_0}}$, $\{Q_n\}_{n\in\mathbb{N}_{n_0}}$, $\{R_n\}_{n\in\mathbb{N}_{n_0}}$ *and* $\{W_n\}_{n\in\mathbb{N}_{n_0}}$ *satisfying* (2.1), (2.2), (2.17), (2.18) *and*

$$
0 \le b_n \le b < 1 \quad \textit{eventually}. \tag{2.26}
$$

Then

(a) *for any $L \in (bM + N, M)$, there exist $\theta \in (0,1)$ and $T \ge n_0 + \tau + \beta$ such that for each $x_0 = \{x_{0n}\}_{n\in\mathbb{N}_\beta} \in A(N,M)$, the Mann iterative sequence $\{x_m\}_{m\in\mathbb{N}_0} = \{\{x_{mn}\}_{n\in\mathbb{N}_\beta}\}_{m\in\mathbb{N}_0}$ generated by the scheme*:

$$
x_{m+1n} = \begin{cases}
(1-\alpha_m)x_{mn} + \alpha_m\{n^2 L - b_n x_{mn-\tau} \\
\quad + \sum_{v=n}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\frac{1}{a_s}[h(s, x_{mh_{1s}}, x_{mh_{2s}}, \ldots, x_{mh_{ks}}) \\
\quad - \sum_{t=s}^{\infty}(f(t, x_{mf_{1t}}, x_{mf_{2t}}, \ldots, x_{mf_{kt}}) - c_t)]\}, \quad m \ge 0, n \ge T, \\
(1-\alpha_m)\frac{n^2}{T^2}x_{mT} + \alpha_m\frac{n^2}{T^2}\{T^2 L - b_T x_{mT-\tau} \\
\quad + \sum_{s=T}^{\infty}\frac{1}{a_s}[h(s, x_{mh_{1s}}, x_{mh_{2s}}, \ldots, x_{mh_{ks}}) \\
\quad - \sum_{t=s}^{\infty}(f(t, x_{mf_{1t}}, x_{mf_{2t}}, \ldots, x_{mf_{kt}}) - c_t)]\}, \quad m \ge 0, \beta \le n < T
\end{cases} \tag{2.27}
$$

converges to a positive solution $w = \{w_n\}_{n\in\mathbb{N}_\beta} \in A(N,M)$ of Eq. (1.1) *with*

$$
\lim_{n\to\infty}\frac{w_n + b_n w_{n-\tau}}{n^2} = L \tag{2.28}
$$

and has the error estimate (2.8), *where $\{\alpha_m\}_{m\in\mathbb{N}_0}$ is an arbitrary sequence in* $[0,1]$ *satisfying* (2.9);

(b) *Equation* (1.1) *possesses uncountably many positive solutions in* $A(N,M)$.

Proof Put $L \in (bM + N, M)$. It follows from (2.17), (2.18) and (2.26) that there exist $\theta \in (0,1)$ and $T \geq n_0 + \tau + \beta$ satisfying

$$\theta = b + \frac{1}{T^2}\sum_{v=T}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\frac{1}{|a_s|}\left(R_sH_s + \sum_{t=s}^{\infty}P_tF_t\right); \tag{2.29}$$

$$\frac{1}{T^2}\sum_{v=T}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\frac{1}{|a_s|}\left(W_s + \sum_{t=s}^{\infty}\bigl(Q_t + |c_t|\bigr)\right) < \min\{M - L, L - bM - N\}; \tag{2.30}$$

$$0 \leq b_n \leq b < 1, \quad \forall n \geq T. \tag{2.31}$$

Define a mapping $S_L : A(N,M) \to l_\beta^\infty$ by

$$S_Lx_n = \begin{cases} n^2L - b_nx_{n-\tau} + \sum_{v=n}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\frac{1}{a_s}\{h(s, x_{h_{1s}}, x_{h_{2s}}, \dots, x_{h_{ks}}) \\ \quad - \sum_{t=s}^{\infty}[f(t, x_{f_{1t}}, x_{f_{2t}}, \dots, x_{f_{kt}}) - c_t]\}, \quad n \geq T, \\ \frac{n^2}{T^2}S_Lx_T, \quad \beta \leq n < T \end{cases} \tag{2.32}$$

for each $x = \{x_n\}_{n\in\mathbb{N}_\beta} \in A(N,M)$. According to (2.1), (2.2) and (2.29)-(2.32), we obtain that for each $x = \{x_n\}_{n\in\mathbb{N}_\beta}$, $y = \{y_n\}_{n\in\mathbb{N}_\beta} \in A(N,M)$

$$\begin{aligned}
&\left|\frac{S_Lx_n}{n^2} - \frac{S_Ly_n}{n^2}\right| \\
&\quad \leq b_n \cdot \frac{(n-\tau)^2}{n^2}\left|\frac{x_{n-\tau} - y_{n-\tau}}{(n-\tau)^2}\right| \\
&\qquad + \frac{1}{n^2}\sum_{v=n}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\frac{1}{|a_s|}\Biggl(\bigl|h(s, x_{h_{1s}}, x_{h_{2s}}, \dots, x_{h_{ks}}) - h(s, y_{h_{1s}}, y_{h_{2s}}, \dots, y_{h_{ks}})\bigr| \\
&\qquad + \sum_{t=s}^{\infty}\bigl|f(t, x_{f_{1t}}, x_{f_{2t}}, \dots, x_{f_{kt}}) - f(t, y_{f_{1t}}, y_{f_{2t}}, \dots, y_{f_{kt}})\bigr|\Biggr) \\
&\quad \leq \left[b + \frac{1}{T^2}\sum_{v=T}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\frac{1}{|a_s|}\left(R_sH_s + \sum_{t=s}^{\infty}P_tF_t\right)\right]\|x - y\| \\
&\quad = \theta\|x - y\|, \quad \forall n \geq T, \\
&\left|\frac{S_Lx_n}{n^2} - \frac{S_Ly_n}{n^2}\right| = \frac{n^2}{T^2}\left|\frac{S_Lx_T}{n^2} - \frac{S_Ly_T}{n^2}\right| \leq \theta\|x - y\|, \quad \beta \leq n < T, \\
&\frac{S_Lx_n}{n^2} \leq L + \frac{1}{n^2}\sum_{v=n}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\frac{1}{|a_s|}\Biggl(\bigl|h(s, x_{h_{1s}}, x_{h_{2s}}, \dots, x_{h_{ks}})\bigr| \\
&\qquad + \sum_{t=s}^{\infty}\bigl[\bigl|f(t, x_{f_{1t}}, x_{f_{2t}}, \dots, x_{f_{kt}})\bigr| + |c_t|\bigr]\Biggr) \\
&\quad \leq L + \frac{1}{T^2}\sum_{v=T}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\frac{1}{|a_s|}\left(W_s + \sum_{t=s}^{\infty}\bigl(Q_t + |c_t|\bigr)\right) \\
&\quad < L + \min\{M - L, L - bM - N\} \\
&\quad \leq M, \quad \forall n \geq T,
\end{aligned}$$

$$
\begin{aligned}
\frac{S_L x_n}{n^2} &= \frac{n^2}{T^2}\cdot\frac{S_L x_T}{n^2} \le M, \quad \beta \le n < T, \\
\frac{S_L x_n}{n^2} &= L - b_n \frac{x_{n-\tau}}{(n-\tau)^2}\cdot\frac{(n-\tau)^2}{n^2} + \frac{1}{n^2}\sum_{v=n}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\frac{1}{a_s}\Biggl(h(s, x_{h_{1s}}, x_{h_{2s}}, \ldots, x_{h_{ks}}) \\
&\quad - \sum_{t=s}^{\infty}\bigl[f(t, x_{f_{1t}}, x_{f_{2t}}, \ldots, x_{f_{kt}}) - c_t\bigr]\Biggr) \\
&\ge L - bM - \frac{1}{n^2}\sum_{v=n}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\frac{1}{|a_s|}\Biggl(\bigl|h(s, x_{h_{1s}}, x_{h_{2s}}, \ldots, x_{h_{ks}})\bigr| \\
&\quad + \sum_{t=s}^{\infty}\bigl[\bigl|f(t, x_{f_{1t}}, x_{f_{2t}}, \ldots, x_{f_{kt}})\bigr| + |c_t|\bigr]\Biggr) \\
&\ge L - bM - \frac{1}{T^2}\sum_{v=T}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\frac{1}{|a_s|}\Biggl(W_s + \sum_{t=s}^{\infty}\bigl(Q_t + |c_t|\bigr)\Biggr) \\
&> L - bM - \min\{M - L, L - bM - N\} \\
&\ge N, \quad \forall n \ge T
\end{aligned}
$$

and

$$
\frac{S_L x_n}{n^2} = \frac{n^2}{T^2}\cdot\frac{S_L x_T}{n^2} \ge N, \quad \beta \le n < T,
$$

which give (2.14), in turns, which implies that S_L is a contraction in $A(N, M)$ and possesses a unique fixed point $w = \{w_n\}_{n\in\mathbb{N}_\beta} \in A(N, M)$, that is,

$$
\begin{aligned}
w_n = S_L w_n = n^2 L - b_n w_{n-\tau} &+ \sum_{v=n}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\frac{1}{a_s}\Biggl\{h(s, w_{h_{1s}}, w_{h_{2s}}, \ldots, w_{h_{ks}}) \\
&- \sum_{t=s}^{\infty}\bigl[f(t, w_{f_{1t}}, w_{f_{2t}}, \ldots, w_{f_{kt}}) - c_t\bigr]\Biggr\}, \quad \forall n \ge T
\end{aligned}
\tag{2.33}
$$

and (2.16) is satisfied. It is easy to verify that (2.33) yields that

$$
\begin{aligned}
\Delta(w_n + b_n w_{n-\tau}) &= (2n+1)L - \sum_{u=n}^{\infty}\sum_{s=u}^{\infty}\frac{1}{a_s}\Biggl\{h(s, w_{h_{1s}}, w_{h_{2s}}, \ldots, w_{h_{ks}}) \\
&\quad - \sum_{t=s}^{\infty}\bigl[f(t, w_{f_{1t}}, w_{f_{2t}}, \ldots, w_{f_{kt}}) - c_t\bigr]\Biggr\}, \quad \forall n \ge T + \tau, \\
\Delta^2(w_n + b_n w_{n-\tau}) &= 2L + \sum_{s=n}^{\infty}\frac{1}{a_s}\Biggl\{h(s, w_{h_{1s}}, w_{h_{2s}}, \ldots, w_{h_{ks}}) \\
&\quad - \sum_{t=s}^{\infty}\bigl[f(t, w_{f_{1t}}, w_{f_{2t}}, \ldots, w_{f_{kt}}) - c_t\bigr]\Biggr\}, \quad \forall n \ge T + \tau, \\
a_n\Delta^3(w_n + b_n w_{n-\tau}) &= -h(n, w_{h_{1n}}, w_{h_{2n}}, \ldots, w_{h_{kn}}) \\
&\quad + \sum_{t=n}^{\infty}\bigl[f(t, w_{f_{1t}}, w_{f_{2t}}, \ldots, w_{f_{kt}}) - c_t\bigr], \quad \forall n \ge T + \tau
\end{aligned}
$$

and

$$\begin{aligned}\Delta\big(a_n\Delta^3(w_n+b_nw_{n-\tau})\big) &= -\Delta h(n,w_{h_{1n}},w_{h_{2n}},\ldots,w_{h_{kn}})\\ &\quad -f(n,w_{f_{1n}},w_{f_{2n}},\ldots,w_{f_{kn}})+c_n,\quad \forall n\ge T+\tau,\end{aligned}$$

that is, $w=\{w_n\}_{n\in\mathbb{N}_\beta}$ is a positive solution of Eq. (1.1) in $A(N,M)$. Making use of (2.17), (2.18) and (2.33), we infer that

$$\begin{aligned}\left|\frac{w_n+b_nw_{n-\tau}}{n^2}-L\right| &\le \frac{1}{n^2}\sum_{v=n}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\frac{1}{|a_s|}\Bigg(\big|h(s,w_{h_{1s}},w_{h_{2s}},\ldots,w_{h_{ks}})\big|\\ &\quad+\sum_{t=s}^{\infty}\big[\big|f(t,w_{f_{1t}},w_{f_{2t}},\ldots,w_{f_{kt}})\big|+|c_t|\big]\Bigg)\\ &\le \frac{1}{n^2}\sum_{v=n}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\frac{1}{|a_s|}\Bigg(W_s+\sum_{t=s}^{\infty}\big(Q_t+|c_t|\big)\Bigg)\\ &\to 0\quad\text{as } n\to\infty,\end{aligned}$$

