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On q-analogs of degenerate Bernoulli polynomials

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Abstract
The degenerate Bernoulli polynomials were introduced by Carlitz and rediscovered later by Ustiniv under the name of Korobov polynomials of the second kind (see Carlitz in Arch. Math. (Basel) 7:28-33, 1956; Util. Math. 15:51-88, 1979). In this paper, we study q-analogs of degenerate Bernoulli polynomials and give some formulas related to these polynomials.

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1 Introduction

Let p be a fixed prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic rational numbers, and the completion of the algebraic closure of $\mathbb{Q}_p$. The p-adic norm is normalized as $|p|_p = \frac{1}{p}$. Let $UD(\mathbb{Z}_p)$ be the space of all $\mathbb{C}_p$-valued uniformly differentiable functions on $\mathbb{Z}_p$, and let q be an indeterminate such that $|1-q|_p < p^{-\frac{1}{p-1}}$. The q-extension of the number x is defined as $[x]_q = \frac{1-q^x}{1-q}$. Note that $\lim_{q\to 1}[x]_q = x$. For $f \in UD(\mathbb{Z}_p)$, the p-adic q-integral on $\mathbb{Z}_p$ is defined by Kim to be

$$I_q(f) = \int_{\mathbb{Z}_p} f(x)\,d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x) q^x \quad \text{(see [1])}. \tag{1.1}$$

The ordinary p-adic invariant integral on $\mathbb{Z}_p$ is given by

$$I_0(f) = \lim_{q\to 1} I_q(f) = \int_{\mathbb{Z}_p} f(x)\,d\mu_0(x) = \lim_{N\to\infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x). \tag{1.2}$$

From (1.1), we can derive the following integral equation:

$$qI_q(f_1) - I_q(f) = (q-1)f(0) + \frac{q-1}{\log q} f'(0) \quad \text{(see [1–9])}, \tag{1.3}$$

where $f_1(x) = f(x+1)$.

For $\lambda, t \in \mathbb{C}$ with $|\lambda t|_p < p^{-\frac{1}{p-1}}$, the degenerate Bernoulli polynomials are defined as

$$\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \beta_n(x \mid \lambda)\frac{t^n}{n!} \quad \text{(see [10, 11])}. \tag{1.4}$$

When $x = 0$, $\beta_n(\lambda) = \beta_n(0 \mid \lambda)$ are called the degenerate Bernoulli numbers. As is well known, the Bernoulli polynomials of the second kind are defined by the generating function:

$$\frac{t}{\log(1+t)}(1+t)^x = \sum_{n=0}^{\infty} b_n(x)\frac{t^n}{n!} \quad \text{(see [3, 4])}. \tag{1.5}$$

When $x = 0$, $b_n = b_n(0)$ are called the Bernoulli numbers of the second kind. The Daehee polynomials are also given by the generating function:

$$\frac{\log(1+t)}{t}(1+t)^x = \sum_{n=0}^{\infty} D_n(x)\frac{t^n}{n!} \quad \text{(see [3, 4, 12, 13])}. \tag{1.6}$$

Now, we define the q-analogs of Bernoulli polynomials of the second kind as follows:

$$\frac{t}{(q-1)+\frac{q-1}{\log q}\log(1+t)}(1+t)^x = \sum_{n=0}^{\infty} b_{n,q}(x)\frac{t^n}{n!}. \tag{1.7}$$

Note that $\lim_{q\to 1} b_{n,q}(x) = b_n(x)$.

The q-analogs of Daehee polynomials are defined by the generating function to be

$$\frac{(q-1)+\frac{q-1}{\log q}\log(1+t)}{t}(1+t)^x = \sum_{n=0}^{\infty} D_{n,q}(x)\frac{t^n}{n!}. \tag{1.8}$$

When $x = 0$, $b_{n,q} = b_{n,q}(0)$ are called the q-analogs of Bernoulli numbers of the second kind and $D_{n,q} = D_{n,q}(0)$ are called the q-analogs of Daehee numbers.

