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A note on (h,q)-Boole polynomials

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Abstract

Kim *et al.* (Appl. Math. Inf. Sci. 9(6):1-6, 2015) consider the q-extensions of Boole polynomials. In this paper, we consider Witt-type formula for the q-Boole polynomials with weights and derive some new interesting identities and properties of those polynomials and numbers from the Witt-type formula which are related to special polynomials and numbers.

Keywords: (h,q)-Euler polynomials; (h,q)-Boole numbers and polynomials; p-adic invariant integral on $\mathbb{Z}_p$

1 Introduction

Let p be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will, respectively, denote the ring of p-adic rational integers, the field of p-adic rational numbers, and the completions of algebraic closure of $\mathbb{Q}_p$. The p-adic norm is defined by $|p|_p = \frac{1}{p}$.

When one talks of q-extension, q is variously considered as an indeterminate, a complex $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally assumes that $|q| < 1$. If $q \in \mathbb{C}_p$, then we assume that $|q - 1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for each $x \in \mathbb{Z}_p$. Throughout this paper, we use the notation

$$[x]_{-q} = \frac{1 - (-q)^x}{1 - (-q)}.$$

Note that $\lim_{q \to -1}[x]_{-q} = x$ for each $x \in \mathbb{Z}_p$.

Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the *p-adic invariant integral on $\mathbb{Z}_p$* is defined by Kim as follows:

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x)\, d\mu_{-q}(x) = \lim_{N \to \infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} f(x)(-q)^x \quad \text{(see [1–5]).} \tag{1.1}$$

Let f_1 be the translation of f with $f_1(x) = f(x + 1)$. Then, by (1.1), we get

$$I_{-q}(f_1) + I_{-q}(f) = [2]_q f(0). \tag{1.2}$$

As is well known, the *Stirling number of the first kind* is defined by

$$(x)_n = x(x-1)\cdots(x-n+1) = \sum_{l=0}^{n} S_1(n,l)x^l, \tag{1.3}$$

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and the *Stirling number of the second kind* is given by the generating function:

$$\left(e^t - 1\right)^m = m! \sum_{l=m}^{\infty} S_2(l, m) \frac{t^l}{l!} \quad \text{(see [6, 7])}. \tag{1.4}$$

It is well known that the (h, q)-*Euler polynomials* are defined by the generating function:

$$\left(\frac{q+1}{q^h e^t + 1}\right) e^{xt} = \sum_{n=0}^{\infty} E_{n,q}(x|h) \frac{t^n}{n!} \quad \text{(see [8])}, \tag{1.5}$$

where h is an integer. When $x = 0$ and $h = 0$, $E_{n,q}(0|h) = E_{n,q}(h)$ are called the *ordinary q-Euler numbers*.

Recently, DS Kim and T Kim introduced the *Changhee polynomials of the first kind* are defined by the generating function:

$$\frac{2}{2+t}(1+t)^x = \sum_{n=0}^{\infty} Ch_n(x) \frac{t^n}{n!} \quad \text{(see [1, 9–11])}, \tag{1.6}$$

and T Kim *et al.* defined the *q-Changhee polynomials* as follows:

$$\frac{[2]_q}{q(1+t)+1}(1+t)^x = \sum_{n=0}^{\infty} Ch_{n,q}(x) \frac{t^n}{n!} \quad \text{(see [9, 11, 12])}. \tag{1.7}$$

As is well known, the *Boole polynomials* are defined by the generating function:

$$\sum_{n=0}^{\infty} Bl_n(x|\lambda) \frac{t^n}{n!} = \frac{1}{1 + (1+t)^\lambda} \quad \text{(see [7, 13])}.$$

When $\lambda = 1$, $2Bl_n(x|1) = Ch_n(x)$ are Changhee polynomials. In [11], Kim *et al.* consider the q-analog of Boole polynomials, and found some new and interesting identities related to special polynomials, and Y Do and D Lim investigated the properties of (h, q)-Daehee numbers and polynomials, which are defined by

$$\int_{\mathbb{Z}_p} q^{-hy}(x+y)_n \, d\mu_q(y) \quad \text{(see [14])}.$$

In this paper, we consider Witt-type formula for the q-Boole polynomials with weights and derive some new interesting identities and properties of those polynomials and numbers from the Witt-type formula which are related to special polynomials and numbers.