which gives (2.28). In light of (2.14), (2.16), (2.27), (2.29) and (2.33), we deduce that

$$\begin{aligned}\frac{|x_{m+1n}-w_n|}{n^2} &= \frac{1}{n^2}\Bigg|(1-\alpha_m)x_{mn}+\alpha_m\Bigg\{n^2L-b_nx_{mn-\tau}\\ &\quad+\sum_{v=n}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\frac{1}{a_s}\Bigg[h(s,x_{mh_{1s}},x_{mh_{2s}},\ldots,x_{mh_{ks}})\\ &\quad-\sum_{t=s}^{\infty}\big[f(t,x_{mf_{1t}},x_{mf_{2t}},\ldots,x_{mf_{kt}})-c_t\big]\Bigg]\Bigg\}-w_n\Bigg|\\ &\le (1-\alpha_m)\frac{|x_{mn}-w_n|}{n^2}+\alpha_m\frac{|S_Lx_{mn}-S_Lw_n|}{n^2}\\ &\le (1-\alpha_m)\|x_m-w\|+\theta\alpha_m\|x_m-w\|=\big[1-(1-\theta)\alpha_m\big]\|x_m-w\|\\ &\le e^{-(1-\theta)\alpha_m}\|x_m-w\|,\quad \forall m\in\mathbb{N}_0, n\ge T\end{aligned}$$

and

$$\begin{aligned}\frac{|x_{m+1n}-w_n|}{n^2} &= \frac{1}{n^2}\Bigg|(1-\alpha_m)\frac{n^2}{T^2}x_{mT}+\alpha_m\frac{n^2}{T^2}\Bigg\{T^2L-b_Tx_{mT-\tau}\\ &\quad+\sum_{v=T}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\frac{1}{a_s}\Bigg[h(s,x_{mh_{1s}},x_{mh_{2s}},\ldots,x_{mh_{ks}})\\ &\quad-\sum_{t=s}^{\infty}\big[f(t,x_{mf_{1t}},x_{mf_{2t}},\ldots,x_{mf_{kt}})-c_t\big]\Bigg]\Bigg\}-w_n\Bigg|\\ &\le (1-\alpha_m)\frac{|x_{mT}-w_T|}{T^2}+\alpha_m\frac{|S_Lx_{mT}-S_Lw_T|}{T^2}\\ &\le \big[1-(1-\theta)\alpha_m\big]\|x_m-w\|\\ &\le e^{-(1-\theta)\alpha_m}\|x_m-w\|,\quad \forall m\in\mathbb{N}_0, \beta\le n<T,\end{aligned}$$

which imply that

$$\|x_{m+1} - w\| \le e^{-(1-\theta)\alpha_m} \|x_m - w\| \le e^{-(1-\theta)\sum_{i=0}^{m} \alpha_i} \|x_0 - w\|, \quad \forall m \in \mathbb{N}_0,$$

that is, (2.8) holds. It follows from (2.8) and (2.9) that $\lim_{m\to\infty} x_m = w$.

Next we show that (b) holds. Let $L_1, L_2 \in (bM + N, M)$ and $L_1 \neq L_2$. Similar to the proof of (a), we get that for each $c \in \{1,2\}$ there exist constants $\theta_c \in (0,1)$, $T_c \ge n_0 + \tau + \beta$ and a mapping S_{L_c} satisfying (2.29)-(2.32), where θ, L and T are replaced by θ_c, L_c and T_c, respectively, and the mapping S_{L_c} has a fixed point $z^c = \{z_n^c\}_{n\in\mathbb{N}_\beta} \in A(N,M)$, which is a positive solution of Eq. (1.1) in $A(N,M)$, that is,

$$\begin{aligned} z_n^c = n^2 L_c - b_n z_{n-\tau}^c + \sum_{v=n}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty} \frac{1}{a_s} \Big\{ & h\big(s, z_{h_{1s}}^c, z_{h_{2s}}^c, \ldots, z_{h_{ks}}^c\big) \\ & - \sum_{t=s}^{\infty} \big[f\big(t, z_{f_{1t}}^c, z_{f_{2t}}^c, \ldots, z_{f_{kt}}^c\big) - c_t \big] \Big\}, \quad \forall n \ge T_c, \end{aligned}$$

which together with (2.1), (2.29) and (2.31) means that

$$\begin{aligned} &\left| \frac{z_n^1}{n^2} - \frac{z_n^2}{n^2} \right| \\ &\quad \ge |L_1 - L_2| - b_n \frac{|z_n^1(n-\tau) - z_n^2(n-\tau)|}{(n-\tau)^2} \cdot \frac{(n-\tau)^2}{n^2} \\ &\qquad - \frac{1}{n^2} \sum_{v=n}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty} \frac{1}{|a_s|} \bigg(\big| h\big(s, z_{h_{1s}}^1, z_{h_{2s}}^1, \ldots, z_{h_{ks}}^1\big) - h\big(s, z_{h_{1s}}^2, z_{h_{2s}}^2, \ldots, z_{h_{ks}}^2\big) \big| \\ &\qquad + \sum_{t=s}^{\infty} \big| f\big(t, z_{f_{1t}}^1, z_{f_{2t}}^1, \ldots, z_{f_{kt}}^1\big) - f\big(t, z_{f_{1t}}^2, z_{f_{2t}}^2, \ldots, z_{f_{kt}}^2\big) \big| \bigg) \\ &\quad \ge |L_1 - L_2| - b \big\| z^1 - z^2 \big\| - \frac{1}{n^2} \sum_{v=n}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty} \frac{1}{|a_s|} \bigg(R_s \max\big\{ \big| z_{h_{ls}}^1 - z_{h_{ls}}^2 \big| : 1 \le l \le k \big\} \\ &\qquad + \sum_{t=s}^{\infty} P_t \max\big\{ \big| z_{f_{lt}}^1 - z_{f_{lt}}^2 \big| : 1 \le l \le k \big\} \bigg) \\ &\quad \ge |L_1 - L_2| - b \big\| z^1 - z^2 \big\| - \frac{\|z^1 - z^2\|}{n^2} \sum_{v=n}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty} \frac{1}{|a_s|} \bigg(R_s H_s + \sum_{t=s}^{\infty} P_t F_t \bigg) \\ &\quad \ge |L_1 - L_2| - b \big\| z^1 - z^2 \big\| - \frac{\|z^1 - z^2\|}{\max\{T_1^2, T_2^2\}} \sum_{v=\max\{T_1,T_2\}}^{\infty} \sum_{u=v}^{\infty}\sum_{s=u}^{\infty} \frac{1}{|a_s|} \bigg(R_s H_s + \sum_{t=s}^{\infty} P_t F_t \bigg) \\ &\quad \ge |L_1 - L_2| - \max\{\theta_1, \theta_2\} \big\| z^1 - z^2 \big\|, \quad \forall n \ge \max\{T_1, T_2\}, \end{aligned}$$

which yields that

$$\big\| z^1 - z^2 \big\| \ge \frac{|L_1 - L_2|}{1 + \max\{\theta_1, \theta_2\}} > 0,$$

that is, $z^1 \neq z^2$. This completes the proof. □

Theorem 2.4 *Assume that there exist constants b, M and N with $(1+b)M > N > 0$ and four nonnegative sequences $\{P_n\}_{n\in\mathbb{N}_{n_0}}$, $\{Q_n\}_{n\in\mathbb{N}_{n_0}}$, $\{R_n\}_{n\in\mathbb{N}_{n_0}}$, $\{W_n\}_{n\in\mathbb{N}_{n_0}}$ satisfying* (2.1), (2.2), (2.17), (2.18) *and*

$$-1 < b \le b_n \le 0 \quad \textit{eventually}. \tag{2.34}$$

Then

(a) *for any $L \in (N, (1+b)M)$, there exist $\theta \in (0,1)$ and $T \ge n_0 + \tau + \beta$ such that for each $x_0 = \{x_{0n}\}_{n\in\mathbb{N}_\beta} \in A(N,M)$, the Mann iterative sequence $\{x_m\}_{m\in\mathbb{N}_0} = \{\{x_{mn}\}_{n\in\mathbb{N}_\beta}\}_{m\in\mathbb{N}_0}$ generated by* (2.27) *converges to a positive solution $w = \{w_n\}_{n\in\mathbb{N}_\beta} \in A(N,M)$ of Eq.* (1.1) *with* (2.28) *and has the error estimate* (2.8), *where $\{\alpha_m\}_{m\in\mathbb{N}_0}$ is an arbitrary sequence in* $[0,1]$ *satisfying* (2.9);

(b) *Equation* (1.1) *possesses uncountably many positive solutions in $A(N,M)$.*

Proof Put $L \in (N, (1+b)M)$. It follows from (2.17), (2.18) and (2.34) that there exist $\theta \in (0,1)$ and $T \ge n_0 + \tau + \beta$ satisfying

$$\theta = -b + \frac{1}{T^2}\sum_{v=T}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\frac{1}{|a_s|}\left(R_sH_s + \sum_{t=s}^{\infty}P_tF_t\right); \tag{2.35}$$

$$\frac{1}{T^2}\sum_{v=T}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\frac{1}{|a_s|}\left(W_s + \sum_{t=s}^{\infty}\big(Q_t + |c_t|\big)\right) < \min\big\{(1+b)M - L, L - N\big\}; \tag{2.36}$$

$$-1 < b \le b_n \le 0, \quad \forall n \ge T. \tag{2.37}$$

Define a mapping $S_L : A(N,M) \to l_\beta^\infty$ by (2.32). By virtue of (2.2), (2.32), (2.36) and (2.37), we easily verify that

$$\begin{aligned}
\frac{S_Lx_n}{n^2} &\le L - b_n\frac{x_{n-\tau}}{(n-\tau)^2}\cdot\frac{(n-\tau)^2}{n^2} + \frac{1}{n^2}\sum_{v=n}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\frac{1}{|a_s|}\Bigg(\big|h(s, x_{h_{1s}}, x_{h_{2s}}, \dots, x_{h_{ks}})\big| \\
&\quad + \sum_{t=s}^{\infty}\big[\big|f(t, x_{f_{1t}}, x_{f_{2t}}, \dots, x_{f_{kt}})\big| + |c_t|\big]\Bigg) \\
&\le L - bM + \frac{1}{T^2}\sum_{v=T}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\frac{1}{|a_s|}\left(W_s + \sum_{t=s}^{\infty}\big(Q_t + |c_t|\big)\right) \\
&< L - bM + \min\big\{(1+b)M - L, L - N\big\} \\
&\le M, \quad \forall n \ge T, \\
\frac{S_Lx_n}{n^2} &= \frac{n^2}{T^2}\cdot\frac{S_Lx_T}{n^2} \le M, \quad \beta \le n < T,
\end{aligned}$$

$$\begin{aligned}
\frac{S_L x_n}{n^2} &\geq L - \frac{1}{n^2}\sum_{v=T}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\frac{1}{|a_s|}\Biggl(\bigl|h(s, x_{h_{1s}}, x_{h_{2s}}, \ldots, x_{h_{ks}})\bigr| \\
&\quad + \sum_{t=s}^{\infty}\bigl[\bigl|f(t, x_{f_{1t}}, x_{f_{2t}}, \ldots, x_{f_{kt}})\bigr| + |c_t|\bigr]\Biggr) \\
&\geq L - \frac{1}{T^2}\sum_{v=T}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\frac{1}{|a_s|}\Biggl(W_s + \sum_{t=s}^{\infty}(Q_t + |c_t|)\Biggr) \\
&> L - \min\bigl\{(1+b)M - L, L - N\bigr\} \\
&\geq N, \quad \forall n \geq T
\end{aligned}$$

and

$$\frac{S_L x_n}{n^2} = \frac{n^2}{T^2}\cdot\frac{S_L x_T}{n^2} \geq N, \quad \beta \leq n < T,$$

which yield that $S_L(A(N,M)) \subseteq A(N,M)$. The rest of the proof is similar to that of Theorem 2.3 and is omitted. This completes the proof. □

