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Nonlinear fractional differential equations with nonlocal integral boundary conditions

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Abstract

This paper concerns the boundary value problem of a class of fractional differential equations involving the Riemann-Liouville fractional derivative with nonlocal integral boundary conditions. By using the properties of the Green's function and the monotone iteration technique, one shows the existence of positive solutions and constructs two successively iterative sequences to approximate the solutions, especially numerically simulates the conclusion by an example.

Keywords: fractional differential equation; integral boundary condition; iterative sequence

1 Introduction

In this paper, we investigate a class of nonlinear fractional differential equations with nonlocal integral boundary value conditions of the form

$$\begin{cases} D_{0+}^{\alpha}u(t) + f(t,u(t)) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = u''(0) = 0, \\ u(1) = \lambda I_{0+}^{\beta}u(\eta) = \lambda \int_0^{\eta} \frac{(\eta-s)^{\beta-1}u(s)}{\Gamma(\beta)}\,ds, \end{cases} \tag{1.1}$$

where $3 < \alpha \leq 4$, $0 < \eta \leq 1$, $\lambda, \beta > 0$, $0 \leq \frac{\lambda\Gamma(\alpha)\eta^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} < 1$, and D_{0+}^{α} is the standard Riemann-Liouville differential operator.

It is well known that fractional order models are more realistic and practical than the classical integer order models (see, *e.g.*, [1–7]). As a result, many mathematicians show strong interest in fractional differential equations and many wonderful results have been obtained. The techniques of nonlinear analysis, as the main method to deal with the problems of nonlinear fractional differential equations, plays an essential role in the research of this field, such as establishing the existence and the uniqueness or the multiplicity of solutions to nonlinear fractional differential equations (see, *e.g.*, [8–16] and the references therein). Among these techniques, the monotone iteration scheme is an interesting and effective way to investigate the existence of solutions to nonlinear fractional problems (see, *e.g.*, [17–19]).

Ahmad and Nieto [20] studied the existence and the uniqueness of solutions to the following nonlinear fractional integro-differential equation:

$$\begin{cases} D_{0+}^{\alpha}u(t) = f(t,u(t),(\phi u)(t),(\psi u)(t)), \quad t \in [0,T], \\ D_{0+}^{\alpha-2}u(0^+) = 0, \qquad D_{0+}^{\alpha-1}u(0^+) = \nu I_{0+}^{\alpha-1}u(\eta), \end{cases}$$

where $1 < \alpha \leq 2$, $0 < \eta < T$, ν is a constant, $f : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous, and

$$(\phi x)(t) = \int_0^t \gamma(t,s)x(s)\,\mathrm{d}s, \qquad (\psi x)(t) = \int_0^t \delta(t,s)x(s)\,\mathrm{d}s$$

with γ and δ being continuous functions on $[0, T] \times [0, T]$. In [21], Ahmad and Agarwal considered the existence and the uniqueness of solutions to a class of Caputo type fractional differential equation of order $q \in (n-1, n]$ with slit-strips type boundary conditions

$$\begin{cases} {}^{\mathrm{c}}D^q u(t) = f(t, u(t)), \quad 0 < t < 1, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \\ u(\eta) = a\int_0^{\xi} u(s)\,\mathrm{d}s + b\int_{\zeta}^1 u(s)\,\mathrm{d}s, \end{cases}$$

where $0 < \xi < \eta < \zeta < 1$, a and b are positive constants. In [22], the authors considered a nonlinear fractional boundary value problem on a half-line given by

$$\begin{cases} D_{0+}^{\alpha} u(t) + f(t, u(t), D_{0+}^{\alpha-1}u(t)) = 0, \quad t > 0, \\ u(0) = 0, \qquad D_{0+}^{\alpha-1}u(\infty) = \beta u(\xi), \end{cases}$$

where $1 < \alpha \leq 2$, $\xi > 0$. The positive extremal solutions and iterative schemes for approximating them were obtained by applying a monotone iterative method.