From (1.7) and (1.8), we have

$$b_{n,q}(x) = \sum_{l=0}^{n} \binom{n}{l}(x)_{n-l} b_{l,q}, \qquad D_{n,q}(x) = \sum_{l=0}^{n} \binom{n}{l}(x)_{n-l} D_{l,q}, \tag{1.9}$$

where $(x)_n = x(x-1)\cdots(x-n+1) = \sum_{l=0}^{n} S_1(n,l)x^l$.

In this paper, we study q-analogs of degenerate Bernoulli polynomials and give some formulas related to these polynomials.

2 q-Analogs of degenerate Bernoulli polynomials

In this section, we assume that $\lambda, t \in \mathbb{C}_p$ with $|\lambda t| < p^{-\frac{1}{p-1}}$. Let us take $f(y) = (1+\lambda t)^{\frac{y}{\lambda}}$. Then by (1.3), we get

$$\int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{x+y}{\lambda}}\, d\mu_q(y) = \frac{(q-1)+\frac{q-1}{\log q}\log(1+\lambda t)^{\frac{1}{\lambda}}}{q(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} D_{n,q}(x \mid \lambda)\frac{t^n}{n!}, \tag{2.1}$$

where $D_{n,q}(x \mid \lambda)$ are called the q-analogs of λ-Daehee polynomials. When $x = 0$, $D_{n,q}(\lambda) = D_{n,q}(0 \mid \lambda)$ are called the q-analogs of λ-Daehee numbers.

From (1.3), we can easily derive the following equation:

$$q^n I_q(f_n) - I_q(f) = (q-1)\sum_{l=0}^{n-1} q^l f(l) + \frac{q-1}{\log q}\sum_{l=0}^{n-1} f'(l) q^l, \tag{2.2}$$

where $f_n(x) = f(x+n)$.

Thus, by (2.2), we get

$$\begin{aligned} &q^n \frac{(q-1) + \frac{q-1}{\log q}\log(1+\lambda t)^{\frac{1}{\lambda}}}{q(1+\lambda t)^{\frac{1}{\lambda}} - 1}(1+\lambda t)^{\frac{n}{\lambda}} \\ &\quad - \frac{(q-1) + \frac{q-1}{\log q}\log(1+\lambda t)^{\frac{1}{\lambda}}}{q(1+\lambda t)^{\frac{1}{\lambda}} - 1} \\ &\quad = (q-1)\sum_{l=0}^{n-1} q^l (1+\lambda t)^{\frac{l}{\lambda}} + \frac{q-1}{\log q}\log(1+\lambda t)^{\frac{1}{\lambda}}\sum_{l=0}^{n-1}(1+\lambda t)^{\frac{l}{\lambda}} q^l. \end{aligned} \tag{2.3}$$

By (2.3), we get

$$\begin{aligned} &q^n \frac{t}{q(1+\lambda t)^{\frac{1}{\lambda}} - 1}(1+\lambda t)^{\frac{n}{\lambda}} - \frac{t}{q(1+\lambda t)^{\frac{1}{\lambda}} - 1} \\ &\quad = t\sum_{l=0}^{n-1} q^l (1+\lambda t)^{\frac{l}{\lambda}}. \end{aligned} \tag{2.4}$$

It is easy to see that

$$(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{l=0}^{\infty} (x \mid \lambda)_l \frac{t^l}{l!},$$

where $(x \mid \lambda)_l = x(x-\lambda)\cdots(x-(l-1)\lambda)$ (see [1–17]).

Now, we define the q-analogs of degenerate Bernoulli polynomials as follows:

$$\frac{t}{q(1+\lambda t)^{\frac{1}{\lambda}} - 1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \beta_{n,q}(x \mid \lambda)\frac{t^n}{n!}. \tag{2.5}$$

When $x = 0$, $\beta_{n,q}(\lambda) = \beta_{n,q}(0 \mid \lambda)$ are called the q-analogs of degenerate Bernoulli numbers.

From (2.4) and (2.5), we have

$$\sum_{m=0}^{\infty}\left\{q^n \beta_{m,q}(n \mid \lambda) - \beta_{m,q}\right\}\frac{t^m}{m!} = \sum_{m=1}^{\infty}\left(m\sum_{l=0}^{n-1}(l \mid \lambda)_{m-1} q^l\right)\frac{t^m}{m!}. \tag{2.6}$$

Therefore, by (2.6), we obtain the following theorem.