Theorem 2.5 *Assume that there exist constants q, b_*, b^*, M and N and four nonnegative sequences $\{P_n\}_{n\in\mathbb{N}_{n_0}}$, $\{Q_n\}_{n\in\mathbb{N}_{n_0}}$, $\{R_n\}_{n\in\mathbb{N}_{n_0}}$, $\{W_n\}_{n\in\mathbb{N}_{n_0}}$ satisfying* (2.1), (2.2), (2.17), (2.18) *and*

$$q^2 b^* < 1 < b_* q, \qquad b^*(Mq+N) < \frac{M}{q} + \frac{N}{qb^*}, \tag{2.38}$$

$$1 < b_* \leq b_n \leq b^*, \quad \text{eventually}. \tag{2.39}$$

Then

(a) *for any $L \in (b^*(Mq+N), \frac{M}{q} + \frac{N}{qb^*})$, there exist $\theta \in (0,1)$ and $T \geq n_0 + \tau + \beta$ such that for each $x_0 = \{x_{0n}\}_{n\in\mathbb{N}_\beta} \in A(N,M)$, the Mann iterative sequence $\{x_m\}_{m\in\mathbb{N}_0} = \{\{x_{mn}\}_{n\in\mathbb{N}_\beta}\}_{m\in\mathbb{N}_0}$ generated by the scheme*:

$$x_{m+1n} = \begin{cases} (1-\alpha_m)x_{mn} + \alpha_m\{\frac{(n+\tau)^2 L}{b_{n+\tau}} - \frac{x_{mn+\tau}}{b_{n+\tau}} \\ \quad + \frac{1}{b_{n+\tau}}\sum_{v=n+\tau}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\frac{1}{a_s}[h(s, x_{mh_{1s}}, x_{mh_{2s}}, \ldots, x_{mh_{ks}}) \\ \quad - \sum_{t=s}^{\infty}(f(t, x_{mf_{1t}}, x_{mf_{2t}}, \ldots, x_{mf_{kt}}) - c_t)]\}, \quad m \geq 0, n \geq T, \\ (1-\alpha_m)\frac{n^2}{T^2}x_{mT} + \alpha_m\frac{n^2}{T^2}\{\frac{(T+\tau)^2 L}{b_{T+\tau}} - \frac{x_{mT+\tau}}{b_{T+\tau}} \\ \quad + \sum_{v=T+\tau}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\frac{1}{a_s}[h(s, x_{mh_{1s}}, x_{mh_{2s}}, \ldots, x_{mh_{ks}}) \\ \quad - \sum_{t=s}^{\infty}(f(t, x_{mf_{1t}}, x_{mf_{2t}}, \ldots, x_{mf_{kt}}) - c_t)]\}, \quad m \geq 0, \beta \leq n < T \end{cases} \tag{2.40}$$

converges to a positive solution $w = \{w_n\}_{n\in\mathbb{N}_\beta} \in A(N,M)$ of Eq. (1.1) *with* (2.28) *and has the error estimate* (2.8), *where $\{\alpha_m\}_{m\in\mathbb{N}_0}$ is an arbitrary sequence in* $[0,1]$ *satisfying* (2.9);

(b) *Equation* (1.1) *possesses uncountably many positive solutions in $A(N,M)$.*

Proof Let $L \in (b^*(Mq+N), \frac{M}{q} + \frac{N}{qb^*})$. It follows from (2.17), (2.18), (2.38) and (2.39) that there exist $\theta \in (0,1)$ and $T \ge n_0 + \tau + \beta$ satisfying

$$\theta = q + \frac{1}{b_* T^2} \sum_{v=T}^{\infty} \sum_{u=v}^{\infty} \sum_{s=u}^{\infty} \frac{1}{|a_s|} \left(R_s H_s + \sum_{t=s}^{\infty} P_t F_t \right); \tag{2.41}$$

$$\begin{aligned} &\frac{1}{b_* T^2} \sum_{v=T}^{\infty} \sum_{u=v}^{\infty} \sum_{s=u}^{\infty} \frac{1}{|a_s|} \left(W_s + \sum_{t=s}^{\infty} \bigl(Q_t + |c_t| \bigr) \right) \\ &\quad < \min\left\{ M - qL + \frac{N}{b^*}, \frac{L}{b^*} - Mq - N \right\}; \end{aligned} \tag{2.42}$$

$$\left(1 + \frac{\tau}{n}\right)^2 < b_* q, \quad 1 < b_* \le b_n \le b^*, \forall n \ge T. \tag{2.43}$$

Define a mapping $S_L : A(N,M) \to l_\beta^\infty$ by

$$S_L x_n = \begin{cases} \frac{(n+\tau)^2 L}{b_{n+\tau}} - \frac{x_{n+\tau}}{b_{n+\tau}} + \frac{1}{b_{n+\tau}} \sum_{v=n+\tau}^{\infty} \sum_{u=v}^{\infty} \sum_{s=u}^{\infty} \frac{1}{a_s} \{ h(s, x_{h_{1s}}, x_{h_{2s}}, \ldots, x_{h_{ks}}) \\ \quad - \sum_{t=s}^{\infty} [f(t, x_{f_{1t}}, x_{f_{2t}}, \ldots, x_{f_{kt}}) - c_t] \}, \quad n \ge T, \\ \frac{n^2}{T^2} S_L x_T, \quad \beta \le n < T \end{cases} \tag{2.44}$$

for each $x = \{x_n\}_{n \in \mathbb{N}_\beta} \in A(N,M)$. On account of (2.1), (2.2) and (2.41)-(2.44), we ensure that for each $x = \{x_n\}_{n \in \mathbb{N}_\beta}, y = \{y_n\}_{n \in \mathbb{N}_\beta} \in A(N,M)$

$$\begin{aligned} &\left| \frac{S_L x_n}{n^2} - \frac{S_L y_n}{n^2} \right| \\ &\quad \le \frac{1}{b_{n+\tau}} \cdot \frac{(n+\tau)^2}{n^2} \cdot \frac{|x_{n+\tau} - y_{n+\tau}|}{(n+\tau)^2} \\ &\qquad + \frac{1}{b_{n+\tau} n^2} \sum_{v=n+\tau}^{\infty} \sum_{u=v}^{\infty} \sum_{s=u}^{\infty} \frac{1}{|a_s|} \Bigl(\bigl| h(s, x_{h_{1s}}, x_{h_{2s}}, \ldots, x_{h_{ks}}) - h(s, y_{h_{1s}}, y_{h_{2s}}, \ldots, y_{h_{ks}}) \bigr| \\ &\qquad + \sum_{t=s}^{\infty} \bigl| f(t, x_{f_{1t}}, x_{f_{2t}}, \ldots, x_{f_{kt}}) - f(t, y_{f_{1t}}, y_{f_{2t}}, \ldots, y_{f_{kt}}) \bigr| \Bigr) \\ &\quad \le \frac{1}{b_*} \left(1 + \frac{\tau}{T}\right)^2 \|x - y\| \\ &\qquad + \frac{1}{b_* T^2} \sum_{v=T}^{\infty} \sum_{u=v}^{\infty} \sum_{s=u}^{\infty} \frac{1}{|a_s|} \Bigl(R_s \max\bigl\{ |x_{h_{ls}} - y_{h_{ls}}| : 1 \le l \le k \bigr\} \\ &\qquad + \sum_{t=s}^{\infty} P_t \max\bigl\{ |x_{f_{lt}} - y_{f_{lt}}| : 1 \le l \le k \bigr\} \Bigr) \\ &\quad \le \left[q + \frac{1}{b_* T^2} \sum_{v=T}^{\infty} \sum_{u=v}^{\infty} \sum_{s=u}^{\infty} \frac{1}{|a_s|} \left(R_s H_s + \sum_{t=s}^{\infty} P_t F_t \right) \right] \|x - y\| \\ &\quad = \theta \|x - y\|, \quad \forall n \ge T, \\ &\left| \frac{S_L x_n}{n^2} - \frac{S_L y_n}{n^2} \right| = \frac{n^2}{T^2} \left| \frac{S_L x_T}{n^2} - \frac{S_L y_T}{n^2} \right| \le \theta \|x - y\|, \quad \beta \le n < T, \end{aligned}$$

$$\begin{aligned}
\frac{S_L x_n}{n^2} &= \left(1+\frac{\tau}{n}\right)^2 \frac{L}{b_{n+\tau}} - \frac{1}{b_{n+\tau}}\left(1+\frac{\tau}{n}\right)^2 \frac{x_{n+\tau}}{(n+\tau)^2} \\
&\quad + \frac{1}{b_{n+\tau} n^2} \sum_{v=n+\tau}^{\infty} \sum_{u=v}^{\infty} \sum_{s=u}^{\infty} \frac{1}{a_s} \Biggl(h(s, x_{h_{1s}}, x_{h_{2s}}, \ldots, x_{h_{ks}}) \\
&\quad - \sum_{t=s}^{\infty} \bigl[f(t, x_{f_{1t}}, x_{f_{2t}}, \ldots, x_{f_{kt}}) - c_t \bigr] \Biggr) \\
&\le \left(1+\frac{\tau}{n}\right)^2 \frac{L}{b_*} - \frac{N}{b^*} + \frac{1}{b_* n^2} \sum_{v=n+\tau}^{\infty} \sum_{u=v}^{\infty} \sum_{s=u}^{\infty} \frac{1}{|a_s|} \Biggl\{ \bigl| h(s, x_{h_{1s}}, x_{h_{2s}}, \ldots, x_{h_{ks}}) \bigr| \\
&\quad + \sum_{t=s}^{\infty} \bigl[\bigl| f(t, x_{f_{1t}}, x_{f_{2t}}, \ldots, x_{f_{kt}}) \bigr| + |c_t| \bigr] \Biggr\} \\
&\le qL - \frac{N}{b^*} + \frac{1}{b_* T^2} \sum_{v=T}^{\infty} \sum_{u=v}^{\infty} \sum_{s=u}^{\infty} \frac{1}{|a_s|} \left(W_s + \sum_{t=s}^{\infty} \bigl(Q_t + |c_t| \bigr) \right) \\
&< qL - \frac{N}{b^*} + \min\left\{ M - qL + \frac{N}{b^*}, \frac{L}{b^*} - Mq - N \right\} \\
&\le M, \quad \forall n \ge T, \\
\frac{S_L x_n}{n^2} &= \frac{n^2}{T^2} \cdot \frac{S_L x_T}{n^2} \le M, \quad \beta \le n < T, \\
\frac{S_L x_n}{n^2} &\ge \frac{L}{b^*} - \frac{M}{b_*}\left(1+\frac{\tau}{n}\right)^2 - \frac{1}{b_* n^2} \sum_{v=n+\tau}^{\infty} \sum_{u=v}^{\infty} \sum_{s=u}^{\infty} \frac{1}{|a_s|} \Biggl(\bigl| h(s, x_{h_{1s}}, x_{h_{2s}}, \ldots, x_{h_{ks}}) \bigr| \\
&\quad + \sum_{t=s}^{\infty} \bigl[\bigl| f(t, x_{f_{1t}}, x_{f_{2t}}, \ldots, x_{f_{kt}}) \bigr| + |c_t| \bigr] \Biggr) \\
&\ge \frac{L}{b^*} - Mq - \frac{1}{b_* T^2} \sum_{v=T}^{\infty} \sum_{u=v}^{\infty} \sum_{s=u}^{\infty} \frac{1}{|a_s|} \left(W_s + \sum_{t=s}^{\infty} \bigl(Q_t + |c_t| \bigr) \right) \\
&> \frac{L}{b^*} - Mq - \min\left\{ M - qL + \frac{N}{b^*}, \frac{L}{b^*} - Mq - N \right\} \\
&\ge N, \quad \forall n \ge T
\end{aligned}$$

and

$$\frac{S_L x_n}{n^2} = \frac{n^2}{T^2} \cdot \frac{S_L x_T}{n^2} \ge N, \quad \beta \le n < T,$$

which mean (2.14). It follows from the Banach fixed point theorem that the contraction mapping S_L possesses a unique fixed point $w = \{w_n\}_{n \in \mathbb{N}_\beta} \in A(N, M)$, that is,

$$\begin{aligned}
w_n = S_L w_n &= \frac{(n+\tau)^2}{b_{n+\tau}} L - \frac{w_{n+\tau}}{b_{n+\tau}} + \frac{1}{b_{n+\tau}} \sum_{v=n+\tau}^{\infty} \sum_{u=v}^{\infty} \sum_{s=u}^{\infty} \frac{1}{a_s} \Biggl\{ h(s, w_{h_{1s}}, w_{h_{2s}}, \ldots, w_{h_{ks}}) \\
&\quad - \sum_{t=s}^{\infty} \bigl[f(t, w_{f_{1t}}, w_{f_{2t}}, \ldots, w_{f_{kt}}) - c_t \bigr] \Biggr\}, \quad \forall n \ge T
\end{aligned} \tag{2.45}$$