Zhang *et al.* [23] studied the existence of positive solutions to the following fractional boundary value problem:

$$\begin{cases} D_{0+}^{\alpha} u(t) + h(t)f(t, u(t)) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = u''(0) = 0, \\ u(1) = \lambda\int_0^{\eta} u(s)\,\mathrm{d}s, \end{cases} \tag{1.2}$$

where $3 < \alpha \leq 4$, $0 < \eta \leq 1$, $0 \leq \frac{\lambda\eta^{\alpha}}{\alpha} < 1$. They got some results as regards the existence of positive solutions by using the properties of the Green's function, the boundedness of u_0, and the fixed point index theory. Jiang *et al.* [24] studied the fractional boundary value problem (1.2); $h(t)f(u(t))$ and $\lambda\int_0^{\eta} u(s)\,\mathrm{d}s$ were replaced by $f(t, u(t))$ and $\int_0^{\eta} u(s)\,\mathrm{d}s$, respectively. The authors obtained the existence of positive solutions to the problem (1.2) by using the monotone iterative method.

Motivated by the works mentioned above, in this article we study the differential equations (1.1) by using the fixed point theorem for increasing operators on the order intervals. We not only obtain the existence of positive solutions, but we also establish two iterative sequences to approximate the solutions. It should be pointed out that our method is different from that in [25]. The first term of the iterative sequence may be taken as a constant function or a simple function.

This paper is arranged as follows. Some lemmas needed below are listed in Section 2. The existence of the positive solutions to the problem (1.1) is proved and two successively iterative sequences to approximate the solutions are constructed in Section 3. Finally, in Section 4, an example is given to numerically simulate our conclusion.

2 Some lemmas

Lemma 2.1 *Assume that* $y(t) \in C([0,1])$, *then the solution to boundary value problem*

$$\begin{cases} D_{0+}^{\alpha} u(t) + y(t) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = u''(0) = 0, \\ u(1) = \lambda I_{0+}^{\beta} u(\eta) = \lambda \int_0^{\eta} \frac{(\eta - s)^{\beta-1} u(s)}{\Gamma(\beta)} \,\mathrm{d}s, \end{cases} \tag{2.1}$$

can be given by

$$u(t) = \int_0^1 G(t,s) y(s) \,\mathrm{d}s,$$

where

$$G(t,s) = \begin{cases} \frac{-P\Gamma(\alpha+\beta)(t-s)^{\alpha-1} + \Gamma(\alpha+\beta)(1-s)^{\alpha-1} t^{\alpha-1} - \Gamma(\alpha)\lambda(\eta-s)^{\alpha+\beta-1} t^{\alpha-1}}{P\Gamma(\alpha)\Gamma(\alpha+\beta)}, & 0 \le s \le t \le 1, s \le \eta, \\ \frac{\Gamma(\alpha+\beta)(1-s)^{\alpha-1} t^{\alpha-1} - \Gamma(\alpha)\lambda(\eta-s)^{\alpha+\beta-1} t^{\alpha-1}}{P\Gamma(\alpha)\Gamma(\alpha+\beta)}, & 0 \le t \le s \le \eta \le 1, \\ \frac{-P\Gamma(\alpha+\beta)(t-s)^{\alpha-1} + \Gamma(\alpha+\beta)(1-s)^{\alpha-1} t^{\alpha-1}}{P\Gamma(\alpha)\Gamma(\alpha+\beta)}, & 0 \le \eta \le s \le t \le 1, \\ \frac{\Gamma(\alpha+\beta)(1-s)^{\alpha-1} t^{\alpha-1}}{P\Gamma(\alpha)\Gamma(\alpha+\beta)}, & 0 \le t \le s \le 1, s \ge \eta, \end{cases}$$

with $P = 1 - \frac{\lambda\Gamma(\alpha)}{\Gamma(\alpha+\beta)} \eta^{\alpha+\beta-1}$. $G(t,s)$ *is called the Green's function of boundary value problem* (2.1). *Obviously,* $G(t,s)$ *is a continuous function on* $[0,1] \times [0,1]$.

Proof It is shown in [1, 2] that problem (2.1) is equivalent to the following integral equation:

$$u(t) = -I_{0^+}^{\alpha} y(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + C_3 t^{\alpha-3} + C_4 t^{\alpha-4}.$$

By $u(0) = u'(0) = u''(0) = 0$, we obtain

$$u(t) = -I_{0^+}^{\alpha} y(t) + C_1 t^{\alpha-1}.$$

It follows from $u(1) = \lambda I_{0+}^{\beta} u(\eta)$, combined with

$$u(1) = -I_{0^+}^{\alpha} y(1) + C_1$$

and

$$\lambda I_{0+}^{\beta} u(\eta) = -\lambda I_{0^+}^{\alpha+\beta} y(\eta) + \lambda C_1 \frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta)} \eta^{\alpha+\beta-1},$$

that

$$C_1 = \frac{1}{1 - \frac{\lambda\Gamma(\alpha)}{\Gamma(\alpha+\beta)} \eta^{\alpha+\beta-1}} \left\{ I_{0^+}^{\alpha} y(1) - \lambda I_{0^+}^{\alpha+\beta} y(\eta) \right\} =: \frac{1}{P} \left\{ I_{0^+}^{\alpha} y(1) - \lambda I_{0^+}^{\alpha+\beta} y(\eta) \right\}.$$

