Xin and Zhao *Advances in Difference Equations* (2015) 2015:209
DOI 10.1186/s13662-015-0524-5

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Existence of periodic solution for generalized neutral Rayleigh equation with variable parameter

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Abstract

In this paper, we consider a generalized neutral Rayleigh equation with variable parameter

$$(x(t)-c(t)x(t-\delta(t)))''+f(t,x'(t))+g(t,x(t-\tau(t)))=e(t),$$

where $|c(t)|\neq 1$, c, $\delta\in C^1(\mathbb{R},\mathbb{R})$ and c, δ are ω-periodic functions for some $\omega>0$. By applications of coincidence degree theory and some analysis skills, sufficient conditions for the existence of periodic solution is established.

Keywords: neutral operator; periodic solution; Rayleigh equation; variable parameter

1 Introduction

In the paper, we consider the generalized neutral Rayleigh differential equation with variable parameter

$$\bigl(x(t)-c(t)x\bigl(t-\delta(t)\bigr)\bigr)''+f\bigl(t,x'(t)\bigr)+g\bigl(t,x\bigl(t-\tau(t)\bigr)\bigr)=e(t), \tag{1.1}$$

where $|c(t)|\neq 1$, $c,\delta\in C^1(\mathbb{R},\mathbb{R})$ and c, δ are ω-periodic functions for some $\omega>0$, $\tau,e\in C[0,\omega]$ and $\int_0^\omega e(t)\,dt=0$; f and g are continuous functions defined on $\mathbb{R}^2$ and periodic in t with $f(t,\cdot)=f(t+\omega,\cdot)$, $g(t,\cdot)=g(t+\omega,\cdot)$, and $f(t,0)=0$.

Neutral differential equations manifest themselves in many fields including biology, mechanics and economics [1–4]. For example, in population dynamics, since a growing population consumes more (or less) food than a matured one, depending on individual species, this leads to neutral equations [2]. These equations also arise in classical 'cobweb' models in economics where current demand depends on price, but supply depends on the previous periodic [4]. The study on neutral differential equations is more intricate than that on ordinary delay differential equations. In recent years, there is a good amount of work on periodic solutions for neutral differential equations (see [5–14] and the references cited therein). For example, in [5], Lu *et al.* considered the following neutral differential equation with deviating arguments:

$$\bigl(x(t)-cx(t-\delta)\bigr)''+f\bigl(x'(t)\bigr)+f\bigl(x\bigl(t-\tau(t)\bigr)\bigr)=p(t).$$

By using the continuation theorem and some analysis techniques, some new results on the existence of periodic solutions are obtained. Afterwards, Du *et al.* [10] investigated the second order neutral equation

$$\bigl(x(t)-c(t)x(t-\delta)\bigr)'' + f\bigl(x(t)\bigr)x'(t) + g\bigl(x\bigl(t-\gamma(t)\bigr)\bigr) = e(t), \tag{1.2}$$

by using Mawhin's continuous theorem, the authors obtained the existence of periodic solution to (1.2). Recently, Ren *et al.* [12] studied the neutral equation with variable delay

$$\bigl(x(t)-cx\bigl(t-\delta(t)\bigr)\bigr)'' = -a(t)x(t) + \lambda b(t)f\bigl(x\bigl(t-\tau(t)\bigr)\bigr), \tag{1.3}$$

by an application of the fixed-point index theorem, the authors obtained sufficient conditions for the existence, multiplicity and nonexistence of positive periodic solutions to (1.3).

Motivated by [5, 10, 12], in this paper, we consider the generalized neutral Rayleigh equation (1.1). Notice that here the neutral operator A is a natural generalization of the familiar operator $A_1 = x(t) - cx(t-\delta)$, $A_2 = x(t) - c(t)x(t-\delta)$, $A_3 = x(t) - cx(t-\delta(t))$. But A possesses a more complicated nonlinearity than A_i, $i = 1, 2, 3$. For example, the neutral operator A_1 is homogeneous in the following sense $(A_1x)'(t) = (A_1x')(t)$, whereas the neutral operator A in general is inhomogeneous. As a consequence, many of the new results for differential equations with the neutral operator A will not be a direct extension of known theorems for neutral differential equations.

The paper is organized as follows. In Section 2, we first analyze qualitative properties of the generalized neutral operator A which will be helpful for further studies of differential equations with this neutral operator; in Section 3, by Mawhin's continuation theorem, we obtain the existence of periodic solutions for the generalized neutral Rayleigh equation with variable parameter. We will give an example to illustrate our results, and an example is also given in this section. Our results improve and extend the results in [5, 10, 12–14].

2 Analysis of the generalized neutral operator with variable parameter

Let

$$c_\infty = \max_{t\in[0,\omega]}\bigl|c(t)\bigr|, \qquad c_0 = \min_{t\in[0,\omega]}\bigl|c(t)\bigr|.$$

Let $X = \{x \in C(\mathbb{R},\mathbb{R}) : x(t+\omega) = x(t), t \in \mathbb{R}\}$ with norm $\|x\| = \max_{t\in[0,\omega]}|x(t)|$. Then $(X, \|\cdot\|)$ is a Banach space. Moreover, define operators $A, B : C_\omega \to C_\omega$ by

$$(Ax)(t) = x(t) - c(t)x\bigl(t-\delta(t)\bigr), \qquad (Bx)(t) = c(t)x\bigl(t-\delta(t)\bigr).$$

Lemma 2.1 *If $|c(t)| \neq 1$, then the operator A has a continuous inverse A^{-1} on C_ω, satisfying*

(1)

$$\bigl(A^{-1}f\bigr)(t) = \begin{cases} f(t) + \sum_{j=1}^{\infty}\prod_{i=1}^{j} c(D_i)x(t-\sum_{i=1}^{j}\delta(D_i)) & \textit{for } |c(t)|<1, \forall f \in C_\omega, \\ -\frac{f(t+\delta(t))}{c(t+\delta(t))} - \sum_{j=1}^{\infty}\frac{f(t+\delta(t)+\sum_{i=1}^{j}\delta(D_i'))}{c(t+\delta(t))\prod_{i=1}^{j}c(D_i')} & \textit{for } |c(t)|>1, \forall f \in C_\omega. \end{cases}$$