and (2.16) is satisfied. It is easy to verify that (2.45) yields that

$$\begin{aligned} w_n + b_n w_{n-\tau} &= n^2 L + \sum_{v=n}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\frac{1}{a_s}\Bigg\{h(s, w_{h_{1s}}, w_{h_{2s}}, \ldots, w_{h_{ks}}) \\ &\quad - \sum_{t=s}^{\infty}\big[f(t, w_{f_{1t}}, w_{f_{2t}}, \ldots, w_{f_{kt}}) - c_t\big]\Bigg\}, \quad \forall n \ge T+\tau, \qquad (2.46) \\ \Delta(w_n + b_n w_{n-\tau}) &= (2n+1)L - \sum_{u=n}^{\infty}\sum_{s=u}^{\infty}\frac{1}{a_s}\Bigg\{h(s, w_{h_{1s}}, w_{h_{2s}}, \ldots, w_{h_{ks}}) \\ &\quad - \sum_{t=s}^{\infty}\big[f(t, w_{f_{1t}}, w_{f_{2t}}, \ldots, w_{f_{kt}}) - c_t\big]\Bigg\}, \quad \forall n \ge T+\tau, \\ \Delta^2(w_n + b_n w_{n-\tau}) &= 2L + \sum_{s=n}^{\infty}\frac{1}{a_s}\Bigg\{h(s, w_{h_{1s}}, w_{h_{2s}}, \ldots, w_{h_{ks}}) \\ &\quad - \sum_{t=s}^{\infty}\big[f(t, w_{f_{1t}}, w_{f_{2t}}, \ldots, w_{f_{kt}}) - c_t\big]\Bigg\}, \quad \forall n \ge T+\tau, \\ a_n\Delta^3(w_n + b_n w_{n-\tau}) &= -h(n, w_{h_{1n}}, w_{h_{2n}}, \ldots, w_{h_{kn}}) \\ &\quad + \sum_{t=n}^{\infty}\big[f(t, w_{f_{1t}}, w_{f_{2t}}, \ldots, w_{f_{kt}}) - c_t\big], \quad \forall n \ge T+\tau \end{aligned}$$

and

$$\begin{aligned} \Delta\big(a_n\Delta^3(w_n + b_n w_{n-\tau})\big) &= -\Delta h(n, w_{h_{1n}}, w_{h_{2n}}, \ldots, w_{h_{kn}}) \\ &\quad - f(n, w_{f_{1n}}, w_{f_{2n}}, \ldots, w_{f_{kn}}) + c_n, \quad \forall n \ge T+\tau, \end{aligned}$$

that is, $w = \{w_n\}_{n\in\mathbb{N}_\beta}$ is a positive solution of Eq. (1.1) in $A(N, M)$. Making use of (2.17), (2.18) and (2.46), we infer that

$$\begin{aligned} \left|\frac{w_n + b_n w_{n-\tau}}{n^2} - L\right| &\le \frac{1}{n^2}\sum_{v=n}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\frac{1}{|a_s|}\Bigg(\big|h(s, w_{h_{1s}}, w_{h_{2s}}, \ldots, w_{h_{ks}})\big| \\ &\quad + \sum_{t=s}^{\infty}\big[\big|f(t, w_{f_{1t}}, w_{f_{2t}}, \ldots, w_{f_{kt}})\big| + |c_t|\big]\Bigg) \\ &\le \frac{1}{n^2}\sum_{v=n}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\frac{1}{|a_s|}\Bigg(W_s + \sum_{t=s}^{\infty}\big(Q_t + |c_t|\big)\Bigg) \\ &\to 0 \quad \text{as } n \to \infty, \end{aligned}$$

which gives (2.28). In terms of (2.14), (2.16), (2.40), (2.44) and (2.45), we deduce that

$$\begin{aligned} \frac{|x_{m+1n} - w_n|}{n^2} &= \frac{1}{n^2}\Bigg|(1-\alpha_m)x_{mn} + \alpha_m\Bigg\{\frac{(n+\tau)^2 L}{b_{n+\tau}} - \frac{x_{n+\tau}}{b_{n+\tau}} \\ &\quad + \frac{1}{b_{n+\tau}}\sum_{v=n+\tau}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\frac{1}{a_s}\Bigg[h(s, x_{mh_{1s}}, x_{mh_{2s}}, \ldots, x_{mh_{ks}}) \end{aligned}$$

$$
\begin{aligned}
&\quad - \sum_{t=s}^{\infty}\left[f(t, x_{mf_{1t}}, x_{mf_{2t}}, \ldots, x_{mf_{kt}}) - c_t\right]\Bigg]\Bigg\} - w_n\Bigg| \\
&\le (1-\alpha_m)\frac{|x_{mn} - w_n|}{n^2} + \alpha_m \frac{|S_L x_{mn} - S_L w_n|}{n^2} \\
&\le (1-\alpha_m)\|x_m - w\| + \theta\alpha_m\|x_m - w\| \\
&= \left[1 - (1-\theta)\alpha_m\right]\|x_m - w\| \\
&\le e^{-(1-\theta)\alpha_m}\|x_m - w\|, \quad \forall m \in \mathbb{N}_0, n \ge T
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{|x_{m+1n} - w_n|}{n^2} &= \frac{1}{n^2}\Bigg|(1-\alpha_m)\frac{n^2}{T^2}x_{mT} + \alpha_m\frac{n^2}{T^2}\Bigg\{\frac{(T+\tau)^2 L}{b_{T+\tau}} - \frac{x_{mT+\tau}}{b_{T+\tau}} \\
&\quad + \sum_{v=T+\tau}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\frac{1}{a_s}\Bigg[h(s, x_{mh_{1s}}, x_{mh_{2s}}, \ldots, x_{mh_{ks}}) \\
&\quad - \sum_{t=s}^{\infty}\left[f(t, x_{mf_{1t}}, x_{mf_{2t}}, \ldots, x_{mf_{kt}}) - c_t\right]\Bigg]\Bigg\} - w_n\Bigg| \\
&\le (1-\alpha_m)\frac{|x_{mT} - w_T|}{T^2} + \alpha_m\frac{|S_L x_{mT} - S_L w_T|}{T^2} \\
&\le \left[1 - (1-\theta)\alpha_m\right]\|x_m - w\| \\
&\le e^{-(1-\theta)\alpha_m}\|x_m - w\|, \quad \forall m \in \mathbb{N}_0, \beta \le n < T,
\end{aligned}
$$

which imply that

$$
\|x_{m+1} - w\| \le e^{-(1-\theta)\alpha_m}\|x_m - w\| \le e^{-(1-\theta)\sum_{i=0}^{m}\alpha_i}\|x_0 - w\|, \quad \forall m \in \mathbb{N}_0,
$$

that is, (2.8) holds. It follows from (2.8) and (2.9) that $\lim_{m\to\infty} x_m = w$.

Next we show that (b) holds. Let $L_1, L_2 \in (b^*(Mq+N), \frac{M}{q} + \frac{N}{qb^*})$ and $L_1 \ne L_2$. Similar to the proof of (a), we get that for each $c \in \{1,2\}$ there exist constants $\theta_c \in (0,1)$, $T_c \ge n_0 + \tau + \beta$ and a mapping S_{L_c} satisfying (2.41)-(2.44), where θ, L and T are replaced by θ_c, L_c and T_c, respectively, and the mapping S_{L_c} has a fixed point $z^c = \{z_n^c\}_{n\in\mathbb{N}_\beta} \in A(N,M)$, which is a positive solution of Eq. (1.1) in $A(N,M)$, that is,

$$
\begin{aligned}
z_n^c &= \frac{(n+\tau)^2}{b_{n+\tau}}L_c - \frac{z_{n+\tau}^c}{b_{n+\tau}} + \frac{1}{b_{n+\tau}}\sum_{v=n}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\frac{1}{a_s}\Bigg\{h\left(s, z_{h_{1s}}^c, z_{h_{2s}}^c, \ldots, z_{h_{ks}}^c\right) \\
&\quad - \sum_{t=s}^{\infty}\left[f\left(t, z_{f_{1t}}^c, z_{f_{2t}}^c, \ldots, z_{f_{kt}}^c\right) - c_t\right]\Bigg\}, \quad \forall n \ge T_c,
\end{aligned}
$$

which together with (2.1), (2.41) and (2.43) means that

$$
\begin{aligned}
&\left|\frac{z_n^1}{n^2} - \frac{z_n^2}{n^2}\right| \\
&\quad \ge \frac{1}{b_{n+\tau}}\left(1 + \frac{\tau}{n}\right)^2|L_1 - L_2| - \frac{1}{b_{n+\tau}}\cdot\frac{(n+\tau)^2}{n^2}\cdot\frac{|z_n^1(n+\tau) - z_n^2(n+\tau)|}{(n+\tau)^2}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{b_{n+\tau}n^2}\sum_{v=n+\tau}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\frac{1}{|a_s|}\biggl(\bigl|h\bigl(s,z^1_{h_{1s}},z^1_{h_{2s}},\dots,z^1_{h_{ks}}\bigr)-h\bigl(s,z^2_{h_{1s}},z^2_{h_{2s}},\dots,z^2_{h_{ks}}\bigr)\bigr| \\
&+\sum_{t=s}^{\infty}\bigl|f\bigl(t,z^1_{f_{1t}},z^1_{f_{2t}},\dots,z^1_{f_{kt}}\bigr)-f\bigl(t,z^2_{f_{1t}},z^2_{f_{2t}},\dots,z^2_{f_{kt}}\bigr)\bigr|\biggr) \\
\ge{}& \frac{|L_1-L_2|}{b^*}-\frac{1}{b_*}\left(1+\frac{\tau}{n}\right)^2\bigl\|z^1-z^2\bigr\| \\
&-\frac{1}{b_*n^2}\sum_{v=n+\tau}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\frac{1}{|a_s|}\biggl(R_s\max\bigl\{\bigl|z^1_{h_{ls}}-z^2_{h_{ls}}\bigr|:1\le l\le k\bigr\} \\
&+\sum_{t=s}^{\infty}P_t\max\bigl\{\bigl|z^1_{f_{lt}}-z^2_{f_{lt}}\bigr|:1\le l\le k\bigr\}\biggr) \\
\ge{}& \frac{|L_1-L_2|}{b^*}-q\bigl\|z^1-z^2\bigr\|-\frac{\|z^1-z^2\|}{b_*\max\{T_1^2,T_2^2\}}\sum_{v=\max\{T_1,T_2\}}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\frac{1}{|a_s|}\left(R_sH_s+\sum_{t=s}^{\infty}P_tF_t\right) \\
\ge{}& \frac{|L_1-L_2|}{b^*}-\max\{\theta_1,\theta_2\}\bigl\|z^1-z^2\bigr\|,\quad \forall n\ge\max\{T_1,T_2\},
\end{aligned}
$$

which yields that

$$\bigl\|z^1-z^2\bigr\|\ge\frac{|L_1-L_2|}{b^*(1+\max\{\theta_1,\theta_2\})}>0,$$

that is, $z^1\ne z^2$. This completes the proof. □

Theorem 2.6 *Assume that there exist constants b_*, b^*, M and N with $N\frac{1+b_*}{1+b^*}>M>N>0$ and four nonnegative sequences $\{P_n\}_{n\in\mathbb{N}_{n_0}}$, $\{Q_n\}_{n\in\mathbb{N}_{n_0}}$, $\{R_n\}_{n\in\mathbb{N}_{n_0}}$ and $\{W_n\}_{n\in\mathbb{N}_{n_0}}$ satisfying* (2.1), (2.2), (2.17), (2.18) *and*

$$b_*\le b_n\le b^*<-1\quad\textit{eventually.}\tag{2.47}$$

Then

(a) *for any $L\in(N(1+b_*),M(1+b^*))$, there exist $\theta\in(0,1)$ and $T\ge n_0+\tau+\beta$ such that for each $x_0=\{x_{0n}\}_{n\in\mathbb{N}_\beta}\in A(N,M)$, the Mann iterative sequence $\{x_m\}_{m\in\mathbb{N}_0}=\{\{x_{mn}\}_{n\in\mathbb{N}_\beta}\}_{m\in\mathbb{N}_0}$ generated by* (2.40) *converges to a positive solution $w=\{w_n\}_{n\in\mathbb{N}_\beta}\in A(N,M)$ of Eq.* (1.1) *with* (2.28) *and has the error estimate* (2.8), *where $\{\alpha_m\}_{m\in\mathbb{N}_0}$ is an arbitrary sequence in $[0,1]$ satisfying* (2.9);