Therefore, the solution to problem (2.1) is

$$\begin{aligned} u(t) = & -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) \,\mathrm{d}s + \frac{t^{\alpha-1}}{P\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} y(s) \,\mathrm{d}s \\ & - \frac{\lambda t^{\alpha-1}}{P\Gamma(\alpha+\beta)} \int_0^{\eta} (\eta - s)^{\alpha+\beta-1} y(s) \,\mathrm{d}s. \end{aligned}$$

For $t \leq \eta$, one has

$$\begin{aligned}
u(t) &= -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)\,\mathrm{d}s + \frac{t^{\alpha-1}}{P\Gamma(\alpha)}\left\{\int_0^t + \int_t^\eta + \int_\eta^1\right\}(1-s)^{\alpha-1}y(s)\,\mathrm{d}s \\
&\quad - \frac{\lambda t^{\alpha-1}}{P\Gamma(\alpha+\beta)}\left\{\int_0^t + \int_t^\eta\right\}(\eta-s)^{\alpha+\beta-1}y(s)\,\mathrm{d}s \\
&= \int_0^t \frac{-P\Gamma(\alpha+\beta)(t-s)^{\alpha-1} + \Gamma(\alpha+\beta)(1-s)^{\alpha-1}t^{\alpha-1} - \Gamma(\alpha)\lambda(\eta-s)^{\alpha+\beta-1}t^{\alpha-1}}{P\Gamma(\alpha)\Gamma(\alpha+\beta)}y(s)\,\mathrm{d}s \\
&\quad + \int_t^\eta \frac{\Gamma(\alpha+\beta)(1-s)^{\alpha-1}t^{\alpha-1} - \Gamma(\alpha)\lambda(\eta-s)^{\alpha+\beta-1}t^{\alpha-1}}{P\Gamma(\alpha)\Gamma(\alpha+\beta)}y(s)\,\mathrm{d}s \\
&\quad + \int_\eta^1 \frac{\Gamma(\alpha+\beta)(1-s)^{\alpha-1}t^{\alpha-1}}{P\Gamma(\alpha)\Gamma(\alpha+\beta)}y(s)\,\mathrm{d}s \\
&= \int_0^1 G(t,s)y(s)\,\mathrm{d}s.
\end{aligned}$$

For $t \geq \eta$, one has

$$\begin{aligned}
u(t) &= -\frac{1}{\Gamma(\alpha)}\left\{\int_0^\eta + \int_\eta^t\right\}(t-s)^{\alpha-1}y(s)\,\mathrm{d}s + \frac{t^{\alpha-1}}{P\Gamma(\alpha)}\left\{\int_0^\eta + \int_\eta^t + \int_t^1\right\}(1-s)^{\alpha-1}y(s)\,\mathrm{d}s \\
&\quad - \frac{\lambda t^{\alpha-1}}{P\Gamma(\alpha+\beta)}\int_0^\eta (\eta-s)^{\alpha+\beta-1}y(s)\,\mathrm{d}s \\
&= \int_0^\eta \frac{-P\Gamma(\alpha+\beta)(t-s)^{\alpha-1} + \Gamma(\alpha+\beta)(1-s)^{\alpha-1}t^{\alpha-1} - \Gamma(\alpha)\lambda(\eta-s)^{\alpha+\beta-1}t^{\alpha-1}}{P\Gamma(\alpha)\Gamma(\alpha+\beta)}y(s)\,\mathrm{d}s \\
&\quad + \int_\eta^t \frac{-P\Gamma(\alpha+\beta)(t-s)^{\alpha-1} + \Gamma(\alpha+\beta)(1-s)^{\alpha-1}t^{\alpha-1}}{P\Gamma(\alpha)\Gamma(\alpha+\beta)}y(s)\,\mathrm{d}s \\
&\quad + \int_t^1 \frac{\Gamma(\alpha+\beta)(1-s)^{\alpha-1}t^{\alpha-1}}{P\Gamma(\alpha)\Gamma(\alpha+\beta)}y(s)\,\mathrm{d}s \\
&= \int_0^1 G(t,s)y(s)\,\mathrm{d}s.
\end{aligned}$$

The proof is finished. □

A careful analysis of the Green's function allows us to deduce the following results.