Theorem 2.1 *For $m \in \mathbb{N}$, we have*

$$\frac{q^n \beta_{m,q}(n \mid \lambda) - \beta_{m,q}}{m} = \sum_{l=0}^{n-1} (l \mid \lambda)_{m-1} q^l.$$

We observe that

$$\begin{aligned} \frac{t}{(q-1) + \frac{q-1}{\log q} \log(1+\lambda t)^{\frac{1}{\lambda}}} \int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{x+y}{\lambda}} \, d\mu_q(y) &= \frac{t}{q(1+\lambda t)^{\frac{1}{\lambda}} - 1} (1+\lambda t)^{\frac{x}{\lambda}} \\ &= \sum_{n=0}^{\infty} \beta_{n,q}(x \mid \lambda) \frac{t^n}{n!}. \end{aligned} \tag{2.7}$$

Now, we define the q-analogs of degenerate Bernoulli polynomials of the second kind as follows:

$$\frac{t}{(q-1) + \frac{q-1}{\log q} \log(1+\lambda t)^{\frac{1}{\lambda}}} (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} b_{n,q}(x \mid \lambda) \frac{t^n}{n!}. \tag{2.8}$$

When $x = 0$, $b_n(\lambda) = b_n(0 \mid \lambda)$ are called the q-analogs of degenerate Bernoulli numbers of the second kind.

Indeed, we note that $\lim_{\lambda \to 1} b_{n,q}(x \mid \lambda) = b_{n,q}(x)$.

By (2.1), we easily get

$$\int_{\mathbb{Z}_p} (x+y \mid \lambda)_n \, d\mu_q(y) = D_{n,q}(x \mid \lambda) \quad (n \geq 0). \tag{2.9}$$

From (2.1) and (2.8), we note that

$$\begin{aligned} &\frac{t}{(q-1) + \frac{q-1}{\log q} \log(1+\lambda t)^{\frac{1}{\lambda}}} \int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{x+y}{\lambda}} \, d\mu_q(y) \\ &\quad = \left(\sum_{l=0}^{\infty} b_{l,q}(\lambda) \frac{t^l}{l!} \right) \left(\sum_{m=0}^{\infty} D_{m,q}(x \mid \lambda) \frac{t^m}{m!} \right) \\ &\quad = \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} b_{l,q}(\lambda) D_{n-l,q}(x \mid \lambda) \binom{n}{l} \right) \frac{t^n}{n!}. \end{aligned} \tag{2.10}$$

Therefore, by (2.7) and (2.10), we obtain the following theorem.

Theorem 2.2 *For $n \geq 0$, we have*

$$\beta_{n,q}(x \mid \lambda) = \sum_{l=0}^{n} \binom{n}{l} b_{l,q}(\lambda) D_{n-l,q}(x \mid \lambda).$$

As is well known, the Apostol-Bernoulli polynomials are defined by the generating function:

$$\frac{t}{qe^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x \mid q) \frac{t^n}{n!}. \tag{2.11}$$

By (2.5) and (2.11), we get $\lim_{\lambda\to 0}\beta_{n,q}(x\mid\lambda)=B_n(x\mid q)$ $(n\geq 0)$.

From (2.8), we can derive the following equation:

$$\begin{aligned}
&\frac{e^{\lambda t}-1}{(q-1)+\frac{q-1}{\log q}t}\frac{1}{\lambda}e^{xt}\\
&\quad=\sum_{n=0}^{\infty}b_{n,q}(x\mid\lambda)\frac{1}{n!}\frac{1}{\lambda^n}\left(e^{\lambda t}-1\right)^n\\
&\quad=\sum_{m=0}^{\infty}b_{m,q}(x\mid\lambda)\sum_{n=m}^{\infty}S_2(n,m)\lambda^{n-m}\frac{t^n}{n!}\\
&\quad=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}b_{m,q}(x\mid\lambda)S_2(n,m)\lambda^{n-m}\right)\frac{t^n}{n!}.
\end{aligned}\tag{2.12}$$