(b) *Equation* (1.1) *possesses uncountably many positive solutions in $A(N,M)$.*

Proof Put $L\in(N(1+b_*),M(1+b^*))$. Observe that

$$
\begin{aligned}
\lim_{n\to\infty}\left[N\left(1+b_*\left(1+\frac{\tau}{n}\right)^{-2}\right)\right]&=N(1+b_*)<L<M\bigl(1+b^*\bigr) \\
&=\lim_{n\to\infty}\left[M\left(1+b^*\left(1-\frac{\tau}{n}\right)^{-2}\right)\right] \\
&=\lim_{n\to\infty}\left[M\left(1+b^*\left(1+\frac{\tau}{n}\right)^{-2}\right)\right],
\end{aligned}
$$

which implies that there exists $T_0 \in \mathbb{N}_{n_0+\tau+\beta}$ satisfying

$$\begin{aligned} L &\in \left(N\left(1+b_*\left(1+\frac{\tau}{n}\right)^{-2}\right), M\left(1+b^*\left(1-\frac{\tau}{n}\right)^{-2}\right)\right) \\ &\subset \left(N(1+b_*), M\left(1+b^*\right)\right) \\ &\subset \left(N(1+b_*), M\left(1+b^*\left(1+\frac{\tau}{n}\right)^{-2}\right)\right), \quad \forall n \in \mathbb{N}_{T_0}. \end{aligned} \tag{2.48}$$

It follows from (2.17), (2.18) and (2.47) that there exist $\theta \in (0,1)$ and $T \geq T_0$ satisfying

$$\theta = -\frac{1}{b^*}\left[\left(1+\frac{\tau}{T}\right)^2 + \frac{1}{T^2}\sum_{v=T}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\frac{1}{|a_s|}\left(R_sH_s + \sum_{t=s}^{\infty}P_tF_t\right)\right]; \tag{2.49}$$

$$\begin{aligned} &-\frac{1}{b^*T^2}\sum_{v=T}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\frac{1}{|a_s|}\left(W_s + \sum_{t=s}^{\infty}\left(Q_t + |c_t|\right)\right) \\ &\quad < \min\left\{M + \left(1+\frac{\tau}{T}\right)^2\frac{M-L}{b^*}, \left(1+\frac{\tau}{T}\right)^2\frac{L-N}{b_*} - N\right\}; \end{aligned} \tag{2.50}$$

$$b_n \leq b < -1, \quad \forall n \geq T. \tag{2.51}$$

Define a mapping $S_L : A(N,M) \to l_\beta^\infty$ by (2.44). Making use of (2.1), (2.2), (2.44) and (2.48)-(2.51), we conclude that for each $x = \{x_n\}_{n\in\mathbb{N}_\beta}, y = \{y_n\}_{n\in\mathbb{N}_\beta} \in A(N,M)$

$$\begin{aligned} &\left|\frac{S_Lx_n}{n^2} - \frac{S_Ly_n}{n^2}\right| \\ &\quad \leq -\frac{1}{b_{n+\tau}}\cdot\frac{(n+\tau)^2}{n^2}\left|\frac{x_{n+\tau}-y_{n+\tau}}{(n+\tau)^2}\right| \\ &\qquad - \frac{1}{b_{n+\tau}n^2}\sum_{v=n+\tau}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\frac{1}{|a_s|}\Bigg(\left|h(s,x_{h_{1s}},x_{h_{2s}},\ldots,x_{h_{ks}}) - h(s,y_{h_{1s}},y_{h_{2s}},\ldots,y_{h_{ks}})\right| \\ &\qquad + \sum_{t=s}^{\infty}\left|f(t,x_{f_{1t}},x_{f_{2t}},\ldots,x_{f_{kt}}) - f(t,y_{f_{1t}},y_{f_{2t}},\ldots,y_{f_{kt}})\right|\Bigg) \\ &\quad \leq -\frac{1}{b^*}\left(1+\frac{\tau}{T}\right)^2\|x-y\| \\ &\qquad - \frac{1}{b^*T^2}\sum_{v=T}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\frac{1}{|a_s|}\Bigg(R_s\max\left\{|x_{h_{ls}}-y_{h_{ls}}| : 1\leq l\leq k\right\} \\ &\qquad + \sum_{t=s}^{\infty}P_t\max\left\{|x_{f_{lt}}-y_{f_{lt}}| : 1\leq l\leq k\right\}\Bigg) \\ &\quad \leq -\frac{1}{b^*}\left[\left(1+\frac{\tau}{T}\right)^2 + \frac{1}{T^2}\sum_{v=T}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\frac{1}{|a_s|}\left(R_sH_s + \sum_{t=s}^{\infty}P_tF_t\right)\right]\|x-y\| \\ &\quad = \theta\|x-y\|, \quad \forall n \geq T, \\ &\left|\frac{S_Lx_n}{n^2} - \frac{S_Ly_n}{n^2}\right| = \left|\frac{n^2}{T^2}\cdot\frac{S_Lx_T}{n^2} - \frac{n^2}{T^2}\cdot\frac{S_Ly_T}{n^2}\right| \leq \theta\|x-y\|, \quad \beta \leq n < T, \end{aligned}$$

$$
\begin{aligned}
\frac{S_L x_n}{n^2} &= \left(1+\frac{\tau}{n}\right)^2 \frac{L}{b_{n+\tau}} - \frac{1}{b_{n+\tau}}\left(1+\frac{\tau}{n}\right)^2 \frac{x_{n+\tau}}{(n+\tau)^2} \\
&\quad + \frac{1}{b_{n+\tau} n^2} \sum_{v=n+\tau}^{\infty} \sum_{u=v}^{\infty} \sum_{s=u}^{\infty} \frac{1}{a_s} \biggl\{ h(s, x_{h_{1s}}, x_{h_{2s}}, \dots, x_{h_{ks}}) \\
&\quad - \sum_{t=s}^{\infty} \bigl[f(t, x_{f_{1t}}, x_{f_{2t}}, \dots, x_{f_{kt}}) - c_t \bigr] \biggr\} \\
&\le \left(1+\frac{\tau}{n}\right)^2 \frac{L}{b^*} - \left(1+\frac{\tau}{n}\right)^2 \frac{M}{b^*} \\
&\quad - \frac{1}{b^* n^2} \sum_{v=n+\tau}^{\infty} \sum_{u=v}^{\infty} \sum_{s=u}^{\infty} \frac{1}{|a_s|} \biggl(\bigl| h(s, x_{h_{1s}}, x_{h_{2s}}, \dots, x_{h_{ks}}) \bigr| \\
&\quad + \sum_{t=s}^{\infty} \bigl[\bigl| f(t, x_{f_{1t}}, x_{f_{2t}}, \dots, x_{f_{kt}}) \bigr| + |c_t| \bigr] \biggr) \\
&\le \left(1+\frac{\tau}{T}\right)^2 \frac{L-M}{b^*} - \frac{1}{b^* T^2} \sum_{v=T}^{\infty} \sum_{u=v}^{\infty} \sum_{s=u}^{\infty} \frac{1}{|a_s|} \left(W_s + \sum_{t=s}^{\infty} \bigl(Q_t + |c_t| \bigr) \right) \\
&< \left(1+\frac{\tau}{T}\right)^2 \frac{L-M}{b^*} \\
&\quad + \min\left\{ M + \left(1+\frac{\tau}{T}\right)^2 \frac{M-L}{b^*}, \left(1+\frac{\tau}{T}\right)^2 \frac{L-N}{b_*} - N \right\} \\
&\le M, \quad \forall n \ge T, \\
\frac{S_L x_n}{n^2} &= \frac{n^2}{T^2} \cdot \frac{S_L x_T}{n^2} \le M, \quad \beta \le n < T, \\
\frac{S_L x_n}{n^2} &\ge \left(1+\frac{\tau}{n}\right)^2 \frac{L}{b_*} - \left(1+\frac{\tau}{n}\right)^2 \frac{N}{b_*} \\
&\quad + \frac{1}{b^* n^2} \sum_{v=n+\tau}^{\infty} \sum_{u=v}^{\infty} \sum_{s=u}^{\infty} \frac{1}{|a_s|} \biggl(\bigl| h(s, x_{h_{1s}}, x_{h_{2s}}, \dots, x_{h_{ks}}) \bigr| \\
&\quad + \sum_{t=s}^{\infty} \bigl[\bigl| f(t, x_{f_{1t}}, x_{f_{2t}}, \dots, x_{f_{kt}}) \bigr| + |c_t| \bigr] \biggr) \\
&\ge \left(1+\frac{\tau}{n}\right)^2 \frac{L-N}{b_*} + \frac{1}{b^* T^2} \sum_{v=T}^{\infty} \sum_{u=v}^{\infty} \sum_{s=u}^{\infty} \frac{1}{|a_s|} \left(W_s + \sum_{t=s}^{\infty} \bigl(Q_t + |c_t| \bigr) \right) \\
&> \left(1+\frac{\tau}{T}\right)^2 \frac{L-N}{b_*} \\
&\quad - \min\left\{ M + \left(1+\frac{\tau}{T}\right)^2 \frac{M-L}{b^*}, \left(1+\frac{\tau}{T}\right)^2 \frac{L-N}{b_*} - N \right\} \\
&\ge N, \quad \forall n \ge T
\end{aligned}
$$

and

$$
\frac{S_L x_n}{n^2} = \frac{n^2}{T^2} \cdot \frac{S_L x_T}{n^2} \ge N, \quad \beta \le n < T,
$$

which yield (2.14). The rest of the proof is similar to that of Theorem 2.5 and is omitted. This completes the proof. □

Theorem 2.7 *Assume that there exist constants b, M and N with $(1-2b)M > N > 0$ and four nonnegative sequences $\{P_n\}_{n\in\mathbb{N}_{n_0}}$, $\{Q_n\}_{n\in\mathbb{N}_{n_0}}$, $\{R_n\}_{n\in\mathbb{N}_{n_0}}$, $\{W_n\}_{n\in\mathbb{N}_{n_0}}$ satisfying* (2.1), (2.2), (2.17), (2.18) *and*

$$|b_n| \le b < \frac{1}{2} \quad \text{eventually.} \tag{2.52}$$

Then

(a) *for any $L \in (N+bM, (1-b)M)$, there exist $\theta \in (0,1)$ and $T \ge n_0 + \tau + \beta$ such that for any $x_0 = \{x_{0n}\}_{n\in\mathbb{N}_\beta} \in A(N,M)$, the Mann iterative sequence $\{x_m\}_{m\in\mathbb{N}_0} = \{\{x_{mn}\}_{n\in\mathbb{N}_\beta}\}_{m\in\mathbb{N}_0}$ generated by* (2.27) *converges to a positive solution $w = \{w_n\}_{n\in\mathbb{N}_\beta} \in A(N,M)$ of Eq.* (1.1) *with* (2.28) *and has the error estimate* (2.8), *where $\{\alpha_m\}_{m\in\mathbb{N}_0}$ is an arbitrary sequence in $[0,1]$ satisfying* (2.9);

(b) *Equation* (1.1) *possesses uncountably many positive solutions in $A(N,M)$.*