Lemma 2.2 *The Green's function $G(t,s)$ has the following properties*:

(1) $G(t,s) > 0$, $\forall t,s \in (0,1)$;
(2) $G(1,s) > 0$, $\forall s \in (0,1)$;
(3) $G(t,s) \leq \frac{(1-s)^{\alpha-1}t^{\alpha-1}}{P\Gamma(\alpha)}$, $\forall t,s \in (0,1)$;
(4) $G(t,s) \geq \frac{\lambda t^{\alpha-1}\eta^{\alpha+\beta-1}}{P\Gamma(\alpha+\beta)}\{(1-s)^{\alpha-1} - (1-s)^{\alpha+\beta-1}\}$, $\forall t,s \in (0,1)$.

Proof Assume at first that $0 \leq s \leq t \leq 1$, $s \leq \eta$, $0 \leq \frac{\lambda\Gamma(\alpha)\eta^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} < 1$, then we have

$$\begin{aligned}
&P\Gamma(\alpha)\Gamma(\alpha+\beta)G(t,s) \\
&\quad = -P\Gamma(\alpha+\beta)(t-s)^{\alpha-1} + \Gamma(\alpha+\beta)(1-s)^{\alpha-1}t^{\alpha-1} - \Gamma(\alpha)\lambda(\eta-s)^{\alpha+\beta-1}t^{\alpha-1}
\end{aligned}$$

$$
\begin{aligned}
&= \lambda\Gamma(\alpha)\eta^{\alpha+\beta-1}(t-s)^{\alpha-1} + \bigl\{-\Gamma(\alpha+\beta)(t-s)^{\alpha-1} + \Gamma(\alpha+\beta)(1-s)^{\alpha-1}t^{\alpha-1}\bigr\} \\
&\quad - \Gamma(\alpha)\lambda(\eta-s)^{\alpha+\beta-1}t^{\alpha-1} \\
&\geq \lambda\Gamma(\alpha)\eta^{\alpha+\beta-1}(t-s)^{\alpha-1} - \lambda\Gamma(\alpha)\eta^{\alpha+\beta-1}(t-s)^{\alpha-1} \\
&\quad + \lambda\Gamma(\alpha)\eta^{\alpha+\beta-1}(1-s)^{\alpha-1}t^{\alpha-1} - \Gamma(\alpha)\lambda(\eta-s)^{\alpha+\beta-1}t^{\alpha-1} \\
&= \lambda\Gamma(\alpha)\eta^{\alpha+\beta-1}(1-s)^{\alpha-1}t^{\alpha-1} - \Gamma(\alpha)\lambda(\eta-s)^{\alpha+\beta-1}t^{\alpha-1} \\
&\geq \lambda\Gamma(\alpha)\eta^{\alpha+\beta-1}t^{\alpha-1}\bigl\{(1-s)^{\alpha-1} - (1-s)^{\alpha+\beta-1}\bigr\}
\end{aligned}
$$

and

$$
\begin{aligned}
&P\Gamma(\alpha)\Gamma(\alpha+\beta)G(t,s) \\
&\quad = -P\Gamma(\alpha+\beta)(t-s)^{\alpha-1} + \Gamma(\alpha+\beta)(1-s)^{\alpha-1}t^{\alpha-1} - \Gamma(\alpha)\lambda(\eta-s)^{\alpha+\beta-1}t^{\alpha-1} \\
&\quad = \lambda\Gamma(\alpha)\eta^{\alpha+\beta-1}(t-s)^{\alpha-1} - \Gamma(\alpha+\beta)(t-s)^{\alpha-1} + \Gamma(\alpha+\beta)(1-s)^{\alpha-1}t^{\alpha-1} \\
&\qquad - \Gamma(\alpha)\lambda(\eta-s)^{\alpha+\beta-1}t^{\alpha-1} \\
&\quad \leq \Gamma(\alpha+\beta)(1-s)^{\alpha-1}t^{\alpha-1} - \Gamma(\alpha)\lambda(\eta-s)^{\alpha+\beta-1}t^{\alpha-1} \\
&\quad \leq \Gamma(\alpha+\beta)(1-s)^{\alpha-1}t^{\alpha-1}.
\end{aligned}
$$