(2)

$$\left|\left(A^{-1}f\right)(t)\right| \le \begin{cases} \frac{\|f\|}{1-c_\infty} & \text{for } c_\infty < 1 \forall f \in C_\omega, \\ \frac{\|f\|}{c_0-1} & \text{for } c_0 > 1 \forall f \in C_\omega. \end{cases}$$

(3)

$$\int_0^\omega \left|\left(A^{-1}f\right)(t)\right| dt \le \begin{cases} \frac{1}{1-c_\infty} \int_0^\omega |f(t)|\,dt & \text{for } c_\infty < 1 \forall f \in C_\omega, \\ \frac{1}{c_0-1} \int_0^\omega |f(t)|\,dt & \text{for } c_0 > 1 \forall f \in C_\omega, \end{cases}$$

where $D_1 = t$, $D_i = t - \sum_{k=1}^{i} \delta(D_k)$, $k = 1, 2, \ldots$, *and* $D_1' = t$, $D_i' = t + \sum_{k=1}^{i} \delta(D_k')$, $k = 1, 2, \ldots$.

Proof *Case* 1: $|c(t)| \le c_\infty < 1$.

Let $t = D_1$ and $D_j = t - \sum_{i=1}^{j} \delta(D_i)$, $j = 1, 2, \ldots$.

$$\begin{aligned} (Bx)(t) &= c(t)x\bigl(t-\delta(t)\bigr) = c(D_1)x\bigl(t-\delta(D_1)\bigr); \\ \bigl(B^2x\bigr)(t) &= c(t)c\bigl(t-\delta(t)\bigr)x\bigl(t-\delta(t)-\delta\bigl(t-\delta(t)\bigr)\bigr) = c(D_1)c(D_2))x\bigl(t-\delta(D_1)-\delta(D_2)\bigr); \\ \bigl(B^3x\bigr)(t) &= c(t)c\bigl(t-\delta(t)\bigr)c\bigl(t-\delta(t)-\delta\bigl(t-\delta(t)\bigr)\bigr)x\bigl(t-\delta(D_1)-\delta(D_2)-\delta(D_3)\bigr) \\ &= c(D_1)c(D_2)c(D_3)x\left(t - \sum_{i=1}^{3} \delta(D_i)\right). \end{aligned}$$

Therefore

$$B^j x(t) = \prod_{i=1}^{j} c(D_i)x\left(t - \sum_{i=1}^{j} \delta(D_i)\right),$$

and

$$\sum_{j=0}^{\infty} \bigl(B^j f\bigr)(t) = f(t) + \sum_{j=1}^{\infty} \prod_{i=1}^{j} c(D_i)x\left(t - \sum_{i=1}^{j} \delta(D_i)\right).$$

Since $A = I - B$, we get from $\|B\| \le c_\infty < 1$ that A has a continuous inverse $A^{-1} : C_\omega \to C_\omega$ with

$$A^{-1} = (I - B)^{-1} = I + \sum_{j=1}^{\infty} B^j = \sum_{j=0}^{\infty} B^j,$$

here $B^0 = I$. Then

$$\bigl(A^{-1}f(t)\bigr) = \sum_{j=0}^{\infty} \bigl[B^j f\bigr](t) = f(t) + \sum_{j=1}^{\infty} \prod_{i=1}^{j} c(D_i)x\left(t - \sum_{i=1}^{j} \delta(D_i)\right),$$

and consequently

$$\begin{aligned} \left|\bigl(A^{-1}f\bigr)(t)\right| &= \left|\sum_{j=0}^{\infty} \bigl[B^j f\bigr](t)\right| \\ &= \left|f(t) + \sum_{j=1}^{\infty} \prod_{i=1}^{j} c(D_i)x\left(t - \sum_{i=1}^{j} \delta(D_i)\right)\right| \end{aligned}$$

$$\begin{aligned}
&\leq \left(1+\sum_{j=1}^{\infty} c_{\infty}^{j}\right)|f|_{\infty} \\
&\leq \frac{|f|_{\infty}}{1-c_{\infty}}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\int_0^{\omega}\left|\left(A^{-1}f\right)(t)\right|dt &= \int_0^{\omega}\left|\sum_{j=0}^{\infty}\left(B^j f\right)(t)\right|dt \\
&\leq \sum_{j=0}^{\infty}\int_0^{\omega}\left|\left(B^j f\right)(t)\right|dt \\
&= \sum_{j=0}^{\infty}\int_0^{\omega}\left|\prod_{i=1}^{j} c(D_i)x\left(t-\sum_{i=1}^{j}\delta(D_i)\right)\right|dt \\
&\leq \frac{1}{1-c_{\infty}}\int_0^{\omega}\left|f(t)\right|dt.
\end{aligned}$$

Case 2: $|c(t)| > c_0 > 1$.