Proof Put $L \in (N+bM, (1-b)M)$. It follows from (2.17), (2.18) and (2.52) that there exist $\theta \in (0,1)$ and $T \ge n_0 + \tau + \beta$ satisfying (2.29),

$$\frac{1}{T^2}\sum_{v=T}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\frac{1}{|a_s|}\left(W_s + \sum_{t=s}^{\infty}(Q_t + |c_t|)\right) < \min\{(1-b)M - L, L - bM - N\}; \tag{2.53}$$

$$|b_n| \le b, \quad \forall n \ge T. \tag{2.54}$$

Define a mapping $S_L : A(N,M) \to l_\beta^\infty$ by (2.32). By virtue of (2.2), (2.32), (2.53) and (2.54), we easily verify that for each $x = \{x_n\}_{n\in\mathbb{N}_\beta}, y = \{y_n\}_{n\in\mathbb{N}_\beta} \in A(N,M)$

$$\begin{aligned}
\frac{S_L x_n}{n^2} &\le L - |b_n|\left(1 - \frac{\tau}{n}\right)^2 \frac{x_{n-\tau}}{(n-\tau)^2} \\
&\quad + \frac{1}{n^2}\sum_{v=n}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\frac{1}{|a_s|}\Bigg(\left|h(s, x_{h_{1s}}, x_{h_{2s}}, \ldots, x_{h_{ks}})\right| \\
&\quad + \sum_{t=s}^{\infty}\left[\left|f(t, x_{f_{1t}}, x_{f_{2t}}, \ldots, x_{f_{kt}})\right| + |c_t|\right]\Bigg) \\
&\le L + bM + \frac{1}{T^2}\sum_{v=T}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\frac{1}{|a_s|}\left(W_s + \sum_{t=s}^{\infty}(Q_t + |c_t|)\right) \\
&< L + bM + \min\{(1-b)M - L, L - bM - N\} \\
&\le M, \quad \forall n \ge T, \\
\frac{S_L x_n}{n^2} &= \frac{n^2}{T^2}\cdot\frac{S_L x_T}{n^2} \le M, \quad \beta \le n < T,
\end{aligned}$$

$$\begin{aligned}
\frac{S_L x_n}{n^2} &\geq L - |b_n|\left(1-\frac{\tau}{n}\right)^2 \frac{x_{n-\tau}}{(n-\tau)^2} \\
&\quad - \frac{1}{n^2}\sum_{v=n}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\frac{1}{|a_s|}\Biggl(\bigl|h(s,x_{h_{1s}},x_{h_{2s}},\ldots,x_{h_{ks}})\bigr| \\
&\quad + \sum_{t=s}^{\infty}\bigl[\bigl|f(t,x_{f_{1t}},x_{f_{2t}},\ldots,x_{f_{kt}})\bigr| + |c_t|\bigr]\Biggr) \\
&\geq L - bM - \frac{1}{T^2}\sum_{v=T}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\frac{1}{|a_s|}\left(W_s + \sum_{t=s}^{\infty}\bigl(Q_t + |c_t|\bigr)\right) \\
&> L - bM - \min\bigl\{(1-b)M - L, L - bM - N\bigr\} \\
&\geq N, \quad \forall n \geq T
\end{aligned}$$

and

$$\frac{S_L x_n}{n^2} = \frac{n^2}{T^2}\cdot\frac{S_L x_T}{n^2} \geq N, \quad \beta \leq n < T,$$

which yield that $S_L(A(N,M)) \subseteq A(N,M)$. The rest of the proof is similar to that of Theorem 2.3 and is omitted. This completes the proof. □

3 Examples

In this section, we suggest seven examples to explain the results presented in Section 2.

Example 3.1 Consider the fourth order neutral delay difference equation

$$\begin{aligned}
&\Delta\bigl((n^2-n+1)\Delta^3(x_n - x_{n-\tau})\bigr) + \Delta\left(\frac{\sin^2(x_{n-3} - n x_{n^2-1})}{n^{18}+3n^6-4n^3+1}\right) \\
&\quad + \frac{3n-\sqrt{n}}{(n^{15}+2n^5-n+1)(1+x_{n^2}^2+x_{n-2}^2)} = \frac{(-1)^n \ln^2 n}{n^{11}+2n^5-n^4+1}, \quad \forall n \geq 4,
\end{aligned} \tag{3.1}$$

where $\tau \in \mathbb{N}$ is fixed. Let $n_0 = 4, k = 2, \beta = \min\{4-\tau, 1\} = 1 \in \mathbb{N}$, M and N be two positive constants with $M > N$ and

$$\begin{aligned}
&a_n = n^2 - n + 1, \qquad b_n = -1, \qquad c_n = \frac{(-1)^n \ln^2 n}{n^{11}+2n^5-n^4+1}, \qquad f_{1n} = n^2, \\
&f_{2n} = n-2, \qquad F_n = n^4, \qquad h_{1n} = n-3, \qquad h_{2n} = n^2-1, \qquad H_n = \bigl(n^2-1\bigr)^2, \\
&f(n,u,v) = \frac{3n-\sqrt{n}}{(n^{15}+2n^5-n+1)(1+u^2+v^2)}, \qquad h(n,u,v) = \frac{\sin^2(u-nv)}{n^{18}+3n^6-4n^3+1}, \\
&P_n = Q_n = \frac{20}{n^{14}}, \qquad R_n = \frac{4}{n^{17}}, \qquad W_n = \frac{1}{n^{18}}, \quad \forall (n,u,v) \in \mathbb{N}_{n_0} \times \mathbb{R}^2.
\end{aligned}$$

It is easy to see that (2.1), (2.2) and (2.5) are satisfied. Note that Lemma 1.1 means that

$$\begin{aligned}
&\frac{1}{n^2}\sum_{i=1}^{\infty}\sum_{v=n+i\tau}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\frac{1}{|a_s|}\max\{R_s H_s, W_s\} \\
&\quad \leq \frac{1}{n^2\tau}\sum_{s=n+\tau}^{\infty}\frac{s^3}{s^2-s+1}\max\left\{\frac{4(s^2-1)^2}{s^{17}}, \frac{1}{s^{18}}\right\} \leq \frac{4}{n^2\tau}\sum_{s=n+\tau}^{\infty}\frac{1}{s^{10}} \to 0 \quad \text{as } n \to \infty
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{n^2}\sum_{i=1}^{\infty}\sum_{v=n+i\tau}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\sum_{t=s}^{\infty}\frac{1}{|a_s|}\max\bigl\{P_tF_t,Q_t,|c_t|\bigr\}\\
&\quad=\frac{1}{n^2}\sum_{i=1}^{\infty}\sum_{v=n+i\tau}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\sum_{t=s}^{\infty}\frac{1}{s^2-s+1}\max\left\{\frac{20}{t^{10}},\frac{20}{t^{14}},\frac{\ln^2 t}{t^{11}+2t^5-t^4+1}\right\}\\
&\quad\le\frac{20}{n^2}\sum_{i=1}^{\infty}\sum_{v=n+i\tau}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\sum_{t=s}^{\infty}\frac{1}{t^8}\\
&\quad\le\frac{20}{n^2\tau}\sum_{t=n+\tau}^{\infty}\frac{1}{t^6}\to 0\quad\text{as } n\to\infty,
\end{aligned}$$

which give that

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{i=1}^{\infty}\sum_{v=n+i\tau}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\frac{1}{|a_s|}\max\{R_sH_s,W_s\}=0$$

and

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{i=1}^{\infty}\sum_{v=n+i\tau}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\sum_{t=s}^{\infty}\frac{1}{|a_s|}\max\bigl\{P_tF_t,Q_t,|c_t|\bigr\}=0.$$

That is, (2.3) and (2.4) hold. Consequently Theorem 2.1 implies that Eq. (3.1) possesses uncountably many positive solutions in $A(N,M)$. Moreover, for each $L\in(N,M)$, there exist $\theta\in(0,1)$ and $T\ge n_0+\tau+\beta$ such that for each $x_0=\{x_{0n}\}_{n\in\mathbb{N}_\beta}\in A(N,M)$, the Mann iterative sequence $\{x_m\}_{m\in\mathbb{N}_0}=\{\{x_{mn}\}_{n\in\mathbb{N}_\beta}\}_{m\in\mathbb{N}_0}$ generated by (2.6) converges to a positive solution $w=\{w_n\}_{n\in\mathbb{N}_\beta}\in A(N,M)$ of Eq. (3.1) with (2.7) and (2.8), where $\{\alpha_m\}_{m\in\mathbb{N}_0}$ is an arbitrary sequence in $[0,1]$ satisfying (2.9).

Example 3.2 Consider the fourth order neutral delay difference equation

$$\begin{aligned}
&\Delta\bigl((-1)^n n^2\Delta^3(x_n+x_{n-\tau})\bigr)+\Delta\left(\frac{\cos^2(n^{14}x_{n-4}-2)}{(n^{34}+28n^{22}-1)(1+x_{2n-3}^4)}\right)\\
&\qquad+\frac{(n^{20}-n^{13}+(-1)^n)(x_{n^2-16}+x_{n^2-20})}{(n^{36}+10n^{28}-\sqrt{n})(1+x_{n^2-16}^2+x_{n^2-20}^2)}\\
&\quad=\frac{(-1)^n n^3+4n^2-\sqrt{\ln n}}{n^{19}+20n^{15}-n^4+1},\quad n\ge 5,
\end{aligned}\tag{3.2}$$

where $\tau\in\mathbb{N}$ is fixed. Let $n_0=5, k=2, \beta=5-\tau\in\mathbb{N}$, M and N be two positive constants with $M>N$ and

$$\begin{aligned}
&a_n=(-1)^n n^2,\qquad b_n=1,\qquad c_n=\frac{(-1)^n n^3+4n^2-\sqrt{\ln n}}{n^{19}+20n^{15}-n^4+1},\qquad f_{1n}=n^2-16,\\
&f_{2n}=n^2-20,\qquad F_n=\bigl(n^2-16\bigr)^2,\qquad h_{1n}=2n-3,\\
&h_{2n}=n-4,\qquad H_n=(2n-3)^2,
\end{aligned}$$

$$\begin{aligned}
&f(n,u,v)=\frac{(n^{20}-n^{13}+(-1)^n)(u+v)}{(n^{36}+10n^{28}-\sqrt{n})(1+u^2+v^2)},\\
&h(n,u,v)=\frac{\cos^2(n^{14}v-2)}{(n^{34}+28n^{22}-1)(1+u^4)},\\
&P_n=Q_n=\frac{4}{n^{12}},\qquad R_n=W_n=\frac{10}{n^{13}},\quad \forall(n,u,v)\in\mathbb{N}_{n_0}\times\mathbb{R}^2.
\end{aligned}$$

It is clear that (2.1), (2.2) and (2.19) are fulfilled. Note that Lemma 1.1 ensures that

$$\begin{aligned}
&\frac{1}{n^2}\sum_{v=n}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\frac{1}{|a_s|}\max\{R_sH_s,W_s\}\\
&\quad\le\frac{1}{n^2}\sum_{s=n}^{\infty}\frac{s^2}{|(-1)^s s^2|}\max\left\{\frac{10(2s-3)^2}{s^{13}},\frac{10}{s^{13}}\right\}\\
&\quad\le\frac{40}{n^2}\sum_{s=n}^{\infty}\frac{1}{s^{11}}\to 0\quad\text{as } n\to\infty
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{n^2}\sum_{v=n}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\sum_{t=s}^{\infty}\frac{1}{|a_s|}\max\bigl\{P_tF_t,Q_t,|c_t|\bigr\}\\
&\quad=\frac{1}{n^2}\sum_{v=n}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\sum_{t=s}^{\infty}\frac{1}{|(-1)^s s^2|}\max\left\{\frac{4|t^2-16|^2}{t^{12}},\frac{4}{t^{12}},\left|\frac{(-1)^t t^3+4t^2-\sqrt{\ln t}}{t^{19}+20t^{15}-t^4+1}\right|\right\}\\
&\quad\le\frac{4}{n^2}\sum_{v=n}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\sum_{t=s}^{\infty}\frac{1}{t^8}\\
&\quad\le\frac{4}{n^2}\sum_{t=n}^{\infty}\frac{1}{t^5}\to 0\quad\text{as } n\to\infty,
\end{aligned}$$

which mean that

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{v=n}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\frac{1}{|a_s|}\max\{R_sH_s,W_s\}=0$$

and

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{v=n}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\sum_{t=s}^{\infty}\frac{1}{|a_s|}\max\bigl\{P_tF_t,Q_t,|c_t|\bigr\}=0.$$

That is, (2.17) and (2.18) hold. Consequently Theorem 2.2 implies that Eq. (3.2) possesses uncountably many positive solutions in $A(N,M)$. Moreover, for each $L\in(N,M)$, there exist $\theta\in(0,1)$ and $T\ge n_0+\tau+\beta$ such that for each $x_0=\{x_{0n}\}_{n\in\mathbb{N}_\beta}\in A(N,M)$, the Mann iterative sequence $\{x_m\}_{m\in\mathbb{N}_0}=\{\{x_{mn}\}_{n\in\mathbb{N}_\beta}\}_{m\in\mathbb{N}_0}$ generated by (2.20) converges to a positive solution $w=\{w_n\}_{n\in\mathbb{N}_\beta}\in A(N,M)$ of Eq. (3.2) with (2.7) and (2.8), where $\{\alpha_m\}_{m\in\mathbb{N}_0}$ is an arbitrary sequence in $[0,1]$ satisfying (2.9).