For $0 \leq t \leq s \leq \eta \leq 1$, we have

$$
\begin{aligned}
&P\Gamma(\alpha)\Gamma(\alpha+\beta)G(t,s) \\
&\quad = \Gamma(\alpha+\beta)(1-s)^{\alpha-1}t^{\alpha-1} - \Gamma(\alpha)\lambda(\eta-s)^{\alpha+\beta-1}t^{\alpha-1} \\
&\quad \geq \lambda\Gamma(\alpha)\eta^{\alpha+\beta-1}(1-s)^{\alpha-1}t^{\alpha-1} - \Gamma(\alpha)\lambda(\eta-s)^{\alpha+\beta-1}t^{\alpha-1} \\
&\quad \geq \lambda\Gamma(\alpha)\eta^{\alpha+\beta-1}t^{\alpha-1}\bigl\{(1-s)^{\alpha-1} - (1-s)^{\alpha+\beta-1}\bigr\}
\end{aligned}
$$

and

$$
\begin{aligned}
&P\Gamma(\alpha)\Gamma(\alpha+\beta)G(t,s) \\
&\quad = \Gamma(\alpha+\beta)(1-s)^{\alpha-1}t^{\alpha-1} - \Gamma(\alpha)\lambda(\eta-s)^{\alpha+\beta-1}t^{\alpha-1} \\
&\quad \leq \Gamma(\alpha+\beta)(1-s)^{\alpha-1}t^{\alpha-1}.
\end{aligned}
$$

For $0 \leq \eta \leq s \leq t \leq 1$, we have

$$
\begin{aligned}
&P\Gamma(\alpha)\Gamma(\alpha+\beta)G(t,s) \\
&\quad = -P\Gamma(\alpha+\beta)(t-s)^{\alpha-1} + \Gamma(\alpha+\beta)(1-s)^{\alpha-1}t^{\alpha-1} \\
&\quad = \lambda\Gamma(\alpha)\eta^{\alpha+\beta-1}(t-s)^{\alpha-1} - \Gamma(\alpha+\beta)(t-s)^{\alpha-1} + \Gamma(\alpha+\beta)(1-s)^{\alpha-1}t^{\alpha-1} \\
&\quad \geq \lambda\Gamma(\alpha)\eta^{\alpha+\beta-1}(t-s)^{\alpha-1} - \lambda\Gamma(\alpha)\eta^{\alpha+\beta-1}(t-s)^{\alpha-1} \\
&\qquad + \lambda\Gamma(\alpha)\eta^{\alpha+\beta-1}(1-s)^{\alpha-1}t^{\alpha-1} \\
&\quad \geq \lambda\Gamma(\alpha)\eta^{\alpha+\beta-1}t^{\alpha-1}\bigl\{(1-s)^{\alpha-1} - (1-s)^{\alpha+\beta-1}\bigr\}
\end{aligned}
$$

and

$$\begin{aligned}
&P\Gamma(\alpha)\Gamma(\alpha+\beta)G(t,s)\\
&\quad = -P\Gamma(\alpha+\beta)(t-s)^{\alpha-1} + \Gamma(\alpha+\beta)(1-s)^{\alpha-1}t^{\alpha-1}\\
&\quad = \lambda\Gamma(\alpha)\eta^{\alpha+\beta-1}(t-s)^{\alpha-1} - \Gamma(\alpha+\beta)(t-s)^{\alpha-1} + \Gamma(\alpha+\beta)(1-s)^{\alpha-1}t^{\alpha-1}\\
&\quad \le \Gamma(\alpha+\beta)(1-s)^{\alpha-1}t^{\alpha-1}.
\end{aligned}$$

For $0 \le t \le s \le 1$, $s \ge \eta$, we have

$$\begin{aligned}
&P\Gamma(\alpha)\Gamma(\alpha+\beta)G(t,s)\\
&\quad = \Gamma(\alpha+\beta)(1-s)^{\alpha-1}t^{\alpha-1}\\
&\quad \ge \lambda\Gamma(\alpha)\eta^{\alpha+\beta-1}t^{\alpha-1}\bigl\{(1-s)^{\alpha-1} - (1-s)^{\alpha+\beta-1}\bigr\}.
\end{aligned}$$

From the above, (3), (4) are complete. Clearly, (1) and (2) are true. The proof is completed. □

From Lemma 2.2, we illustrate the following lemma without proof.