2 *q*-Analog of Boole polynomials with weight

In this section, we assume that $t \in \mathbb{C}_p$ with $|t|_p < p^{-\frac{1}{p-1}}$, $\lambda \in \mathbb{Z}_p$ with $\lambda \neq 0$ and $h \in \mathbb{Z}$. From (1.2), we have

$$\int_{\mathbb{Z}_p} q^{(h-1)y}(1+t)^{x+\lambda y} \, d\mu_{-q}(y) = \frac{1+q}{q^h(1+t)^\lambda + 1}(1+t)^x = \sum_{n=0}^{\infty} [2]_q Bl_{n,q}(x|h, \lambda) \frac{t^n}{n!}, \tag{2.1}$$

where $Bl_{n,q}(x|h,\lambda)$ are the (h,q)-Boole polynomials which are defined by

$$\frac{1}{q^h(1+t)^\lambda+1}(1+t)^x = \sum_{n=0}^\infty Bl_{n,q}(x|h,\lambda)\frac{t^n}{n!}. \qquad (2.2)$$

By (2.1), we can derive the following equation:

$$\int_{\mathbb{Z}_p} q^{(h-1)y}\binom{x+\lambda y}{n}\,d\mu_{-q} = \frac{1+q}{n!}Bl_{n,q}(x|h,\lambda). \qquad (2.3)$$

In the special case $x=0$, $Bl_{n,q}(0|h,\lambda) = Bl_{n,q}(h,\lambda)$ are called the (h,q)-*Boole numbers*.

Note that

$$(1+t)^{x+\lambda y} = e^{(x+\lambda y)\log(1+t)}$$

$$= \sum_{n=0}^\infty \frac{(x+\lambda y)^n}{n!}\big(\log(1+t)\big)^n$$

$$= \sum_{n=0}^\infty \frac{(x+\lambda y)^n}{n!}m!\sum_{m=n}^\infty S_1(m,n)\frac{t^m}{m!}$$

$$= \sum_{n=0}^\infty \left\{\sum_{m=0}^n (x+\lambda y)^m S_1(n,m)\right\}\frac{t^n}{n!}. \qquad (2.4)$$

The (h,q)-*Euler polynomials* are defined by the generating function:

$$\frac{1+q}{q^h e^t+1}e^{xt} = \sum_{n=0}^\infty E_{n,q}(x|h)\frac{t^n}{n!}. \qquad (2.5)$$

Note that $\lim_{q\to 1}E_{n,q}(x|1) = E_n(x)$. When $x=0$, $E_n(0|h) = E_{n,q}(h)$ are called the (h,q)-*Euler numbers*.

By (1.2), we can derive easily the following equation:

$$\int_{\mathbb{Z}_p} q^{(h-1)y}e^{(x+y)t}\,d\mu_{-q}(y) = \frac{1+q}{q^h e^t+1}e^{xt} = \sum_{n=0}^\infty E_{n,q}(x|h)\frac{t^n}{n!}. \qquad (2.6)$$

Since

$$\int_{\mathbb{Z}_p} q^{(h-1)y}e^{(x+y)t}\,d\mu_{-q}(y) = \sum_{n=0}^\infty \int_{\mathbb{Z}_p} q^{(h-1)y}(x+y)^n\,d\mu_{-q}(y)\frac{t^n}{n!},$$

by (2.5), we have

$$\int_{\mathbb{Z}_p} q^{(h-1)y}(x+y)^n\,d\mu_{-q}(y) = E_{n,q}(x|h) \quad (n\geq 0). \qquad (2.7)$$