Example 3.3 Consider the fourth order neutral delay difference equation

$$\begin{aligned}
&\Delta\left(\sqrt{n^5+1}\Delta^3\left(x_n+\frac{3n^3-2}{4n^3+3}x_{n-\tau}\right)\right)\\
&\quad+\Delta\left(\frac{\sin(n^8|x_{n-1}|-\sqrt{n})}{n^{24}+n^4-\sqrt{n}+1}-\frac{n^5-(-1)^n n+1}{(n^{19}+6n^8-n^2+1)2^{|x_{2n-1}|}}\right)\\
&\quad+\frac{(-1)^n n^9-3n^4+2n^2+1}{(n^{17}+n^5+1)(1+x_{2n-4}^2)}-\frac{n^{15}\sin^5(3n^8-1)+n^3-1}{(n^{25}+4n^{24}+n^7-1)(1+x_{n-3}^2)}\\
&=\frac{(-1)^n n^{21}-n^7+2n^3-1}{n^{28}+8n^{14}-2n^7+1},\quad \forall n\ge 7,
\end{aligned}\tag{3.3}$$

where $\tau\in\mathbb{N}$ is fixed. Let $n_0=7, k=2, b=\frac{3}{4}, \beta=\min\{7-\tau,5\}\in\mathbb{N}$, M and N be two positive constants with $M>4N$ and

$$\begin{aligned}
&a_n=\sqrt{n^5+1},\qquad b_n=\frac{3n^3-2}{4n^3+3},\qquad c_n=\frac{(-1)^n n^{21}-n^7+2n^3-1}{n^{28}+8n^{14}-2n^7+1},\qquad f_{1n}=2n-4,\\
&f_{2n}=n-3,\qquad F_n=(2n-4)^2,\qquad h_{1n}=n-1,\qquad h_{2n}=2n-1,\qquad H_n=(2n-1)^2,\\
&f(n,u,v)=\frac{(-1)^n n^9-3n^4+2n^2+1}{(n^{17}+n^5+1)(1+u^2)}-\frac{n^{15}\sin^5(3n^8-1)+n^3-1}{(n^{25}+4n^{24}+n^7-1)(1+v^2)},\\
&h(n,u,v)=\frac{\sin(n^8|u|-\sqrt{n})}{n^{24}+n^4-\sqrt{n}+1}-\frac{n^5-(-1)^n n+1}{(n^{19}+6n^8-n^2+1)2^{|v|}},\\
&P_n=Q_n=\frac{3}{n^8},\qquad R_n=W_n=\frac{2}{n^{10}},\qquad \forall(n,u,v)\in\mathbb{N}_{n_0}\times\mathbb{R}^2.
\end{aligned}$$

It is not difficult to verify that (2.1), (2.2) and (2.26) are fulfilled. Note that Lemma 1.1 implies that

$$\begin{aligned}
&\frac{1}{n^2}\sum_{v=n}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\frac{1}{|a_s|}\max\{R_sH_s,W_s\}\\
&\quad\le\frac{1}{n^2}\sum_{s=n}^{\infty}\frac{s^2}{\sqrt{s^5+1}}\max\left\{\frac{2|2s-1|^2}{s^{10}},\frac{2}{s^{10}}\right\}\\
&\quad\le\frac{8}{n^2}\sum_{s=n}^{\infty}\frac{1}{s^6}\to 0\quad\text{as } n\to\infty
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{n^2}\sum_{v=n}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\sum_{t=s}^{\infty}\frac{1}{|a_s|}\max\bigl\{P_tF_t,Q_t,|c_t|\bigr\}\\
&\quad=\frac{1}{n^2}\sum_{v=n}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\sum_{t=s}^{\infty}\frac{1}{|\sqrt{s^5+1}|}\max\left\{\frac{3|2t-4|^2}{t^8},\frac{3}{t^8},\frac{|(-1)^t t^{21}-t^7+2t^3-1|}{t^{28}+8t^{14}-2t^7+1}\right\}\\
&\quad\le\frac{12}{n^2}\sum_{v=n}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\sum_{t=s}^{\infty}\frac{1}{t^6}\\
&\quad\le\frac{12}{n^2}\sum_{t=n}^{\infty}\frac{1}{t^3}\to 0\quad\text{as } n\to\infty,
\end{aligned}$$

which mean that

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{v=n}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\frac{1}{|a_s|}\max\{R_sH_s,W_s\}=0$$

and

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{v=n}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\sum_{t=s}^{\infty}\frac{1}{|a_s|}\max\{P_tF_t,Q_t,|c_t|\}=0.$$

That is, (2.17) and (2.18) hold. Consequently Theorem 2.3 implies that Eq. (3.3) possesses uncountably many positive solutions in $A(N,M)$. Moreover, for each $L\in(bM+N,M)$, there exist $\theta\in(0,1)$ and $T\geq n_0+\tau+\beta$ such that for each $x_0=\{x_{0n}\}_{n\in\mathbb{N}_\beta}\in A(N,M)$, the Mann iterative sequence $\{x_m\}_{m\in\mathbb{N}_0}=\{\{x_{mn}\}_{n\in\mathbb{N}_\beta}\}_{m\in\mathbb{N}_0}$ generated by (2.27) converges to a positive solution $w=\{w_n\}_{n\in\mathbb{N}_\beta}\in A(N,M)$ of Eq. (3.3) with (2.28) and (2.7), where $\{\alpha_m\}_{m\in\mathbb{N}_0}$ is an arbitrary sequence in $[0,1]$ satisfying (2.9).

Example 3.4 Consider the fourth order neutral delay difference equation

$$\begin{aligned}&\Delta\left((-1)^n\ln^3(n+2)\Delta^3\left(x_n+\frac{2-7\ln^9 n}{3+8\ln^9 n}x_{n-\tau}\right)\right)+\Delta\left(\frac{-3n^2+\ln^2 n-1}{(n^9+6n^6+1)(1+x_{3n-7}^4)}\right)\\&\quad+\frac{\sin^2(n^{12}x_{2n^2-1}-3n^4+1)}{2n^{26}+3n^8+1}=\frac{(-1)^n n^3+n-2}{n^9+9n^6-3n^3+1},\quad\forall n\geq 9,\end{aligned}\tag{3.4}$$

where $\tau\in\mathbb{N}$ is fixed. Let $n_0=9,k=1,b=-\frac{7}{8},\beta=9-\tau\in\mathbb{N}$, M and N be two positive constants with $M>8N$ and

$$\begin{aligned}&a_n=(-1)^n\ln^3(n+2),\qquad b_n=\frac{2-7\ln^9 n}{3+8\ln^9 n},\\&c_n=\frac{(-1)^n n^3+n-2}{n^9+9n^6-3n^3+1},\qquad f_{1n}=2n^2-1,\\&F_n=\left(2n^2-1\right)^2,\qquad f(n,u)=\frac{\sin^2(n^{12}u-3n^4+1)}{2n^{26}+3n^8+1},\\&h(n,u)=\frac{-3n^2+\ln^2 n-1}{(n^9+6n^6+1)(1+u^4)},\\&h_{1n}=3n-7,\qquad H_n=(3n-7)^2,\qquad P_n=Q_n=\frac{3}{n^{11}},\\&R_n=W_n=\frac{5}{n^7},\quad\forall(n,u)\in\mathbb{N}_{n_0}\times\mathbb{R}.\end{aligned}$$

Obviously, (2.1), (2.2) and (2.34) are satisfied. Note that

$$\begin{aligned}&\frac{1}{n^2}\sum_{v=n}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\frac{1}{|a_s|}\max\{R_sH_s,W_s\}\\&\quad\leq\frac{1}{n^2}\sum_{s=n}^{\infty}\frac{s^2}{|(-1)^s\ln^3(s+2)|}\max\left\{\frac{5|3s-7|^2}{s^7},\frac{5}{s^7}\right\}\\&\quad\leq\frac{45}{n^2}\sum_{s=n}^{\infty}\frac{1}{s^3}\to 0\quad\text{as }n\to\infty\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{n^2}\sum_{v=n}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\sum_{t=s}^{\infty}\frac{1}{|a_s|}\max\bigl\{P_tF_t, Q_t, |c_t|\bigr\} \\
&\quad = \frac{1}{n^2}\sum_{v=n}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\sum_{t=s}^{\infty}\frac{1}{|(-1)^s\ln^3(s+2)|}\max\left\{\frac{3(2t^2-1)^2}{t^{11}}, \frac{3}{t^{11}}, \frac{|(-1)^t t^3+t-2|}{t^9+9t^6-3t^3+1}\right\} \\
&\quad \le \frac{12}{n^2}\sum_{v=n}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\sum_{t=s}^{\infty}\frac{1}{t^7} \le \frac{12}{n^2}\sum_{t=n}^{\infty}\frac{1}{t^4} \to 0 \quad \text{as } n\to\infty,
\end{aligned}$$

which yield that

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{v=n}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\frac{1}{|a_s|}\max\{R_sH_s, W_s\} = 0$$

and

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{v=n}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\sum_{t=s}^{\infty}\frac{1}{|a_s|}\max\bigl\{P_tF_t, Q_t, |c_t|\bigr\} = 0.$$

That is, (2.17) and (2.18) hold. Thus Theorem 2.4 shows that Eq. (3.4) possesses uncountably many positive solutions in $A(N,M)$. Moreover, for each $L \in (N,(1+b)M)$, there exist $\theta \in (0,1)$ and $T \ge n_0+\tau+\beta$ such that for each $x_0 = \{x_{0n}\}_{n\in\mathbb{N}_\beta} \in A(N,M)$, the Mann iterative sequence $\{x_m\}_{m\in\mathbb{N}_0} = \{\{x_{mn}\}_{n\in\mathbb{N}_\beta}\}_{m\in\mathbb{N}_0}$ generated by (2.27) converges to a positive solution $w = \{w_n\}_{n\in\mathbb{N}_\beta} \in A(N,M)$ of Eq. (3.4) with (2.28) and (2.8), where $\{\alpha_m\}_{m\in\mathbb{N}_0}$ is an arbitrary sequence in $[0,1]$ satisfying (2.9).

Example 3.5 Consider the fourth order neutral delay difference equation

$$\begin{aligned}
&\Delta\left(\left(n^3-n^2+1\right)\Delta^3\left(x_n+\left(3+\frac{3}{n}\right)x_{n-\tau}\right)\right) \\
&\quad + \Delta\left(\frac{n^2-3n+\arctan^2 n}{(n^{17}+9n^2+1)(1+|\cos(n^4x_{2n-1}-n)|)}\right) \\
&\quad + \frac{n\cos(n^3x_{n-2})-1}{n^{18}+2n^{16}+\ln^3 n} = \frac{(-1)^{n-1}n^4-2n^3+\sqrt{n+1}}{n^{21}+3n^{15}-2n^{11}+1}, \quad \forall n\ge 3,
\end{aligned} \tag{3.5}$$

where $\tau \in \mathbb{N}$ is fixed. Let $n_0 = 3$, $k = 1$, $b^* = 4$, $b_* = 3$, $q = \frac{\sqrt{2}}{3}$, $\beta = \min\{3-\tau, 1\} = 1$, $M = 300$, $N = 1$ and

$$\begin{aligned}
&a_n = n^3-n^2+1, \qquad b_n = 3+\frac{3}{n}, \qquad c_n = \frac{(-1)^{n-1}n^4-2n^3+\sqrt{n+1}}{n^{21}+3n^{15}-2n^{11}+1}, \\
&f_{1n} = n-2, \qquad F_n = (n-2)^2, \qquad h_{1n} = 2n-1, \qquad H_n = (2n-1)^2, \\
&f(n,u) = \frac{n\cos(n^3u)-1}{n^{18}+2n^{16}+\ln^3 n}, \qquad h(n,u) = \frac{n^2-3n+\arctan^2 n}{(n^{17}+9n^2+1)(1+|\cos(n^4u-n)|)}, \\
&P_n = Q_n = \frac{1}{n^{14}}, \qquad R_n = W_n = \frac{2}{n^9}, \quad \forall (n,u) \in \mathbb{N}_{n_0}\times\mathbb{R}.
\end{aligned}$$