Let $D_1' = t$, $D_j' = t + \sum_{i=1}^{j}\delta(D_i')$, $j = 1, 2, \ldots$. And set

$$\begin{aligned}
&E : C_{\omega} \to C_{\omega}, \quad (Ex)(t) = x(t) - \frac{1}{c(t)}x\left(t+\delta(t)\right), \\
&B_1 : C_{\omega} \to C_{\omega}, \quad (B_1 x)(t) = \frac{1}{c(t)}x\left(t+\delta(t)\right).
\end{aligned}$$

By definition of the linear operator B_1, we have

$$\left(B_1^j f\right)(t) = \frac{1}{\prod_{i=1}^{j} c(D_i')} f\left(t+\sum_{i=1}^{j}\delta\left(D_i'\right)\right),$$

here D_i is defined as in Case 1. Summing over j yields

$$\sum_{j=0}^{\infty}\left(B_1^j f\right)(t) = f(t) + \sum_{j=1}^{\infty}\frac{1}{\prod_{i=1}^{j} c(D_i')} f\left(t+\sum_{i=1}^{j}\delta\left(D_i'\right)\right).$$

Since $\|B_1\| < 1$, we obtain that the operator E has a bounded inverse E^{-1},

$$E^{-1} : C_{\omega} \to C_{\omega}, \quad E^{-1} = (I - B_1)^{-1} = I + \sum_{j=1}^{\infty} B_1^j,$$

and $\forall f \in C_{\omega}$ we get

$$\left(E^{-1}f\right)(t) = f(t) + \sum_{j=1}^{\infty}\left(B_1^j f\right)(t).$$

On the other hand, from $(Ax)(t) = x(t) - c(t)x(t - \delta(t))$, we have

$$(Ax)(t) = x(t) - c(t)x\big(t - \delta(t)\big) = -c(t)\left[x\big(t - \delta(t)\big) - \frac{1}{c(t)}x(t)\right],$$

i.e.,

$$(Ax)(t) = -c(t)(Ex)\big(t - \delta(t)\big).$$

Let $f \in C_\omega$ be arbitrary. We are looking for x such that

$$(Ax)(t) = f(t),$$

i.e.,

$$-c(t)(Ex)\big(t - \delta(t)\big) = f(t).$$

Therefore

$$(Ex)(t) = -\frac{f(t + \delta(t))}{c(t + \delta(t))} =: f_1(t),$$

and hence

$$x(t) = \big(E^{-1}f_1\big)(t) = f_1(t) + \sum_{j=1}^{\infty}\big(B_1^j f_1\big)(t) = -\frac{f(t + \delta(t))}{c(t + \delta(t))} - \sum_{j=1}^{\infty} B_1^j \frac{f(t + \delta(t))}{c(t + \delta(t))},$$

proving that A^{-1} exists and satisfies

$$\begin{aligned}\big[A^{-1}f\big](t) &= -\frac{f(t + \delta(t))}{c(t + \delta(t))} - \sum_{j=1}^{\infty} B_1^j \frac{f(t + \delta(t))}{c(t + \delta(t))} \\ &= -\frac{f(t + \delta(t))}{c(t + \delta(t))} - \sum_{j=1}^{\infty} \frac{f(t + \delta(t) + \sum_{i=1}^{j}\delta(D_i'))}{c(t + \delta(t))\prod_{i=1}^{j} c(D_i')}\end{aligned}$$

and

$$\big|\big[A^{-1}f\big](t)\big| = \left|-\frac{f(t + \delta(t))}{c(t + \delta(t))} - \sum_{j=1}^{\infty} \frac{f(t + \delta(t) + \sum_{i=1}^{j}\delta(D_i'))}{c(t + \delta(t))\prod_{i=1}^{j} c(D_i')}\right| \le \frac{\|f\|}{c_0 - 1}.$$

Statements (1) and (2) are proved. From the above proof, (3) can easily be deduced. □

3 Periodic solution for (1.1)

We first recall Mawhin's continuation theorem which our study is based upon. Let X and Y be real Banach spaces and $L : D(L) \subset X \to Y$ be a Fredholm operator with index zero, here $D(L)$ denotes the domain of L. This means that $\operatorname{Im} L$ is closed in Y and $\dim \operatorname{Ker} L = \dim(Y/\operatorname{Im} L) < +\infty$. Consider supplementary subspaces X_1, Y_1, of X, Y, respectively, such that $X = \operatorname{Ker} L \oplus X_1$, $Y = \operatorname{Im} L \oplus Y_1$, and let $P_1 : X \to \operatorname{Ker} L$ and $Q_1 : Y \to Y_1$ denote the natural projections. Clearly, $\operatorname{Ker} L \cap (D(L) \cap X_1) = \{0\}$, thus the restriction $L_{P_1} := L|_{D(L)\cap X_1}$ is invertible. Let $L_{P_1}^{-1}$ denote the inverse of L_{P_1}.

Let Ω be an open bounded subset of X with $D(L) \cap \Omega \neq \emptyset$. A map $N : \overline{\Omega} \to Y$ is said to be L-compact in $\overline{\Omega}$ if $Q_1 N(\overline{\Omega})$ is bounded and the operator $L_{P_1}^{-1}(I - Q_1)N : \overline{\Omega} \to X$ is compact.

Lemma 3.1 (Gaines and Mawhin [15]) *Suppose that X and Y are two Banach spaces, and $L : D(L) \subset X \to Y$ is a Fredholm operator with index zero. Furthermore, $\Omega \subset X$ is an open bounded set and $N : \overline{\Omega} \to Y$ is L-compact on $\overline{\Omega}$. Assume that the following conditions hold*:

(1) $Lx \neq \lambda Nx$, $\forall x \in \partial\Omega \cap D(L)$, $\lambda \in (0, 1)$;

(2) $Nx \notin \operatorname{Im} L$, $\forall x \in \partial\Omega \cap \operatorname{Ker} L$;

(3) $\deg\{JQ_1N, \Omega \cap \operatorname{Ker} L, 0\} \neq 0$, *where $J : \operatorname{Im} Q_1 \to \operatorname{Ker} L$ is an isomorphism.*

Then the equation $Lx = Nx$ has a solution in $\overline{\Omega} \cap D(L)$.

In order to use Mawhin's continuation theorem to study the existence of ω-periodic solutions for (1.1), we rewrite (1.1) in the following form:

$$\begin{cases} (Ax_1)'(t) = x_2(t), \\ x_2'(t) = -f(t, x_1'(t)) - g(t, x_1(t - \tau(t))) + e(t). \end{cases} \tag{3.1}$$

Clearly, if $x(t) = (x_1(t), x_2(t))^\top$ is an ω-periodic solution to (3.1), then $x_1(t)$ must be an ω-periodic solution to (1.1). Thus, the problem of finding an ω-periodic solution for (1.1) reduces to finding one for (3.1).