From (2.4) and (2.7), we get

$$\int_{\mathbb{Z}_p} q^{(h-1)y}(1+t)^{x+\lambda y}\, d\mu_{-q}(y)$$

$$= \sum_{n=0}^{\infty}\left\{\sum_{m=0}^{n}\int_{\mathbb{Z}_p} q^{(h-1)y}(x+\lambda y)^m\, d\mu_{-q}(y)S_1(n,m)\right\}\frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty}\left\{\sum_{m=0}^{n}\lambda^m E_{m,q}\left(\frac{x}{\lambda}\Big|h\right)S_1(n,m)\right\}\frac{t^n}{n!}. \tag{2.8}$$

Thus, by (2.2), (2.3), and (2.8), we obtain the following theorem.

Theorem 2.1 *For $n \geq 0$, we have*

$$Bl_{n,q}(x|h,\lambda) = \frac{1}{[2]_q}\sum_{m=0}^{n}\lambda^m E_{m,q}\left(\frac{x}{\lambda}\Big|h\right)S_1(n,m)$$

and

$$\int_{\mathbb{Z}_p} q^{(h-1)y}\binom{x+\lambda y}{n}\, d\mu_{-q} = \frac{[2]_q}{n!}Bl_{n,q}(x|h,\lambda).$$

By Theorem 2.1, we note that

$$Bl_{n,q}(x|h,\lambda) = \frac{1}{[2]_q}\int_{\mathbb{Z}_p} q^{(h-1)y}(x+\lambda y)_n\, d\mu_{-q}(y),$$

where $(x)_n = x(x-1)\cdots(x-n+1)$. When $\lambda = 1$ and $h = 0$, we have

$$Bl_{n,q}(x|0,1) = \frac{1}{[2]_q}\int_{\mathbb{Z}_p} q^{-1}(x+y)^n\, d\mu_{-q}(y). \tag{2.9}$$

In [13], Arici *et al.* defined the *q-analog of Changhee polynomials* by the generating function:

$$\sum_{n=0}^{\infty} Ch_n(x|q)\frac{t^n}{n!} = \frac{[2]_q}{[2]_t + 1}(1+t)^x. \tag{2.10}$$

By (2.10), we have

$$\int_{\mathbb{Z}_p} q^{-y}(1+t)^{x+y}\, d\mu_{-q}(y) = \frac{[2]_q}{[2]_t + 1}(1+t)^x = \sum_{n=0}^{\infty} Ch_n(x|q)\frac{t^n}{n!}. \tag{2.11}$$

By (1.6) and (2.10), we note that

$$\frac{[2]_q}{2}Ch_n(x) = Ch_n(x|q). \tag{2.12}$$

From (2.11), we get

$$\int_{\mathbb{Z}_p} q^{-1}(x+y)_n\, d\mu_{-q}(y) = Ch_n(x|q). \tag{2.13}$$

By (2.9), (2.12), and (2.13), we have

$$Bl_{n,q}(x|0,1) = \frac{1}{[2]_q} Ch_n(x|q) = \frac{1}{2} Ch_n(x).$$

By replacing t as $e^t - 1$ in (2.1), we derive the following equations:

$$
\begin{aligned}
\frac{1+q}{q^h e^{\lambda t} + 1} e^{xt} &= \sum_{n=0}^{\infty} [2]_q Bl_{n,q}(x|h,\lambda) \frac{1}{n!} \left(e^t - 1\right)^n \\
&= \sum_{n=0}^{\infty} [2]_q Bl_{n,q}(x|h,\lambda) \frac{1}{n!} n! \sum_{m=n}^{\infty} S_2(m,n) \frac{t^m}{m!} \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{n} [2]_q Bl_{m,q}(x|h,\lambda) S_2(n,m) \frac{t^n}{n!}
\end{aligned}
\tag{2.14}
$$

and

$$\frac{1+q}{q^h e^{\lambda t} + 1} e^{xt} = \frac{1+q}{q^h e^{\lambda t} + 1} e^{(\frac{x}{\lambda})\lambda t} = \sum_{n=0}^{\infty} E_{n,q}\left(\frac{x}{\lambda}\Big|h\right) \lambda^m \frac{t^m}{m!}. \tag{2.15}$$

Hence, by (2.14) and (2.15), we obtain the following theorem.