Clearly, (2.1), (2.2) and (2.39) are satisfied. Note that Lemma 1.1 yields that

$$\begin{aligned}
&\frac{1}{n^2}\sum_{v=n}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\frac{1}{|a_s|}\max\{R_sH_s, W_s\}\\
&\quad\le \frac{1}{n^2}\sum_{s=n}^{\infty}\frac{s^2}{s^3-s^2+1}\max\left\{\frac{2(2s-1)^2}{s^9}, \frac{2}{s^9}\right\}\\
&\quad\le \frac{8}{n^2}\sum_{s=n}^{\infty}\frac{1}{s^5}\to 0 \quad \text{as } n\to\infty
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{n^2}\sum_{v=n}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\sum_{t=s}^{\infty}\frac{1}{|a_s|}\max\bigl\{P_tF_t, Q_t, |c_t|\bigr\}\\
&\quad= \frac{1}{n^2}\sum_{v=n}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\sum_{t=s}^{\infty}\frac{1}{s^3-s^2+1}\max\left\{\frac{|t-2|^2}{t^{14}}, \frac{1}{t^{14}}, \frac{|(-1)^{t-1}t^4-2t^3+\sqrt{t+1}|}{t^{21}+3t^{15}-2t^{11}+1}\right\}\\
&\quad\le \frac{1}{n^2}\sum_{v=n}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\sum_{t=s}^{\infty}\frac{1}{t^{12}}\\
&\quad\le \frac{1}{n^2}\sum_{t=n}^{\infty}\frac{1}{t^9}\to 0 \quad \text{as } n\to\infty,
\end{aligned}$$

which mean that

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{v=n}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\frac{1}{|a_s|}\max\{R_sH_s, W_s\} = 0$$

and

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{v=n}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\sum_{t=s}^{\infty}\frac{1}{|a_s|}\max\bigl\{P_tF_t, Q_t, |c_t|\bigr\} = 0.$$

That is, (2.17) and (2.18) hold. Thus Theorem 2.5 shows that Eq. (3.5) possesses uncountably many positive solutions in $A(N,M)$. Moreover, for each $L\in(b^*(Mq+N), \frac{M}{q}+\frac{N}{qb^*})$, there exist $\theta\in(0,1)$ and $T\ge n_0+\tau+\beta$ such that for each $x_0=\{x_{0n}\}_{n\in\mathbb{N}_\beta}\in A(N,M)$, the Mann iterative sequence $\{x_m\}_{m\in\mathbb{N}_0}=\{\{x_{mn}\}_{n\in\mathbb{N}_\beta}\}_{m\in\mathbb{N}_0}$ generated by (2.40) converges to a positive solution $w=\{w_n\}_{n\in\mathbb{N}_\beta}\in A(N,M)$ of Eq. (3.5) with (2.28) and (2.8), where $\{\alpha_m\}_{m\in\mathbb{N}_0}$ is an arbitrary sequence in $[0,1]$ satisfying (2.9).

Example 3.6 Consider the fourth order neutral delay difference equation

$$\begin{aligned}
&\Delta\left(n^4\Delta^3\left(x_n-\frac{2n^{12}+9n^{11}-1}{n^{12}+3n^{11}+2}x_{n-\tau}\right)\right)+\Delta\left(\frac{(-1)^n\cos(n^{30}-2\sqrt{n+1})}{(n+3)^9\sqrt{n|x_{n-2}|+1}}\right)\\
&\quad+\frac{n^4-\ln^3 n}{n^{15}+2n^2+\sin(n^3x_{n-1})}=\frac{(-1)^{n-1}n^4+5\ln^5 n-1}{n^{13}+12n^{11}+1}, \quad \forall n\ge 6,
\end{aligned}\tag{3.6}$$

where $\tau \in \mathbb{N}$ is fixed. Let $n_0 = 6, k = 1, b^* = -2, b_* = -3, \beta = \min\{6-\tau, 4\} \in \mathbb{N}$, M and N be two positive constants with $2N > M > N$ and

$$\begin{aligned}
&a_n = n^4, \qquad b_n = -\frac{2n^{12}+9n^{11}-1}{n^{12}+3n^{11}+2}, \qquad c_n = \frac{(-1)^{n-1}n^4+5\ln^5 n-1}{n^{13}+12n^{11}+1},\\
&f_{1n} = n-1, \qquad F_n = (n-1)^2, \qquad h_{1n} = n-2, \qquad H_n = (n-2)^2,\\
&f(n,u) = \frac{n^4-\ln^3 n}{n^{15}+2n^2+\sin(n^3u)}, \qquad h(n,u) = \frac{(-1)^n\cos(n^{30}-2\sqrt{n+1})}{(n+3)^9\sqrt{n|u|+1}},\\
&P_n = Q_n = R_n = W_n = \frac{1}{n^8}, \quad \forall (n,u) \in \mathbb{N}_{n_0}\times\mathbb{R}.
\end{aligned}$$

Obviously, (2.1), (2.2) and (2.47) are satisfied. Note that Lemma 1.1 guarantees that

$$\begin{aligned}
&\frac{1}{n^2}\sum_{v=n}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\frac{1}{|a_s|}\max\{R_sH_s, W_s\}\\
&\quad\leq \frac{1}{n^2}\sum_{s=n}^{\infty}\frac{s^2}{s^4}\max\left\{\frac{|s-2|^2}{s^8}, \frac{1}{s^8}\right\}\\
&\quad\leq \frac{1}{n^2}\sum_{s=n}^{\infty}\frac{1}{s^8} \to 0 \quad \text{as } n\to\infty
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{n^2}\sum_{v=n}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\sum_{t=s}^{\infty}\frac{1}{|a_s|}\max\bigl\{P_tF_t, Q_t, |c_t|\bigr\}\\
&\quad= \frac{1}{n^2}\sum_{v=n}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\sum_{t=s}^{\infty}\frac{1}{s^4}\max\left\{\frac{(t-1)^2}{t^8}, \frac{1}{t^8}, \frac{|(-1)^{t-1}t^4+5\ln^5 t-1|}{t^{13}+12t^{11}+1}\right\}\\
&\quad\leq \frac{1}{n^2}\sum_{v=n}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\sum_{t=s}^{\infty}\frac{1}{t^6}\\
&\quad\leq \frac{1}{n^2}\sum_{t=n}^{\infty}\frac{1}{t^3} \to 0 \quad \text{as } n\to\infty,
\end{aligned}$$

which imply that

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{v=n}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\frac{1}{|a_s|}\max\{R_sH_s, W_s\} = 0$$

and

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{v=n}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\sum_{t=s}^{\infty}\frac{1}{|a_s|}\max\bigl\{P_tF_t, Q_t, |c_t|\bigr\} = 0.$$

That is, (2.17) and (2.18) hold. Thus Theorem 2.6 shows that Eq. (3.6) possesses uncountably many positive solutions in $A(N,M)$. Moreover, for each $L \in (N(1+b_*), M(1+b^*))$, there exist $\theta \in (0,1)$ and $T \geq n_0 + \tau + \beta$ such that for each $x_0 = \{x_{0n}\}_{n\in\mathbb{N}_\beta} \in A(N,M)$, the

Mann iterative sequence $\{x_m\}_{m\in\mathbb{N}_0} = \{\{x_{mn}\}_{n\in\mathbb{N}_\beta}\}_{m\in\mathbb{N}_0}$ generated by (2.40) converges to a positive solution $w = \{w_n\}_{n\in\mathbb{N}_\beta} \in A(N,M)$ of Eq. (3.6) with (2.28) and (2.8), where $\{\alpha_m\}_{m\in\mathbb{N}_0}$ is an arbitrary sequence in $[0,1]$ satisfying (2.9).

Example 3.7 Consider the fourth order neutral delay difference equation

$$\begin{aligned}
&\Delta\left(n\ln^2(n+3)\Delta^3\left(x_n + \frac{2(-1)^n n^8 - n + 1}{5n^8 + 3n - 1}x_{n-\tau}\right)\right) + \Delta\left(\frac{\sin x_{n-6}}{n^{12} + nx^2_{n^2-2}}\right) \\
&\quad + \frac{(-1)^{n-1}n^3\cos^3(4n^9 - 3\ln^2 n)}{n^{15} + \ln^8 n + |nx_{3n-1} - x_{2n-3}|} = \frac{(-1)^n n^8 - 5n^7 - 4n^3 + 1}{n^{25} + 30n^{16} - 2n^7 + 1}, \quad \forall n \geq 8, \qquad (3.7)
\end{aligned}$$

where $\tau \in \mathbb{N}$ is fixed. Let $n_0 = 4, k = 2, b = \frac{2}{5}, \beta = \min\{4-\tau, 2\} \in \mathbb{N}$, M and N be two positive constants with $M > 5N$ and

$$\begin{aligned}
&a_n = n\ln^2(n+3), \qquad b_n = \frac{2(-1)^n n^8 - n + 1}{5n^8 + 3n - 1}, \qquad c_n = \frac{(-1)^n n^8 - 5n^7 - 4n^3 + 1}{n^{25} + 30n^{16} - 2n^7 + 1}, \\
&f_{1n} = 3n - 1, \qquad f_{2n} = 2n - 3, \qquad F_n = (3n-1)^2, \\
&h_{1n} = n^2 - 2, \qquad h_{2n} = n - 4, \qquad H_n = \left(n^2 - 2\right)^2, \\
&f(n,u,v) = \frac{(-1)^{n-1}n^3\cos^3(4n^9 - 3\ln^2 n)}{n^{15} + \ln^8 n + |nu - v|}, \qquad h(n,u,v) = \frac{\sin u}{n^{12} + nv^2}, \\
&P_n = Q_n = R_n = W_n = \frac{4}{n^{11}}, \quad \forall (n,u,v) \in \mathbb{N}_{n_0} \times \mathbb{R}^2.
\end{aligned}$$

It is not difficult to verify that (2.1), (2.2) and (2.52) are fulfilled. Note that Lemma 1.1 gives that

$$\begin{aligned}
&\frac{1}{n^2}\sum_{v=n}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\frac{1}{|a_s|}\max\{R_sH_s, W_s\} \\
&\quad \leq \frac{1}{n^2}\sum_{s=n}^{\infty}\frac{s^2}{s\ln^2(s+3)}\max\left\{\frac{4(s^2-2)^2}{s^{11}}, \frac{4}{s^{11}}\right\} \\
&\quad \leq \frac{4}{n^2}\sum_{s=n}^{\infty}\frac{1}{s^6} \to 0 \quad \text{as } n \to \infty
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{n^2}\sum_{v=n}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\sum_{t=s}^{\infty}\frac{1}{|a_s|}\max\left\{P_tF_t, Q_t, |c_t|\right\} \\
&\quad = \frac{1}{n^2}\sum_{v=n}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\sum_{t=s}^{\infty}\frac{1}{s\ln^2(s+3)}\max\left\{\frac{4(3t-1)^2}{t^{11}}, \frac{4}{t^{11}}, \frac{|(-1)^t t^8 - 5t^7 - 4t^3 + 1|}{t^{25} + 30t^{16} - 2t^7 + 1}\right\} \\
&\quad \leq \frac{36}{n^2}\sum_{v=n}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\sum_{t=s}^{\infty}\frac{1}{t^9} \\
&\quad \leq \frac{36}{n^2}\sum_{t=n}^{\infty}\frac{1}{t^6} \to 0 \quad \text{as } n \to \infty,
\end{aligned}$$

which mean that

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{v=n}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\frac{1}{|a_s|}\max\{R_sH_s,W_s\}=0$$

and

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{v=n}^{\infty}\sum_{u=v}^{\infty}\sum_{s=u}^{\infty}\sum_{t=s}^{\infty}\frac{1}{|a_s|}\max\bigl\{P_tF_t,Q_t,|c_t|\bigr\}=0.$$

That is, (2.17) and (2.18) hold. Consequently Theorem 2.7 implies that Eq. (3.7) possesses uncountably many positive solutions in $A(N,M)$. Moreover, for each $L\in(N+bM,(1-b)M)$, there exist $\theta\in(0,1)$ and $T\geq n_0+\tau+\beta$ such that for each $x_0=\{x_{0n}\}_{n\in\mathbb{N}_\beta}\in A(N,M)$, the Mann iterative sequence $\{x_m\}_{m\in\mathbb{N}_0}=\{\{x_{mn}\}_{n\in\mathbb{N}_\beta}\}_{m\in\mathbb{N}_0}$ generated by (2.27) converges to a positive solution $w=\{w_n\}_{n\in\mathbb{N}_\beta}\in A(N,M)$ of Eq. (3.7) with (2.28) and (2.8), where $\{\alpha_m\}_{m\in\mathbb{N}_0}$ is an arbitrary sequence in $[0,1]$ satisfying (2.9).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Acknowledgements
This research was supported by the Science Research Foundation of Educational Department of Liaoning Province (L2012380) and the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Science, ICT & Future Planning (2013R1A1A2057665).

Received: 26 January 2015 Accepted: 19 May 2015 Published online: 12 June 2015

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