Recall that $C_\omega = \{\phi \in C(\mathbb{R}, \mathbb{R}) : \phi(t + \omega) \equiv \phi(t)\}$ with norm $\|\phi\| = \max_{t\in[0,\omega]} |\phi(t)|$. Define $X = Y = C_\omega \times C_\omega = \{x = (x_1(\cdot), x_2(\cdot)) \in C(\mathbb{R}, \mathbb{R}^2) : x(t) = x(t + \omega), t \in \mathbb{R}\}$ with norm $\|x\| = \max\{\|x_1\|, \|x_2\|\}$. Clearly, X and Y are Banach spaces. Moreover, define

$$L : D(L) = \{x \in C^1(\mathbb{R}, \mathbb{R}^2) : x(t + \omega) = x(t), t \in \mathbb{R}\} \subset X \to Y$$

by

$$(Lx)(t) = \begin{pmatrix} (Ax_1)'(t) \\ x_2'(t) \end{pmatrix}$$

and $N : X \to Y$ by

$$(Nx)(t) = \begin{pmatrix} x_2(t) \\ -f(t, x_1'(t)) - g(t, x_1(t - \tau(t))) + e(t) \end{pmatrix}. \tag{3.2}$$

Then (3.1) can be converted to the abstract equation $Lx = Nx$. From the definition of L, one can easily see that

$$\operatorname{Ker} L \cong \mathbb{R}^2, \qquad \operatorname{Im} L = \left\{ y \in Y : \int_0^\omega \begin{pmatrix} y_1(s) \\ y_2(s) \end{pmatrix} ds = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}.$$

So L is a Fredholm operator with index zero. Let $P_1 : X \to \operatorname{Ker} L$ and $Q_1 : Y \to \operatorname{Im} Q_1 \subset \mathbb{R}^2$ be defined by

$$P_1 x = \begin{pmatrix} (Ax_1)(0) \\ x_2(0) \end{pmatrix}; \qquad Q_1 y = \frac{1}{\omega} \int_0^\omega \begin{pmatrix} y_1(s) \\ y_2(s) \end{pmatrix} ds,$$

then $\operatorname{Im} P_1 = \operatorname{Ker} L$, $\operatorname{Ker} Q_1 = \operatorname{Im} L$. Set $L_{P_1} = L|_{D(L)\cap \operatorname{Ker} P_1}$ and $L_{P_1}^{-1} : \operatorname{Im} L \to D(L)$ denotes the inverse of L_{P_1}, then

$$\begin{aligned}
&\left[L_{P_1}^{-1}y\right](t) = \begin{pmatrix} (A^{-1}Fy_1)(t) \\ (Fy_2)(t) \end{pmatrix}, \\
&[Fy_1](t) = \int_0^t y_1(s)\,ds, \qquad [Fy_2](t) = \int_0^t y_2(s)\,ds.
\end{aligned} \tag{3.3}$$

From (3.2) and (3.3), it is clear that Q_1N and $L_{P_1}^{-1}(I - Q_1)N$ are continuous, and $Q_1N(\overline{\Omega})$ is bounded, and then $L_{P_1}^{-1}(I - Q_1)N(\overline{\Omega})$ is compact for any open bounded $\Omega \subset X$, which means N is L-compact on $\bar{\Omega}$.

For the sake of convenience, we list the following assumptions which will be used repeatedly in the sequel:

(H_1) there exists a positive constant K_1 such that $|f(t,u)| \leq K_1$ for $(t,u) \in \mathbb{R} \times \mathbb{R}$;
(H_2) there exists a positive constant D such that $x \cdot g(t,x) > 0$ and $|g(x,x)| > K_1$ for $|x| > D$;
(H_3) there exists a positive constant M and $M > \|e\|$ such that $g(t,x) \geq -M$ for $x \leq -D$ and $t \in \mathbb{R}$;
(H_4) there exists a positive constant M and $M > \|e\|$ such that $g(t,x) \leq M$ for $x \geq D$ and $t \in \mathbb{R}$;

Now we give our main results on periodic solutions for (1.1).

Theorem 3.1 *Assume that conditions* (H_1)-(H_3) *hold. Suppose that one of the following conditions is satisfied*:
(i) *If* $c_\infty < 1$ *and* $1 - c_\infty - \delta_1 c_\infty - \frac{1}{2}c_1\omega > 0$;
(ii) *If* $c_0 > 1$ *and* $c_0 - 1 - \delta_1 c_\infty - \frac{1}{2}c_1\omega > 0$;
where $\delta_1 = \max_{t\in[0,\omega]} |\delta'(t)|$, $c_1 = \max_{t\in[0,\omega]} |c'(t)|$.
Then (1.1) *has at least one solution with period* ω.

Proof By construction, (3.1) has an ω-periodic solution if and only if the following operator equation

$$Lx = Nx$$

has an ω-periodic solution. From (3.3), we see that N is L-compact on $\bar{\Omega}$, where Ω is any open, bounded subset of C_ω. For $\lambda \in (0,1]$, define

$$\Omega_1 = \{x \in C_\omega : Lx = \lambda Nx\}.$$

Then $x = (x_1,x_2)^\top \in \Omega_1$ satisfies

$$\begin{cases} (Ax_1)'(t) = \lambda x_2(t), \\ x_2'(t) = -\lambda f(t,x_1'(t)) - \lambda g(t,x_1(t-\tau(t))) + \lambda e(t). \end{cases} \tag{3.4}$$

Substituting $x_2(t) = \frac{1}{\lambda}(Ax_1)'(t)$ into the second equation of (3.4) yields

$$\left(\frac{1}{\lambda}(Ax_1)(t)\right)'' = -\lambda f\bigl(t,x_1'(t)\bigr) - \lambda g\bigl(t,x_1\bigl(t-\tau(t)\bigr)\bigr) + \lambda e(t),$$

i.e.,

$$\big((Ax_1)(t)\big)'' = -\lambda^2 f\big(t, x_1'(t)\big) - \lambda^2 g\big(t, x_1\big(t-\tau(t)\big)\big) + \lambda^2 e(t). \tag{3.5}$$