Theorem 2.2 *For $n \geq 0$, we have*

$$\sum_{m=0}^{n} Bl_{m,q}(x|h,\lambda) S_2(n,m) = \frac{\lambda^m}{q+1} E_{n,q}\left(\frac{x}{\lambda}\Big|h\right).$$

From now on, we define the $(h_1,\ldots,h_r,q)$-*Boole numbers of the first kind* as follows:

$$
\begin{aligned}
&[2]_q^r Bl_{n,q}^{(h_1,\ldots,h_r)}(\lambda) \\
&= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{h_1+\cdots+h_r-r} \left(\lambda(x_1+\cdots+x_r)\right)_n d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r) \quad (n \geq 0).
\end{aligned}
\tag{2.16}
$$

By (2.16), we have

$$
\begin{aligned}
&[2]_q^r \sum_{n=0}^{\infty} Bl_{n,q}^{(h_1,\ldots,h_r)}(\lambda) \frac{t^n}{n!} \\
&= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \sum_{n=0}^{\infty} q^{h_1+\cdots+h_r-r} \binom{\lambda(x_1+\cdots+x_r)}{n} t^n \, d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r) \\
&= \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{h_1+\cdots+h_r-r} (1+t)^{\lambda(x_1+\cdots+x_k)} \, d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r) \\
&= \prod_{i=1}^{r} \left(\frac{1+q}{q^{h_i}(1+t)^{\lambda}+1}\right) \\
&= (1+q)^r \sum_{n=0}^{\infty} \left(\sum_{l_1+\cdots+l_r=n} \binom{n}{l_1,\ldots,l_r} B_{i_1,q}(h,\lambda)\cdots B_{i_r,q}(h,\lambda)\right) \frac{t^n}{n!}.
\end{aligned}
\tag{2.17}
$$

Thus, by (2.17), we obtain the following corollary.

Corollary 2.3 *For $n \geq 0$, we have*

$$Bl_{n,q}^{(h_1,\ldots,h_r)}(\lambda) = \sum_{l_1+\cdots+l_r=n} \binom{n}{l_1,\ldots,l_r} B_{i_1,q}(h,\lambda)\cdots B_{i_r,q}(h,\lambda).$$

The $(h_1,\ldots,h_r,q)$-*Euler polynomials* are defined by the generating function to be

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{h_1+\cdots+h_r-r}e^{(x_1+\cdots+x_r+x)t}\,d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r)$$

$$= \prod_{i=1}^{r}\left(\frac{1+q}{q^{h_i}e^t+1}\right)e^{xt}$$

$$= \sum_{n=0}^{\infty} E_{n,q}(x|h_1,\ldots,h_r)\frac{t^n}{n!}. \tag{2.18}$$

By (2.18), we have

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{h_1+\cdots+h_r-r}(x_1+\cdots+x_r+x)^n\,d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r) = E_{n,q}(x|h_1,\ldots,h_r).$$

In the special case $x = 0$, $E_{n,q}(0|h_1,\ldots,h_r) = E_{n,q}(h_1,\ldots,h_r)$ are called the $(h_1,\ldots,h_r,q)$-*Euler numbers*.