Lemma 2.3 *The Green's function $G(t,s)$ satisfies*

$$t^{\alpha-1}w_1(s) \le G(t,s) \le t^{\alpha-1}w_2(s), \quad \forall t,s \in (0,1),$$

where

$$w_1(s) = \frac{\lambda\eta^{\alpha+\beta-1}}{P\Gamma(\alpha+\beta)}\bigl\{(1-s)^{\alpha-1} - (1-s)^{\alpha+\beta-1}\bigr\}, \qquad w_2(s) = \frac{(1-s)^{\alpha-1}}{P\Gamma(\alpha)}.$$

3 Main results

Let Banach space $E = C([0,1])$ be endowed with the norm $\|u\|_\infty = \max_{0\le t\le 1}|u(t)|$. A closed cone $K \subset E$ by $K = \{u \in E : u \succeq 0\}$, where 0 is the zero function, and the cone K is normal.

Set $K_a = \{u \in K : \|u\| \le a\}$. Define the operator $T : K_a \to E$ as

$$(Tu)(t) = \int_0^1 G(t,s)f\bigl(s,u(s)\bigr)\,\mathrm{d}s, \quad t \in [0,1]. \tag{3.1}$$

It is not hard to see that the fixed points of operator T coincide with the solutions to the problem (1.1).

Lemma 3.1 ([26]) *Let X be a Banach space ordered by a normal cone $K \subset X$. Assume that $T : [x_1,x_2] \to X$ is a completely continuous increasing operator such that $x_1 \preceq Tx_1$, $x_2 \succeq Tx_2$. Then T has a minimal fixed point x_* and a maximal fixed point x^* such that $x_1 \preceq x_* \preceq x^* \preceq x_2$. Moreover, $x_* = \lim_{n\to\infty} T^n x_1$, $x^* = \lim_{n\to\infty} T^n x_2$, where $\{T^n x_1\}_{n=1}^\infty$ is an increasing sequence, $\{T^n x_2\}_{n=1}^\infty$ is a decreasing sequence.*

For the forthcoming analysis, we need the following assumptions:

(A_1) $f:[0,1]\times[0,a]\to[0,\infty)$ is continuous and $f(t,0)\not\equiv 0$;
(A_2) there exists a nonnegative function $j\in C[0,1]\subseteq L^1[0,1]$ such that $|f(t,u)|\le j(t)$, $(t,u)\in[0,1]\times[0,a]$;
(A_3) $f(t,\underline{u})\le f(t,\overline{u})$, $t\in[0,1]$, $0\le\underline{u}\le\overline{u}\le a$.

Lemma 3.2 *Assume that* (A_1)-(A_3) *hold, then the operator* T *defined in* (3.1) *is a completely continuous increasing operator.*

Proof Firstly, the operator T is continuous in view of the continuity of functions $f(t,u(t))$ and $G(t,s)$.

Secondly, we will show that $T(K_a)$ is bounded. Let

$$L=\int_0^1 j(t)\,\mathrm{d}t<\infty.$$

Then, for any $u\in K_a$, we have

$$\|Tu(t)\|=\max_{t\in[0,1]}\int_0^1 G(t,s)\big|f\big(s,u(s)\big)\big|\,\mathrm{d}s\le\frac{L}{P\Gamma(\alpha)},\quad t\in[0,1].$$

For each $u\in K_a$, one can show that

$$\begin{aligned}\big|(Tu)'(t)\big| &= \bigg|-\frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2}f\big(s,u(s)\big)\,\mathrm{d}s\\ &\quad+\frac{t^{\alpha-2}}{P\Gamma(\alpha-1)}\int_0^1(1-s)^{\alpha-1}f\big(s,u(s)\big)\,\mathrm{d}s\\ &\quad-\frac{(\alpha-1)\lambda t^{\alpha-2}}{P\Gamma(\alpha+\beta)}\int_0^\eta(\eta-s)^{\alpha+\beta-1}f\big(s,u(s)\big)\,\mathrm{d}s\bigg|\\ &\le\frac{1}{\Gamma(\alpha-1)}\int_0^t(t-s)^{\alpha-2}\big|f\big(s,u(s)\big)\big|\,\mathrm{d}s\\ &\quad+\frac{t^{\alpha-2}}{P\Gamma(\alpha-1)}\int_0^1(1-s)^{\alpha-1}\big|f\big(s,u(s)\big)\big|\,\mathrm{d}s\\ &\quad+\frac{(\alpha-1)\lambda t^{\alpha-2}}{P\Gamma(\alpha+\beta)}\int_0^\eta(\eta-s)^{\alpha+\beta-1}\big|f\big(s,u(s)\big)\big|\,\mathrm{d}s\\ &\le\frac{L}{\Gamma(\alpha-1)}+\frac{L}{P\Gamma(\alpha-1)}+\frac{(\alpha-1)\lambda L}{P\Gamma(\alpha+\beta)}:=\overline{L}.\end{aligned}$$