We first claim that there is a constant $\xi \in \mathbb{R}$ such that

$$\big|x_1(\xi)\big| \le D. \tag{3.6}$$

Integrating both sides of (3.5) over $[0,\omega]$, we have

$$\int_0^\omega \big[f\big(t, x_1'(t)\big) + g\big(t, x_1\big(t-\tau(t)\big)\big)\big]\,dt = 0, \tag{3.7}$$

which yields that there at least exists a point t_1 such that

$$f\big(t_1, x_1'(t_1)\big) + g\big(t_1, x_1\big(t_1-\tau(t_1)\big)\big) = 0,$$

then by (H_1) we have

$$\big|g\big(t_1, x_1\big(t_1-\tau(t_1)\big)\big)\big| = \big|-f\big(t_1, x_1'(t_1)\big)\big| \le K,$$

and in view of (H_2) we get that $|x_1(t_1-\tau(t_1))| \le D$. Since $x_1(t)$ is periodic with periodic ω. So $t_1 - \tau(t_1) = n\omega + \xi$, $\xi \in [0,\omega]$, where n is some integer, then $|x_1(\xi)| \le D$. Equation (3.6) is proved.

Then we have

$$\big|x(t)\big| = \left|x(\xi) + \int_\xi^t x'(s)\,ds\right| \le D + \int_\xi^t \big|x'(s)\big|\,ds, \quad t \in [\xi, \xi+\omega],$$

and

$$\big|x(t)\big| = \big|x(t-\omega)\big| = \left|x(\xi) - \int_{t-\omega}^\xi x'(s)\,ds\right| \le D + \int_{t-\omega}^\xi \big|x'(s)\big|\,ds, \quad t \in [\xi, \xi+\omega].$$

Combining the above two inequalities, we obtain

$$\begin{aligned} |x|_0 = \max_{t\in[0,\omega]}\big|x(t)\big| &= \max_{t\in[\xi,\xi+\omega]}\big|x(t)\big| \le \max_{t\in[\xi,\xi+\omega]}\left\{D + \frac{1}{2}\left(\int_\xi^t \big|x'(s)\big|\,ds + \int_{t-\omega}^\xi \big|x'(s)\big|\,ds\right)\right\} \\ &\le D + \frac{1}{2}\int_0^\omega \big|x'(s)\big|\,ds. \end{aligned} \tag{3.8}$$

On the other hand, multiplying both sides of (3.5) by $(Ax_1)(t)$ and integrating over $[0,\omega]$, we get

$$\begin{aligned} \int_0^\omega \big((Ax_1)(t)\big)''\big(Ax_1(t)\big)\,dt = -\int_0^\omega \big|(Ax_1)'(t)\big|^2\,dt = {}& -\lambda^2 \int_0^\omega f\big(t, x_1'(t)\big)(Ax_1)(t)\,dt \\ & - \lambda^2 \int_0^\omega g\big(t, x_1\big(t-\tau(t)\big)\big)(Ax_1)(t)\,dt \\ & + \lambda^2 \int_0^\omega e(t)(Ax_1)(t)\,dt. \end{aligned}$$

Using (H_1), we have

$$\begin{aligned}\int_0^\omega \left|(Ax_1)'(t)\right|^2 dt &\leq \int_0^\omega \left|f\left(t,x_1'(t)\right)\right|\left|\left[x_1(t)-c(t)x_1\left(t-\delta(t)\right)\right]\right| dt \\ &\quad+\int_0^\omega \left|g\left(t,x_1\left(t-\tau(t)\right)\right)\right|\left|\left[x_1(t)-c(t)x_1\left(t-\delta(t)\right)\right]\right| dt \\ &\quad+\int_0^\omega \left|e(t)\right|\left|\left[x_1(t)-c(t)x_1\left(t-\delta(t)\right)\right]\right| dt \\ &\leq (1+c_\infty)\|x_1\|\left[K_1\omega+\int_0^\omega \left|g\left(t,x_1\left(t-\tau(t)\right)\right)\right| dt+\omega\|e\|\right]. \end{aligned} \tag{3.9}$$

Besides, we can assert that there exists some positive constant N_1 such that

$$\int_0^\omega \left|g\left(t,x_1\left(t-\tau(t)\right)\right)\right| dt \leq 2\omega N_1+K_1\omega. \tag{3.10}$$

In fact, in view of condition (H_1) and (3.7), we have

$$\begin{aligned}&\int_0^\omega \left\{g\left(t,x_1\left(t-\tau(t)\right)\right)-K_1\right\} dt \\ &\quad\leq \int_0^\omega \left\{g\left(t,x_1\left(t-\tau(t)\right)\right)-\left|f\left(t,x_1'(t)\right)\right|\right\} dt \\ &\quad\leq \int_0^\omega \left\{g\left(t,x_1\left(t-\tau(t)\right)\right)+f\left(t,x_1'(t)\right)\right\} dt \\ &\quad= 0.\end{aligned}$$

Define

$$\begin{aligned}&E_1=\left\{t\in[0,\omega]:x_1\left(t-\tau(t)\right)>D\right\}; \\ &E_2=\left\{t\in[0,\omega]:\left|x_1\left(t-\tau(t)\right)\right|\leq D\right\}\cup\left\{t\in[0,\omega]:x_1\left(t-\tau(t)\right)<-D\right\}.\end{aligned}$$