By replacing t by $\frac{1}{\lambda}(e^{\lambda t}-1)$ in (2.7), we get

$$\begin{aligned}
&\sum_{n=0}^{\infty}\beta_{n,q}(x\mid\lambda)\frac{1}{\lambda^n}\frac{1}{n!}\left(e^{\lambda t}-1\right)^n\\
&\quad=\frac{e^{\lambda t}-1}{(q-1)+\frac{q-1}{\log q}t}\frac{1}{\lambda}\int_{\mathbb{Z}_p}e^{(x+y)t}\,d\mu_q(y)\\
&\quad=\left(\sum_{k=0}^{\infty}\left(\sum_{m=0}^{k}b_{m,q}(\lambda)S_2(k,m)\lambda^{k-m}\right)\frac{t^k}{k!}\right)\left(\sum_{l=0}^{\infty}\int_{\mathbb{Z}_p}(x+y)^l\,d\mu_q(y)\frac{t^l}{l!}\right)\\
&\quad=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}b_{m,q}(\lambda)S_2(k,m)\lambda^{k-m}B_{n-k,q}(x)\right)\frac{t^n}{n!},
\end{aligned}\tag{2.13}$$

where $B_{n,q}(x)$ are the q-Bernoulli polynomials which are given by the generating function:

$$\frac{(q-1)+\frac{q-1}{\log q}t}{qe^t-1}e^{xt}=\sum_{n=0}^{\infty}B_{n,q}(x)\frac{t^n}{n!}.\tag{2.14}$$

On the other hand,

$$\begin{aligned}
\sum_{m=0}^{\infty}\beta_{m,q}(x\mid\lambda)\frac{1}{\lambda^m}\frac{1}{m!}\left(e^{\lambda t}-1\right)^m&=\sum_{m=0}^{\infty}\beta_{m,q}(x\mid\lambda)\lambda^{-m}\sum_{n=m}^{\infty}S_2(n,m)\lambda^n\frac{t^n}{n!}\\
&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\beta_{m,q}(x\mid\lambda)\lambda^{n-m}S_2(n,m)\right)\frac{t^n}{n!}.
\end{aligned}\tag{2.15}$$

Therefore, by (2.13) and (2.15), we obtain the following theorem.

Theorem 2.3 *For $n\geq 0$, we have*

$$\sum_{m=0}^{n}\beta_{m,q}(x\mid\lambda)\lambda^{n-m}S_2(n,m)=\sum_{k=0}^{n}\binom{n}{k}\sum_{m=0}^{k}b_{m,q}(\lambda)S_2(k,m)\lambda^{k-m}B_{n-k,q}(x).$$

From (2.7), we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}\beta_{n,q}(x\mid\lambda)\frac{t^n}{n!}\\
&\quad=\frac{t}{(q-1)+\frac{q-1}{\log q}\log(1+\lambda t)^{\frac{1}{\lambda}}}\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{x+y}{\lambda}}\,d\mu_q(y)\\
&\quad=\left(\sum_{l=0}^{\infty}b_{l,q}(\lambda)\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}\int_{\mathbb{Z}_p}(x+y\mid\lambda)_m\,d\mu_q(y)\frac{t^m}{m!}\right)\\
&\quad=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}b_{n-m,q}(\lambda)\int_{\mathbb{Z}_p}(x+y\mid\lambda)_m\,d\mu_q(y)\right)\frac{t^n}{n!}.
\end{aligned}\tag{2.16}$$