From (1.5) and (2.16), we note that

$$(1+q)^r Bl_{n,q}^{(h_1,\ldots,h_r)}(\lambda)$$

$$= \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{h_1+\cdots+h_r-r}\big(\lambda(x_1+\cdots+x_r)\big)_n\,d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r)$$

$$= \sum_{l=0}^{n} S_1(n,l)\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{h_1+\cdots+h_r-r}\lambda^l(x_1+\cdots+x_r)^l\,d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r)$$

$$= \sum_{l=0}^{n} S_1(n,l)\lambda^l E_{l,q}(h_1,\ldots,h_r). \tag{2.19}$$

Therefore, by (2.19), we obtain the following theorem.

Theorem 2.4 *For $n \geq 0$, we get*

$$Bl_{n,q}^{(h_1,\ldots,h_r)}(\lambda) = \frac{1}{(1+q)^r}\sum_{l=0}^{n} S_1(n,l)\lambda^l E_{l,q}(h_1,\ldots,h_r).$$

By replacing t by $e^t - 1$ in (2.17), we have

$$[2]_q^r \sum_{n=0}^{\infty} Bl_{n,q}^{(h_1,\ldots,h_r)}(\lambda)\frac{(e^t-1)^n}{n!} = \prod_{i=1}^{r}\left(\frac{1+q}{q^{h_i}e^{\lambda t}+1}\right)$$

$$= \sum_{n=0}^{\infty} E_{n,q}(h_1,\ldots,h_r)\lambda^n\frac{t^n}{n!} \tag{2.20}$$

and

$$[2]_q^r \sum_{n=0}^{\infty} Bl_{n,q}^{(h_1,\ldots,h_r)}(\lambda) \frac{1}{n!} (e^t - 1)^n = [2]_q^r \sum_{n=0}^{\infty} Bl_{n,q}^{(h_1,\ldots,h_r)}(\lambda) \sum_{m=n}^{\infty} S_2(m,n) \frac{t^m}{m!}$$

$$= [2]_q^r \sum_{n=0}^{\infty} \left\{ \sum_{m=0}^{n} Bl_{m,q}^{(h_1,\ldots,h_r)}(\lambda) S_2(n,m) \right\} \frac{t^n}{n!}. \qquad (2.21)$$

Hence, by (2.20) and (2.21), we obtain the following theorem.

Theorem 2.5 *For $n \geq 0$, we have*

$$\frac{\lambda^n}{[2]_q^r} E_{n,q}(h_1,\ldots,h_r) = \sum_{m=0}^{n} Bl_{m,q}^{(h_1,\ldots,h_r)}(\lambda) S_2(n,m).$$

Let us define the $(h_1,\ldots,h_r,q)$-*Boole polynomials of the first kind* as follows:

$$[2]_q^r Bl_{n,q}^{(h_1,\ldots,h_r)}(x|\lambda)$$

$$= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{h_1+\cdots+h_r-r} \big(\lambda(x_1+\cdots+x_r)+x\big)_n \, d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_r), \qquad (2.22)$$

where $n \geq 0$ and $r \in \mathbb{N}$. By (2.22), we can derive the generating function of the $(h_1,\ldots,h_r,q)$-Boole polynomials of the first kind as follows:

$$[2]_q^r \sum_{n=0}^{\infty} Bl_{n,q}^{(h_1,\ldots,h_r)}(x|\lambda) \frac{t^n}{n!}$$

$$= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{h_1+\cdots+h_r-r} (1+t)^{\lambda(x_1+\cdots+x_r)+x} \, d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_r)$$

$$= \prod_{i=1}^{r} \left(\frac{1+q}{q^{h_i}(1+t)^\lambda + 1} \right) (1+t)^x. \qquad (2.23)$$