With these sets we get

$$\begin{aligned}&\int_{E_2} \left|g\left(t,x_1\left(t-\tau(t)\right)\right)\right| dt \leq \omega\max\left\{M,\sup_{t\in[0,\omega],|x_1(t-\tau(t))|\leq D}\left|g(t,x_1)\right|\right\}, \\ &\int_{E_1} \left\{\left|g\left(t,x_1\left(t-\tau(t)\right)\right)\right|-K_1\right\} dt = \int_{E_1} \left\{g\left(t,x_1\left(t-\tau(t)\right)\right)-K_1\right\} dt \\ &\qquad\leq -\int_{E_2} \left\{g\left(t,x_1\left(t-\tau(t)\right)\right)-K_1\right\} dt \\ &\qquad\leq \int_{E_2} \left\{\left|g\left(t,x_1\left(t-\tau(t)\right)\right)\right|+K_1\right\} dt,\end{aligned}$$

which yields

$$\begin{aligned}\int_{E_1} \left|g\left(t,x_1\left(t-\tau(t)\right)\right)\right| dt &\leq \int_{E_2} \left|g\left(t,x_1\left(t-\tau(t)\right)\right)\right| dt+\int_{E_1\cup E_2} K_1\, dt \\ &= \int_{E_2} \left|g\left(t,x_1\left(t-\tau(t)\right)\right)\right| dt+K_1\omega.\end{aligned}$$

That is,

$$\begin{aligned}\int_0^\omega \bigl|g\bigl(t,x_1\bigl(t-\tau(t)\bigr)\bigr)\bigr|\,dt &= \int_{E_1} \bigl|g\bigl(t,x_1\bigl(t-\tau(t)\bigr)\bigr)\bigr|\,dt + \int_{E_2} \bigl|g\bigl(x_1\bigl(t-\tau(t)\bigr)\bigr)\bigr|\,dt \\ &\le 2\int_{E_2} \bigl|g\bigl(t,x_1\bigl(t-\tau(t)\bigr)\bigr)\bigr|\,dt + K_1\omega \\ &\le 2\omega \max\Bigl\{M, \sup_{t\in[0,\omega],|x_1(t-\tau(t))|<D} \bigl|g(t,x_1)\bigr|\Bigr\} + K_1\omega \\ &= 2\omega N_1 + K_1\omega,\end{aligned}$$

where $N_1 = \max\{M, \sup_{t\in[0,\omega],|x_1(t-\tau(t))|<D} |g(t,x_1)|\}$, proving (3.10).

Substituting (3.10) into (3.9) and recalling (3.8), we get

$$\begin{aligned}\int_0^\omega \bigl|(Ax_1)'(t)\bigr|^2\,dt &\le (1+c_\infty)\|x_1\|\bigl(2K_1\omega + 2\omega N_1 + \omega\|e\|\bigr) \\ &\le (1+c_\infty)\bigl(2K_1\omega + 2\omega N_1 + \omega\|e\|\bigr)\left(D + \frac{1}{2}\int_0^\omega \bigl|x_1'(t)\bigr|\,dt\right) \\ &= (1+c_\infty)N_2 D + (1+c_\infty)\frac{N_2}{2}\int_0^\omega \bigl|x_1'(t)\bigr|\,dt,\end{aligned} \tag{3.11}$$

where $N_2 = 2K_1\omega + 2\omega N_1 + \omega\|e\|$. Since $(Ax)(t) = x(t) - c(t)x(t-\delta(t))$, we have

$$\begin{aligned}(Ax_1)'(t) &= \bigl(x_1(t) - c(t)x_1\bigl(t-\delta(t)\bigr)\bigr)' \\ &= x_1'(t) - c'(t)x_1\bigl(t-\delta(t)\bigr) - c(t)x_1'\bigl(t-\delta(t)\bigr)\bigl(1-\delta'(t)\bigr) \\ &= x_1'(t) - c'(t)x_1\bigl(t-\delta(t)\bigr) - c(t)x_1'\bigl(t-\delta(t)\bigr) + c(t)x_1'\bigl(t-\delta(t)\bigr)\delta'(t) \\ &= \bigl(Ax_1'\bigr)(t) - c'(t)x_1\bigl(t-\delta(t)\bigr) + c(t)x_1'\bigl(t-\delta(t)\bigr)\delta'(t),\end{aligned}$$

and

$$\bigl(Ax_1'\bigr)(t) = (Ax_1)'(t) + c'(t)x_1\bigl(t-\delta(t)\bigr) - c(t)x_1'\bigl(t-\delta(t)\bigr)\delta'(t).$$

Case (i): If $c_\infty < 1$, by applying Lemma 2.1, we have

$$\begin{aligned}\int_0^\omega \bigl|x_1'(t)\bigr|\,dt &= \int_0^\omega \bigl|\bigl(A^{-1}Ax_1'\bigr)(t)\bigr|\,dt \\ &\le \frac{\int_0^\omega |(Ax_1')(t)|\,dt}{1-c_\infty} \\ &= \frac{\int_0^\omega |(Ax_1)'(t) + c'(t)x_1(t-\delta(t)) - c(t)x_1'(t-\delta(t))\delta'(t)|\,dt}{1-c_\infty} \\ &\le \frac{\int_0^\omega |(Ax_1')(t)|\,dt + \int_0^\omega |c'(t)x_1(t-\delta(t))|\,dt + \int_0^\omega |c(t)x_1'(t-\delta(t))\delta'(t)|\,dt}{1-c_\infty} \\ &\le \frac{\int_0^\omega |(Ax_1')(t)|\,dt + c_1\omega\|x_1\| + c_\infty\delta_1\int_0^\omega |x_1'(t)|\,dt}{1-c_\infty} \\ &\le \frac{\int_0^\omega |(Ax_1')(t)|\,dt + c_1\omega D + (\frac{1}{2}c_1\omega + c_\infty\delta_1)\int_0^\omega |x_1'(t)|\,dt}{1-c_\infty},\end{aligned}$$

where $c_1 = \max_{t\in[0,\omega]} |c'(t)|$, $\delta_1 = \max_{t\in[0,\omega]} |\delta'(t)|$. Since $1 - c_\infty - \frac{1}{2}c_1\omega - c_\infty\delta_1 > 0$, so we get

$$\int_0^\omega \left|x_1'(t)\right| dt \le \frac{\int_0^\omega \left|(Ax_1)'(t)\right| dt + c_1\omega D}{1 - c_\infty - \frac{1}{2}c_1\omega - c_\infty\delta_1} \le \frac{\omega^{\frac{1}{2}}(\int_0^\omega \left|(Ax_1)'(t)\right|^2 dt)^{\frac{1}{2}} + c_1\omega D}{1 - c_\infty - \frac{1}{2}c_1\omega - c_\infty\delta_1}. \tag{3.12}$$