Note that

$$(x+y\mid\lambda)_m=\lambda^m\sum_{l=0}^{m}S_1(m,l)\lambda^{-l}(x+y)^l\tag{2.17}$$

and

$$\int_{\mathbb{Z}_p}(x+y)^l\,d\mu_q(y)=B_{n,q}(x)\quad(n\geq 0).\tag{2.18}$$

By (2.16), (2.17), and (2.18), we get

$$\begin{aligned}
&\beta_{n,q}(x\mid\lambda)\\
&\quad=\sum_{m=0}^{n}\binom{n}{m}b_{n-m,q}(\lambda)\lambda^m\sum_{l=0}^{m}S_1(m,l)\lambda^{-l}\int_{\mathbb{Z}_p}(x+y)^l\,d\mu_q(y)\\
&\quad=\sum_{m=0}^{n}\binom{n}{m}b_{n-m,q}(\lambda)\sum_{l=0}^{m}S_1(m,l)\lambda^{m-l}B_{l,q}(x)\\
&\quad=\sum_{m=0}^{n}\sum_{l=0}^{m}\binom{n}{m}b_{n-m,q}(\lambda)S_1(m,l)\lambda^{m-l}B_{l,q}(x).
\end{aligned}\tag{2.19}$$

Therefore, by (2.19), we obtain the following theorem.

Theorem 2.4 *For $n\geq 0$, we have*

$$\beta_{n,q}(x\mid\lambda)=\sum_{m=0}^{n}\sum_{l=0}^{m}\binom{n}{m}b_{n-m,q}(\lambda)S_1(m,n)\lambda^{m-l}B_{l,q}(x).$$

For $k\in\mathbb{N}$, we define the q-analogs of degenerate Bernoulli polynomials of order k as follows:

$$\left(\frac{t}{q(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^k(1+\lambda t)^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}\beta_{n,q}^{(k)}(x\mid\lambda)\frac{t^n}{n!}.\tag{2.20}$$

When $x=0$, $\beta_{n,q}(\lambda)=\beta_{n,q}(0\mid\lambda)$ are called the q-analogs of degenerate Bernoulli numbers of order k.

From (2.20), we note that

$$\begin{aligned}
&\sum_{n=0}^{\infty} \lim_{\lambda\to 0} \beta_{n,q}^{(k)}(x \mid \lambda)\frac{t^n}{n!} \\
&\quad = \lim_{\lambda\to 0}\left(\frac{t}{q(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^k (1+\lambda t)^{\frac{x}{\lambda}} \\
&\quad = \left(\frac{t}{qe^t-1}\right)^k e^{xt} \\
&\quad = \sum_{n=0}^{\infty} B_n^{(k)}(x \mid q)\frac{t^n}{n!},
\end{aligned} \tag{2.21}$$

where $B_n^{(k)}(x \mid q)$ are called the higher-order Apostol-Bernoulli polynomials.

Thus, by (2.21), we get $\lim_{\lambda\to 0} \beta_{n,q}^{(k)}(x \mid \lambda) = B_n^{(k)}(x \mid q)$ $(n \geq 0)$.

For $k \in \mathbb{N}$, by (2.20), we get

$$\begin{aligned}
&\left(\frac{t}{(q-1)+\frac{q-1}{\log q}\log(1+\lambda t)^{\frac{1}{\lambda}}}\right)^k \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{x_1+\cdots+x_k+x}{\lambda}}\,d\mu_q(x_1)\cdots d\mu_q(x_k) \\
&\quad = \left(\frac{t}{q(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^k (1+\lambda t)^{\frac{x}{\lambda}} \\
&\quad = \sum_{n=0}^{\infty} \beta_{n,q}^{(k)}(x \mid \lambda)\frac{t^n}{n!}.
\end{aligned} \tag{2.22}$$

Now, we define the q-analogs of higher-order degenerate Bernoulli polynomials of the second kind as follows:

$$\left(\frac{t}{(q-1)+\frac{q-1}{\log q}\log(1+\lambda t)^{\frac{1}{\lambda}}}\right)^k (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} b_{n,q}^{(k)}(x \mid \lambda)\frac{t^n}{n!}. \tag{2.23}$$

When $x = 0$, $b_{n,q}^{(k)}(\lambda) = b_{n,q}^{(k)}(0 \mid \lambda)$ are called the q-analogs of higher-order degenerate Bernoulli numbers of the second kind. Note that $\lim_{\lambda\to 1} b_{n,q}^{(k)}(x \mid \lambda) = b_{n,q}^{(k)}(x)$, and $\lim_{q\to 1} b_{n,q}^{(k)}(x) = b_n^{(k)}(x)$.