By (2.23), we can see easily

$$\prod_{i=1}^{r} \left(\frac{1+q}{q^{h_i}(1+t)^\lambda + 1} \right) (1+t)^x$$

$$= [2]_q^r \left(\sum_{n=0}^{\infty} Bl_{n,q}^{(h_1,\ldots,h_r)}(\lambda) \frac{t^n}{n!} \right) \left(\sum_{m=0}^{\infty} \binom{x}{m} t^m \right)$$

$$= [2]_q^r \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} m! \binom{x}{m} \frac{n!}{(n-m)!m!} Bl_{n-m,q}^{(h_1,\ldots,h_r)}(\lambda) \right) \frac{t^n}{n!}$$

$$= [2]_q^r \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} m! \binom{x}{m} \binom{n}{m} Bl_{n-m,q}^{(h_1,\ldots,h_r)}(\lambda) \right) \frac{t^n}{n!}$$

$$= [2]_q^r \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \binom{n}{m} Bl_{n-m,q}^{(h_1,\ldots,h_r)}(\lambda)(x)_m \right) \frac{t^n}{n!}. \qquad (2.24)$$

By (2.23) and (2.24), we obtain the following theorem.

Theorem 2.6 *For $n \geq 0$, we have*

$$Bl_{n,q}^{(h_1,\ldots,h_r)}(x|\lambda) = \sum_{m=0}^{n} \binom{n}{m} Bl_{n-m,q}^{(h_1,\ldots,h_r)}(\lambda)(x)_m.$$

Replacing t as $e^t - 1$ in (2.23), we get

$$[2]_q^r \sum_{n=0}^{\infty} Bl_{n,q}^{(h_1,\ldots,h_r)}(x|\lambda)\frac{1}{n!}\left(e^t - 1\right)^n = \prod_{i=1}^{n}\left(\frac{1+q}{q^{h_i}e^{\lambda t} + 1}\right)e^{xt}$$

$$= \sum_{n=0}^{\infty} E_{n,q}^{(h_1,\ldots,h_r)}\left(\frac{x}{\lambda}\right)\lambda^n\frac{t^n}{n!} \tag{2.25}$$

and

$$[2]_q^r \sum_{n=0}^{\infty} Bl_{n,q}^{(h_1,\ldots,h_r)}(x|\lambda)\frac{(e^t - 1)^n}{n!}$$

$$= [2]_q^r \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} Bl_{m,q}^{(h_1,\ldots,h_r)}(x|\lambda)S_2(n,m)\right)\frac{t^n}{n!}. \tag{2.26}$$

Hence, by (2.25) and (2.26), we obtain the following theorem.

Theorem 2.7 *For $n \geq 0$, we have*

$$\sum_{m=0}^{n} Bl_{m,q}^{(h_1,\ldots,h_r)}(x|\lambda)S_2(n,m) = \frac{\lambda^n}{[2]_q^r} E_{n,q}^{(h_1,\ldots,h_r)}\left(\frac{x}{\lambda}\right).$$

From (2.23), we get

$$[2]_q^r Bl_{n,q}^{(h_1,\ldots,h_r)}(x|\lambda)$$

$$= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{h_1+\cdots+h_r-r}\left(\lambda(x_1 + \cdots + x_r) + x\right)_n d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r)$$

$$= \sum_{l=0}^{n} S_1(n,l) \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{h_1+\cdots+h_r-r}\left(\lambda(x_1 + \cdots + x_r) + x\right)^l d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r)$$

$$= \sum_{l=0}^{n} S_1(n,l)\lambda^l E_{n,q}^{(h_1,\ldots,h_r)}\left(\frac{x}{\lambda}\right). \tag{2.27}$$

Thus, by (2.27), we obtain the following theorem.