Applying the inequality $(a+b)^k \le a^k + b^k$ for $a, b > 0$, $0 < k < 1$, it follows from (3.11) and (3.12) that

$$\begin{aligned}\int_0^\omega \left|x_1'(t)\right| dt \le{}& \frac{\omega^{\frac{1}{2}}}{1 - c_\infty - \frac{1}{2}c_1\omega - c_\infty\delta_1}\left[(1+c_\infty)^{\frac{1}{2}}\left(\frac{N_2}{2}\right)^{\frac{1}{2}}\left(\int_0^\omega \left|x_1'(t)\right| dt\right)^{\frac{1}{2}}\right.\\ &\left. + (1+c_\infty)^{\frac{1}{2}}(N_2 D)^{\frac{1}{2}} + c_1\omega^{\frac{1}{2}} D\right].\end{aligned}$$

It is easy to see that there exists a constant $M_1 > 0$ (independent of λ) such that

$$\int_0^\omega \left|x_1'(t)\right| dt \le M_1.$$

It follows from (3.8) that

$$\|x_1\| \le D + \frac{1}{2}\int_0^\omega \left|x_1'(t)\right| dt \le D + \frac{1}{2}M_1 := M_2.$$

Case (ii): If $c_0 > 1$,we have

$$\begin{aligned}\int_0^\omega \left|x_1'(t)\right| dt &= \int_0^\omega \left|\left(A^{-1}Ax_1'\right)(t)\right| dt\\ &\le \frac{\int_0^\omega |(Ax_1')(t)| dt}{c_0 - 1}\\ &= \frac{\int_0^\omega |(Ax_1)'(t) + c'(t)x_1(t-\delta(t)) - c(t)x_1'(t-\delta(t))\delta'(t)| dt}{c_0 - 1}\\ &\le \frac{\int_0^\omega |(Ax_1')(t)| dt + \int_0^\omega |c'(t)x_1(t-\delta(t))| dt + \int_0^\omega |c(t)x_1'(t-\delta(t))\delta'(t)| dt}{c_0 - 1}\\ &\le \frac{\int_0^\omega |(Ax_1')(t)| dt + c_1\omega D + (\frac{1}{2}c_1\omega + c_\infty\delta_1)\int_0^\omega |x_1'(t)| dt}{c_0 - 1}.\end{aligned}$$

Since $c_0 - 1 - \frac{1}{2}c_1\omega - c_\infty\delta_1 > 0$, so we get

$$\int_0^\omega \left|x_1'(t)\right| dt \le \frac{\int_0^\omega |(Ax_1)'(t)| dt + c_1\omega D}{c_0 - 1 - \frac{1}{2}c_1\omega - c_\infty\delta_1} \le \frac{\omega^{\frac{1}{2}}(\int_0^\omega |(Ax_1)'(t)|^2 dt)^{\frac{1}{2}} + c_1\omega D}{c_0 - 1 - \frac{1}{2}c_1\omega - c_\infty\delta_1}. \tag{3.13}$$

Similarly, we can get $\|x_1\| \le M_2$.

By the first equation of (3.4) we have $\int_0^\omega x_2(t)\, dt = \int_0^\omega (Ax_1)'(t)\, dt = 0$, which implies that there is a constant $t_1 \in [0, \omega]$ such that $x_2(t_1) = 0$, hence $\|x_2\| \le \int_0^\omega |x_2'(t)| dt$. By the second equation of (3.4) we obtain

$$x_2'(t) = -\lambda f\left(t, x_1'(t)\right) - \lambda g\left(x_1\left(t - \tau(t)\right)\right) + \lambda e(t).$$

So, from (H_1) and (3.10), we have

$$\begin{aligned}\|x_2\| &\le \int_0^\omega \bigl|f\bigl(t,x_1'(t)\bigr)\bigr|\,dt + \int_0^\omega \bigl|g\bigl(t,x_1\bigl(t-\tau(t)\bigr)\bigr)\bigr|\,dt + \int_0^\omega \bigl|e(t)\bigr|\,dt \\ &\le 2K_1\omega + 2\omega N_1 + \omega\|e\| := M_3.\end{aligned}$$

Let $M_4 = \sqrt{M_2^2 + M_3^2} + 1$, $\Omega = \{x = (x_1,x_2)^\top : \|x_1\| < M_4, \|x_2\| < M_4\}$, then $\forall x \in \partial\Omega \cap \operatorname{Ker} L$

$$Q_1Nx = \frac{1}{\omega}\int_0^\omega \begin{pmatrix} x_2(t) \\ -f(t,x_1'(t)) - g(t,x_1(t-\tau(t))) + e(t)\end{pmatrix} dt.$$

If $Q_1Nx = 0$, then $x_2(t) = 0$, $x_1 = M_4$ or $-M_4$. But if $x_1(t) = M_4$, we know

$$0 = \int_0^\omega g(t,M_4)\,dt,$$

there exists a point t_2 such that $g(t_2,M_4) = 0$. From assumption (H_2), we know $M_4 \le D$, which yields a contradiction. Similarly if $x_1 = -M_4$. We also have $Q_1Nx \ne 0$, *i.e.*, $\forall x \in \partial\Omega \cap \operatorname{Ker} L$, $x \notin \operatorname{Im} L$, so conditions (1) and (2) of Lemma 3.1 are both satisfied. Define the isomorphism $J : \operatorname{Im} Q_1 \to \operatorname{Ker} L$ as follows:

$$J(x_1,x_2)^\top = (-x_2,x_1)^\top.$$

Let $H(\mu,x) = \mu x + (1-\mu)JQ_1Nx$, $(\mu,x) \in [0,1] \times \Omega$, then $\forall(\mu,x) \in (0,1) \times (\partial\Omega \cap \operatorname{Ker} L)$,

$$H(\mu,x) = \begin{pmatrix} \mu x_1(t) + \frac{1-\mu}{\omega}\int_0^\omega [f(t,x_1'(t)) + g(t,x_1(t-\tau(t))) - e(t)]\,dt \\ (\mu + (1-\mu))x_2(t)\end{pmatrix}.$$