Therefore, for any $t_1,t_2\in[0,1]$ with $t_1<t_2$, we have

$$\big|(Tu)(t_2)-(Tu)(t_1)\big|\le\int_{t_1}^{t_2}\big|(Tu)'(s)\big|\,\mathrm{d}s\le\overline{L}(t_2-t_1),$$

that is, $T(K_a)$ is equicontinuous.

The Arzela-Ascoli theorem implies that the operator $T:K_a\to E$ is completely continuous.

The assumption (A_3) provides that the operator $T:K_a\to E$ is an increasing operator.

The proof is completed. □

Theorem 3.1 *Assume that* (A_1)-(A_3) *hold, and*

$$\int_0^1 w_1(s)f(s,0)\,\mathrm{d}s \geq 0, \qquad \int_0^1 w_2(s)f\bigl(s, as^{\alpha-1}\bigr)\,\mathrm{d}s \leq a, \quad s \in [0,1],$$

then the problem (1.1) *has two positive solutions* u^*, v^* *satisfying* $0 < u^* \leq v^* \leq a$. *Moreover, there exist a non-decreasing iterative sequence* $\{u_n\}_{n=0}^{\infty}$ *with*

$$\lim_{n\to\infty} u_n = u^*, \qquad u_0 = 0, \qquad u_{n+1} = Tu_n, \quad n = 0,1,2,\ldots,$$

and a non-increasing iterative sequence $\{v_n\}_{n=0}^{\infty}$ *with*

$$\lim_{n\to\infty} v_n = v^*, \qquad v_0 = at^{\alpha-1}, \qquad v_{n+1} = Tv_n, \quad n = 0,1,2,\ldots, \ t \in [0,1].$$

Proof We only need to prove that $Tu_0 \geq u_0$ and $Tv_0 \leq v_0$:

$$\begin{aligned} Tu_0 &= \int_0^1 G(t,s)f(s,u_0)\,\mathrm{d}s = \int_0^1 G(t,s)f(s,0)\,\mathrm{d}s \\ &\geq t^{\alpha-1}\int_0^1 w_1(s)f(s,0)\,\mathrm{d}s \geq 0 = u_0, \quad t \in [0,1], \end{aligned}$$

implies $u_1 \geq u_0$, which combined with (A_3) gives

$$\begin{aligned} u_2 = Tu_1 &= \int_0^1 G(t,s)f(s,u_1)\,\mathrm{d}s \\ &\geq \int_0^1 G(t,s)f(s,u_0)\,\mathrm{d}s = u_1, \quad t \in [0,1]. \end{aligned}$$

Similarly, we have

$$\begin{aligned} v_1 = Tv_0 &= \int_0^1 G(t,s)f(s,v_0)\,\mathrm{d}s \\ &\leq t^{\alpha-1}\int_0^1 w_2(s)f\bigl(s,at^{\alpha-1}\bigr)\,\mathrm{d}s \\ &\leq at^{\alpha-1} = v_0, \quad t \in [0,1]. \end{aligned}$$

By induction, one can prove that $u_{n+1} \geq u_n$ and $v_{n+1} \leq u_n$.

Lemma 3.1 shows that the operator T has a minimal fixed point u^* and a maximal fixed point v^* satisfying $0 \leq u^* \leq v^* \leq a$.

From (A_1), we find that the zero function is not the solution to the problem (1.1). Thus $0 < u^* \leq v^* \leq a$. The proof is finished. □

Remark 3.1 The iterative sequences in Theorem 3.1 starting with a simple function is helpful for calculating.