From (2.23), we can derive the following equation:

$$\begin{aligned}
&\left(\frac{t}{(q-1)+\frac{q-1}{\log q}\log(1+\lambda t)^{\frac{1}{\lambda}}}\right)^k \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{x_1+\cdots+x_k+x}{\lambda}}\,d\mu_q(x_1)\cdots d\mu_q(x_k) \\
&\quad = \left(\sum_{m=0}^{\infty} b_{m,q}^{(k)}(\lambda)\frac{t^m}{m!}\right)\left(\sum_{l=0}^{\infty}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1+\cdots+x_k \mid \lambda)_l\,d\mu_q(x_1)\cdots d\mu_q(x_k)\frac{t^l}{l!}\right) \\
&\quad = \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l} b_{n-l,q}^{(k)}(\lambda)\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1+\cdots+x_k+x \mid \lambda)_l\,d\mu_q(x_1)\cdots d\mu_q(x_k)\right)\frac{t^n}{n!}.
\end{aligned} \tag{2.24}$$

It is easy to show that

$$\begin{aligned}
&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} e^{(x_1+\cdots+x_k+x)t}\,d\mu_q(x_1)\cdots d\mu_q(x_k) \\
&= \left(\frac{q-1+\frac{q-1}{\log q}t}{qe^t-1}\right)^k e^{xt} \\
&= \sum_{n=0}^{\infty} B_{n,q}^{(k)}(x)\frac{t^n}{n!},
\end{aligned} \tag{2.25}$$

where $B_{n,q}^{(k)}(x)$ are called the q-Bernoulli polynomials of order k.

Thus, by (2.25), we get

$$\begin{aligned}
&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} (x_1+\cdots+x_k+x\mid\lambda)_l\,d\mu_q(x_1)\cdots d\mu_q(x_k) \\
&= \lambda^l\sum_{m=0}^{l}\lambda^{-m}S_1(l,m)\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1+\cdots+x_k+x)^m\,d\mu_q(x_1)\cdots d\mu_q(x_k) \\
&= \sum_{m=0}^{l}\lambda^{l-m}S_1(l,m)B_{m,q}^{(k)}(x).
\end{aligned} \tag{2.26}$$

Therefore, by (2.22), (2.24), and (2.26), we obtain the following theorem.

Theorem 2.5 *For $n \geq 0$, we have*

$$\beta_{n,q}^{(k)}(x\mid\lambda) = \sum_{l=0}^{n}\sum_{m=0}^{l}\binom{n}{l}b_{n-l,q}^{(k)}(\lambda)\lambda^{l-m}S_1(l,m)B_{m,q}^{(k)}(x).$$

Remark We define the q-analogs of λ-Daehee polynomials of order k as follows:

$$\left(\frac{q-1+\frac{q-1}{\log q}\log(1+\lambda t)^{\frac{1}{\lambda}}}{q(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^k(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty}D_{n,q}^{(k)}(x\mid\lambda)\frac{t^n}{n!}. \tag{2.27}$$

From (2.27), we have

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{x_1+\cdots+x_k+x}{\lambda}}\,d\mu_q(x_1)\cdots d\mu_q(x_k) = \sum_{n=0}^{\infty}D_{n,q}^{(k)}(x\mid\lambda)\frac{t^n}{n!}. \tag{2.28}$$

Thus, by (2.28), we get

$$D_{n,q}^{(k)}(x\mid\lambda) = \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1+\cdots+x_k+x\mid\lambda)_n\,d\mu_q(x_1)\cdots d\mu_q(x_k)\quad(n\geq 0). \tag{2.29}$$

From (2.22), (2.24), and (2.29), we have

$$\beta_{n,q}^{(k)}(x\mid\lambda) = \sum_{l=0}^{n}\binom{n}{l}b_{n-l,q}^{(k)}(\lambda)D_{l,q}^{(k)}(x\mid\lambda). \tag{2.30}$$

From (2.1), we can derive the following equation:

$$\begin{aligned}\int_{\mathbb{Z}_p}(1-\lambda t)^{-\frac{x+y}{\lambda}}\,d\mu_q(y) &= \frac{(q-1)+\frac{q-1}{\log q}\log(1-\lambda t)^{-\frac{1}{\lambda}}}{q(1-\lambda t)^{-\frac{1}{\lambda}}-1}(1-\lambda t)^{-\frac{x}{\lambda}}\\ &= \sum_{n=0}^{\infty} D_{n,q}(x\mid -\lambda)\frac{t^n}{n!}. \end{aligned}\tag{2.31}$$

Note that $D_n(1-x\mid\lambda)=(-1)^n D_n(x\mid -\lambda)$ $(n\geq 0)$.