Theorem 2.8 *For $n \geq 0$, we have*

$$Bl_{n,q}^{(h_1,\ldots,h_r)}(x|\lambda) = \frac{1}{[2]_q^r} \sum_{l=0}^{n} S_1(n,l)\lambda^l E_{n,q}^{(h_1,\ldots,h_r)}\left(\frac{x}{\lambda}\right).$$

Now, we define the (h,q)-*Boole polynomials of the second kind* as follows:

$$\widehat{Bl}_{n,q}(x|h,\lambda) = \frac{1}{[2]_q} \int_{\mathbb{Z}_p} q^{(h-1)y}(-\lambda y + x)_n d\mu_{-q}(y) \quad (n \geq 0). \tag{2.28}$$

By (2.28), we have

$$\widehat{Bl}_{n,q}(x|h,\lambda) = \frac{1}{[2]_q} \sum_{l=0}^{n} (-\lambda)^l S_1(n,l) \int_{\mathbb{Z}_p} \left(y - \frac{x}{\lambda} \right)^l d\mu_{-q}(y)$$

$$= \frac{1}{[2]_q} \sum_{l=0}^{n} (-\lambda)^l S_1(n,l) E_{l,q}\left(-\frac{x}{\lambda} \right). \tag{2.29}$$

In the special case $x = 0$, $\widehat{Bl}_{n,q}(0|h,\lambda) = \widehat{Bl}_{n,q}(h,\lambda)$ are called the (h,q)-*Boole numbers of the second kind*. From (2.29), we can derive the generating function of $\widehat{Bl}_{n,q}(x|\lambda)$ as follows:

$$\sum_{n=0}^{\infty} \widehat{Bl}_{n,q}(x|h,\lambda) \frac{t^n}{n!} = \frac{1}{[2]_q} \int_{\mathbb{Z}_p} q^{(h-1)y} (1+t)^{-\lambda y + x} d\mu_{-q}(y)$$

$$= \frac{(1+t)^\lambda}{q^h + (1+t)^\lambda} (1+t)^x. \tag{2.30}$$

By replacing t by $e^t - 1$ in (2.30), we have

$$\sum_{n=0}^{\infty} \widehat{Bl}_{n,q}(x|h,\lambda) \frac{(e^t - 1)^n}{n!} = \frac{e^{\lambda t}}{q^h + e^{\lambda t}} e^{xt}$$

$$= \frac{1}{1+q} \sum_{n=0}^{\infty} (-\lambda)^n E_{n,q}\left(-\frac{\lambda}{x} \Big| h \right) \frac{t^n}{n!} \tag{2.31}$$

and

$$\sum_{n=0}^{\infty} \widehat{Bl}_{n,q}(x|h,\lambda) \frac{(e^t - 1)^n}{n!} = \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \widehat{Bl}_{m,q}(x|h,\lambda) S_2(n,m) \right) \frac{t^n}{n!}. \tag{2.32}$$

By (2.31) and (2.32), we obtain the following theorem.

Theorem 2.9 *For $n \geq 0$, we have*

$$\widehat{Bl}_{n,q}(x|h,\lambda) = \frac{1}{[2]_q} \sum_{l=0}^{n} (-\lambda)^l S_1(n,l) E_{l,q}\left(-\frac{x}{\lambda} \right)$$

and

$$\frac{1}{[2]_q} (-\lambda)^n E_{n,q}\left(-\frac{\lambda}{x} \Big| h \right) = \sum_{m=0}^{n} \widehat{Bl}_{m,q}(x|h,\lambda) S_2(n,m).$$

For $h_1, \ldots, h_r \in \mathbb{Z}$, we define the $(h_1, \ldots, h_r, q)$-*Boole polynomials of the second kind* as follows:

$$\widehat{Bl}_{n,q}^{(h_1,\ldots,h_r)}(x|\lambda)$$

$$= \frac{1}{q+1} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{(h_1+\cdots+h_r-r)y} \left(-\lambda(x_1+\cdots+x_r) + x \right)_n d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_r). \tag{2.33}$$