4 An example

Consider the following boundary value problem:

$$\begin{cases} D_{0+}^{7/2}u(t) + 100t^5 + 0.001 + \frac{\sin t}{400} - u^2(t) + 20u(t) = 0, & 0 < t < 1, \\ u(0) = u'(0) = u''(0) = 0, & u(1) = 5I_{0+}^{3/2}u(\frac{1}{2}), \end{cases} \tag{4.1}$$

where $\alpha = \frac{7}{2}$, $\beta = \frac{3}{2}$, $\eta = \frac{1}{2}$, $\lambda = 5$, $0 \leq \frac{\lambda\Gamma(\alpha)\eta^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \approx 0.0433 < 1$, and

$$f\bigl(t,u(t)\bigr) = 100t^5 + \frac{\sin t}{400} + 0.001 - u^2(t) + 20u(t).$$

We take $a = 10$ for calculating conveniently. Then the assumptions (A_1)-(A_3) hold, and

$$f(t,0) = 0.001 + 100t^5 + \frac{\sin t}{400},$$
$$f\bigl(t,10t^{5/2}\bigr) = \frac{\sin t}{400} + 200t^{5/2} + 0.001.$$

A simple calculation leads to

$$w_1(s) = 0.0136\bigl\{(1-s)^{5/2} - (1-s)^4\bigr\},$$
$$w_2(s) = 0.3146(1-s)^{5/2},$$
$$\int_0^1 w_1(s)f(s,0)\,\mathrm{d}s \approx 0.0033589 \geq 0,$$
$$\int_0^1 w_2(s)f\bigl(s,as^{\alpha-1}\bigr)\,\mathrm{d}s \approx 0.965319 \leq 10.$$

By Theorem 3.1, the problem (4.1) has two nontrivial solutions u^*, v^* with $0 < u^* \leq v^* \leq 10$, and the two monotone iterative sequences $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ can be taken as

$$u_0 = 0, \qquad u_{n+1} = Tu_n, \qquad v_0 = 10t^{\alpha-1}, \qquad v_{n+1} = Tv_n, \quad n = 0,1,2,\ldots.$$

Using MATLAB, the iterative sequences are computed and are depicted in Figure 1 and Figure 2.

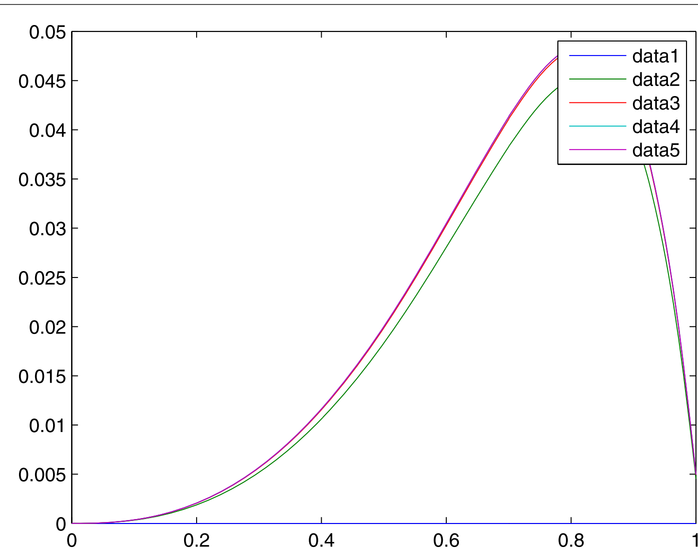

Figure 1 The non-decreasing sequence $\{u_n(t)\}_{n=0}^{4}$ in the interval [0, 1].

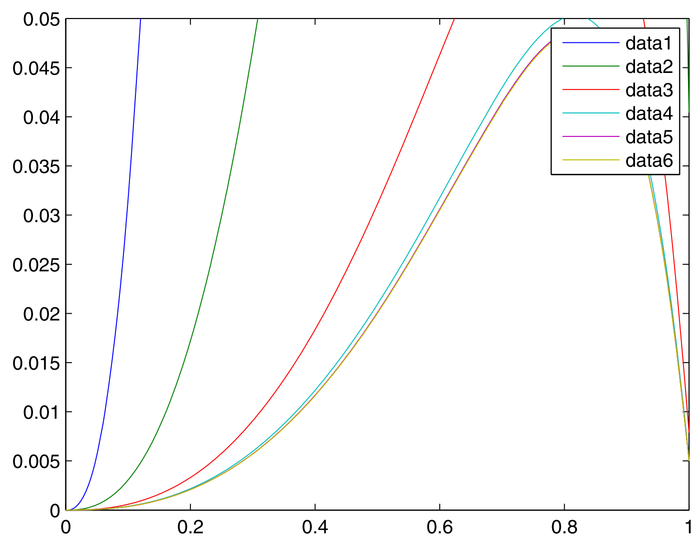

Figure 2 The non-increasing sequence $\{v_n(t)\}_{n=0}^{5}$ in the interval [0, 1].

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript. All authors read and approved the final manuscript.

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