By (2.31), we get

$$\int_{\mathbb{Z}_p}\langle x+y|\lambda\rangle_n\,d\mu_q(y)=D_{n,q}(x\mid -\lambda)\quad (n\geq 0),\tag{2.32}$$

where $\langle x|\lambda\rangle_n = x(x+\lambda)\cdots(x+(n-1)\lambda)$.

Note that

$$\begin{aligned}\langle x+y|\lambda\rangle_n &= \lambda^n\left(\frac{x+y}{\lambda}\right)\left(\frac{x+y}{\lambda}+1\right)\cdots\left(\frac{x+y}{\lambda}+n-1\right)\\ &= \sum_{l=0}^{n}\bigl|S_1(n,l)\bigr|(x+y)^l\lambda^{n-l}. \end{aligned}\tag{2.33}$$

From (2.32) and (2.33), we have

$$\begin{aligned}D_{n,q}(x\mid -\lambda) &= \sum_{l=0}^{n}\bigl|S_1(n,l)\bigr|\lambda^{n-l}\int_{\mathbb{Z}_p}(x+y)^l\,d\mu_q(y)\\ &= \sum_{l=0}^{n}\bigl|S_1(n,l)\bigr|\lambda^{n-l}B_{l,q}(x). \end{aligned}\tag{2.34}$$

By (2.7), we get

$$\begin{aligned}&\frac{t}{(q-1)+\frac{q-1}{\log q}\log(1-\lambda t)^{-\frac{1}{\lambda}}}\int_{\mathbb{Z}_p}(1-\lambda t)^{-\frac{x+y}{\lambda}}\,d\mu_q(y)\\ &\quad= \frac{t}{q(1-\lambda t)^{-\frac{1}{\lambda}}-1}(1-\lambda t)^{-\frac{x}{\lambda}}\\ &\quad= \sum_{n=0}^{\infty}\beta_{n,q}(x\mid -\lambda)\frac{t^n}{n!}. \end{aligned}\tag{2.35}$$

By (2.8), we get

$$\begin{aligned}&\frac{t}{(q-1)+\frac{q-1}{\log q}\log(1-\lambda t)^{-\frac{1}{\lambda}}}\int_{\mathbb{Z}_p}(1-\lambda t)^{-\frac{x+y}{\lambda}}\,d\mu_q(y)\\ &\quad= \left(\sum_{m=0}^{\infty} b_{m,q}(-\lambda)\frac{t^m}{m!}\right)\left(\sum_{l=0}^{\infty}\int_{\mathbb{Z}_p}\langle x+y|\lambda\rangle_l\,d\mu_q(y)\frac{t^l}{l!}\right)\\ &\quad= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}b_{n-l,q}(-\lambda)\int_{\mathbb{Z}_p}\langle x+y|\lambda\rangle_l\,d\mu_q(y)\right)\frac{t^n}{n!}. \end{aligned}\tag{2.36}$$

Therefore, by (2.32), (2.33), (2.35), and (2.36), we get

$$\begin{aligned}
&\beta_{n,q}(x \mid -\lambda) \\
&\quad = \sum_{l=0}^{n} \binom{n}{l} b_{n-l,q}(-\lambda) D_{l,q}(x \mid -\lambda) \\
&\quad = \sum_{l=0}^{n} \sum_{m=0}^{l} \binom{n}{l} \bigl|S_1(l,m)\bigr| \lambda^{l-m} B_{m,q}(x) b_{n-l,q}(-\lambda).
\end{aligned} \tag{2.37}$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to this work. All authors read and approved the final manuscript.

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