By (2.33), we can derive the generating function of the $(h_1, \ldots, h_r, q)$-Boole polynomials of the second kind as follows:

$$
\begin{aligned}
\sum_{n=0}^{\infty} \widehat{Bl}_{n,q}^{(h_1,\ldots,h_r)}(x|\lambda) \frac{t^n}{n!} \\
&= \frac{1}{(1+q)^r} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+t)^{-\lambda x_1 - \cdots - \lambda x_r + x} \, d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_r) \\
&= \prod_{i=1}^{r} \left(\frac{(1+t)^{\lambda}}{q^{h_i} + (1+t)^{\lambda}} \right) (1+t)^x \\
&= \prod_{i=1}^{r} \left(\frac{1}{q^{h_i}(1+t)^{-\lambda} + 1} \right) (1+t)^x \\
&= \sum_{n=0}^{\infty} Bl_{n,q}^{(h_1,\ldots,h_r)}(x|-\lambda) \frac{t^n}{n!}.
\end{aligned}
\tag{2.34}
$$

Hence, by (2.34), we obtain the following proposition.

Proposition 2.10 *For $n \geq 0$, we have*

$$
\widehat{Bl}_{n,q}^{(h_1,\ldots,h_r)}(x|\lambda) = Bl_{n,q}^{(h_1,\ldots,h_r)}(x|-\lambda).
$$

Note that

$$
\begin{aligned}
\frac{(-1)^n [2]_q}{n!} Bl_{n,q}(x|h,\lambda) &= (-1)^n \int_{\mathbb{Z}_p} q^{(h-1)y} \binom{x + \lambda y}{n} \, d\mu_{-q}(y) \\
&= \int_{\mathbb{Z}_p} q^{(h-1)y} \binom{-x - \lambda y + n - 1}{n} \, d\mu_{-q}(y) \\
&= \int_{\mathbb{Z}_p} q^{(h-1)y} \sum_{m=0}^{n} \binom{-x - \lambda y}{m} \binom{n-1}{n-m} \, d\mu_{-q}(y) \\
&= \sum_{m=0}^{n} \binom{n-1}{n-m} \int_{\mathbb{Z}_p} q^{(h-1)y} \binom{-x - \lambda y}{m} \, d\mu_{-q}(y) \\
&= [2]_q \sum_{m=0}^{n} \binom{n-1}{n-m} \frac{\widehat{Bl}_{m,q}(-x|h,\lambda)}{m!},
\end{aligned}
\tag{2.35}
$$

and, by a similar method, we get

$$
\begin{aligned}
\frac{(-1)^n [2]_q}{n!} \widehat{Bl}_{n,q}(x|h,\lambda) &= (-1)^n \int_{\mathbb{Z}_p} q^{(h-1)y} \binom{x - \lambda y}{n} \, d\mu_{-q}(y) \\
&= [2]_q \sum_{m=0}^{n} \binom{n-1}{n-m} \frac{Bl_{m,q}(-x|h,\lambda)}{m!}.
\end{aligned}
\tag{2.36}
$$

By (2.35) and (2.36), we obtain the following theorem.

Theorem 2.11 *For $n \geq 0$, we have*

$$\frac{(-1)^n}{n!} Bl_{n,q}(x|h,\lambda) = \sum_{m=0}^{n} \binom{n-1}{n-m} \frac{\widehat{Bl}_{m,q}(-x|h,\lambda)}{m!}$$

and

$$\frac{(-1)^n}{n!} \widehat{Bl}_{n,q}(x|h,\lambda) = \sum_{m=0}^{n} \binom{n-1}{n-m} \frac{Bl_{m,q}(-x|h,\lambda)}{m!}.$$

By Theorem 2.11, we obtain the following corollary.

Corollary 2.12 *For $n \geq 0$, we have*

$$Bl_{n,q}(x|h,\lambda) = \sum_{m=0}^{n} \sum_{k=0}^{m} (-1)^{n+m} \binom{n}{n-m, m-k, k} (n-1)_{l-1} Bl_{k,q}(x|h,\lambda)$$

where $\binom{n}{p,q,r} = \frac{n!}{p!q!r!}, p+q+r = n.$

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