We have $\int_0^\omega e(t)\,dt = 0$. So, we can get

$$H(\mu,x) = \begin{pmatrix} \mu x_1(t) + \frac{1-\mu}{\omega}\int_0^\omega [f(t,x_1'(t)) + g(t,x_1(t-\tau(t)))]\,dt \\ (\mu + (1-\mu))x_2(t)\end{pmatrix}, \quad \forall(\mu,x) \in (0,1) \times (\partial\Omega \cap \operatorname{Ker} L).$$

From (H_2), it is obvious that $x^\top H(\mu,x) > 0$, $\forall(\mu,x) \in (0,1) \times (\partial\Omega \cap \operatorname{Ker} L)$. Hence

$$\begin{aligned}\deg\{JQ_1N, \Omega \cap \operatorname{Ker} L, 0\} &= \deg\bigl\{H(0,x), \Omega \cap \operatorname{Ker} L, 0\bigr\} \\ &= \deg\bigl\{H(1,x), \Omega \cap \operatorname{Ker} L, 0\bigr\} \\ &= \deg\{I, \Omega \cap \operatorname{Ker} L, 0\} \ne 0.\end{aligned}$$

So condition (3) of Lemma 3.1 is satisfied. By applying Lemma 3.1, we conclude that equation $Lx = Nx$ has a solution $x = (x_1,x_2)^\top$ on $\bar\Omega \cap D(L)$, *i.e.*, (1.1) has an ω-periodic solution $x_1(t)$. □

By using a similar argument, we can obtain the following theorem.

Theorem 3.2 *Assume that conditions* (H_1), (H_2), (H_4) *hold. Suppose one of the following conditions is satisfied*:

(i) *If* $c_\infty < 1$ *and* $1 - c_\infty - \delta_1 c_\infty - \frac{1}{2}c_1\omega > 0$;

(ii) *If* $c_0 > 1$ *and* $c_0 - 1 - \delta_1 c_\infty - \frac{1}{2}c_1\omega > 0$.

Then (1.1) *has at least one solution with period* ω.

Remark 3.1 If $\int_0^\omega e(t)\,dt \neq 0$ and $f(t,0) \neq 0$, the problem of existence of ω-periodic solutions to (1.1) can be converted to the existence of ω-periodic solutions to the equation

$$\bigl(x(t) - c(t)x\bigl(t-\delta(t)\bigr)\bigr)'' + f_1\bigl(t,x'(t)\bigr) + g_1\bigl(t,x\bigl(t-\tau(t)\bigr)\bigr) = e_1(t), \tag{3.14}$$

where $f_1(t,x) = f(t,x) - f(t,0)$, $g_1(t,x) = g(t,x) + \frac{1}{\omega}\int_0^\omega e(t)\,dt + f(t,0)$ and $e_1(t) = e(t) - \frac{1}{\omega}\int_0^\omega e(t)\,dt$. Clearly, $\int_0^\omega e_1(t)\,dt = 0$ and $f_1(t,0) = 0$, (3.14) can be discussed by using Theorem 3.1 (or Theorem 3.2).

Example 3.1 Consider the following equation:

$$\begin{aligned}&\left(x(t) - \frac{1}{150}\sin(16t)x\left(t - \frac{1}{160}\sin 16t\right)\right)'' + \cos 16t \sin x'(t) \\ &\quad + \arctan\left(\frac{x(t-\sin 16t)}{1+\cos^2(8t)}\right) = \cos 16t.\end{aligned} \tag{3.15}$$

Comparing (3.15) to (1.1), we have $\omega = \frac{\pi}{8}$, $f(t,u) = \cos 16t \sin u$, $g(t,x) = \arctan\frac{x}{1+\cos^2(8t)}$, $c(t) = \frac{1}{150}\sin 16t$, $\delta(t) = \frac{1}{160}\sin 16t$, $\tau(t) = \sin 16t$, $e(t) = \cos 16t$ and $\delta_1 = \max_{t\in[0,\frac{\pi}{8}]}|\frac{1}{10}\times \cos 16t| = \frac{1}{10}$, $c_\infty = \max_{t\in[0,\frac{\pi}{8}]}|\frac{1}{150}\sin 16t| = \frac{1}{150} < 1$, $c_1 = \max_{t\in[0,\frac{\pi}{8}]}|\frac{16}{150}\cos 16t| = \frac{8}{75}$. We can easily choose $K_1 = 1$, $D > \frac{\pi}{8}$ and $M = \frac{\pi}{8}$ such that (H_1)-(H_3) hold. And

$$1 - c_\infty - \delta_1 c_\infty - \frac{1}{2}c_1\omega = 1 - \frac{1}{150} - \frac{1}{10}\times\frac{1}{150} - \frac{1}{2}\times\frac{8}{75}\times\frac{\pi}{8} > 0.$$

Hence, by Theorem 3.1, (3.15) has at least one $\frac{\pi}{8}$-periodic solution.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

YX and SZ worked together in the derivation of the mathematical results. Both authors read and approved the final manuscript.

Acknowledgements

YX and SZ would like to thank the referee for invaluable comments and insightful suggestions. This work was supported by NSFC project (No. 11326124) and the Fundamental Research Funds for the Universities of Henan Province (NSFRF140142).

Received: 13 November 2014 Accepted: 29 May 2015 Published online: 09 July 2015

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