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The general solution of differential equations with Caputo-Hadamard fractional derivatives and impulsive effect

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Abstract
In this paper, the formula of general solution for nonlinear systems with Caputo-Hadamard fractional derivatives and impulsive effect is found by analysis of the limit case (as impulse tends to zero), and it shows that the deviation caused by impulse for the fractional-order nonlinear systems is undetermined. Next, an example is given to illustrate the result.

MSC: 34A08; 34A37

Keywords: fractional differential equations; Caputo-Hadamard fractional derivative; impulse; general solution

1 Introduction

Fractional calculus was utilized as a powerful tool to reveal the hidden aspects of the dynamics of complex or hypercomplex systems [1–3]. And the subject of fractional differential equations is gaining much attention [4–11].

The Hadamard approach to fractional integral was based on the generalization of the nth integral [12],

$$(\mathcal{J}_{a^+}^{n}f)(x) = \int_a^x \frac{ds_1}{s_1}\int_a^{s_1}\frac{ds_2}{s_2}\cdots\int_a^{s_{n-1}} f(s_n)\frac{ds_n}{s_n} = \frac{1}{(n-1)!}\int_a^x \left(\ln\frac{x}{s}\right)^{q-1} f(s)\frac{ds}{s},$$

and the works in [13–15] were important to develop the fractional calculus within the frame of the Hadamard fractional derivative. Recently, Klimek investigated existence and uniqueness of the solution of sequential fractional differential equations with Hadamard derivative by using the contraction principle in [16]. Ahmad and Ntouyas studied two-dimensional fractional differential systems with Hadamard derivative in [17]. Thiramanus *et al.* studied existence and uniqueness of solutions for a kind of Hadamard-type fractional differential equations with nonlocal fractional integral boundary conditions in [18].

Next, Jarad *et al.* suggested a Caputo-type modification of the Hadamard fractional derivative in [12] (by the Caputo-Hadamard fractional derivative, we refer to this modified fractional derivative) and presented the fundamental theorem of fractional calculus in the Caputo-Hadamard setting in [12, 19].

Furthermore, impulsive effects exist widely in many processes in which their states can be described by impulsive differential equations. There have appeared a number of papers to research the subject of impulsive differential equations with Caputo fractional derivative [20–25], and impulsive fractional partial differential equations were considered in [26–31].

Recently, we found the formula of general solution for impulsive systems with Caputo fractional derivatives of order $q \in (0,1)$ in [32]. Motivated by the above-mentioned works, we will consider the following system with Caputo-Hadamard fractional derivative and impulsive effect:

$$\begin{cases} {}_{C-H}D^q_{a^+}u(t) = f(t,u(t)), \quad t \in (a,T] \text{ and } t \neq t_k \ (k = 1,2,\dots,m), \\ \Delta u|_{t=t_k} = u(t_k^+) - u(t_k^-) = \Delta_k(u(t_k^-)) \in \mathbb{C}, \quad k = 1,2,\dots,m, \\ u(a) = u_a, \quad u_a \in \mathbb{C}, \end{cases} \tag{1.1}$$

where $q \in \mathbb{C}$ and $\Re(q) \in (0,1)$, ${}_{C-H}D^q_{a^+}$ denotes the left-sided Caputo-Hadamard fractional derivative of order q with the low limit a (> 0), $a = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $f : (a,T] \times \mathbb{C} \to \mathbb{C}$ is an appropriate continuous function. Here $u(t_k^+) = \lim_{\varepsilon\to 0^+} u(t_k + \varepsilon)$ and $u(t_k^-) = \lim_{\varepsilon\to 0^-} u(t_k + \varepsilon)$ represent the right and left limits of $u(t)$ at $t = t_k$, respectively.

The rest of this paper is organized as follows. In Section 2, some preliminaries are presented. In Section 3, we give the formula of general solution for impulsive differential equations with Caputo-Hadamard fractional derivatives. In Section 4, an example is provided to expound the main result in this paper.

2 Preliminaries

In this section, we shall introduce some basic definitions, notations and lemmas which are used throughout this paper.

Definition 2.1 ([2], p.110) Let $0 \leq a \leq b \leq \infty$ be finite or infinite interval of the half-axis $\mathbb{R}^+$. The left-sided Hadamard fractional integral of order $\alpha \in \mathbb{C}$ of function $\varphi(x)$ is defined by

$$\left({}_H\mathcal{J}^\alpha_{a^+}\varphi\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(\ln\frac{x}{s}\right)^{\alpha-1} \varphi(s)\frac{ds}{s} \quad (a < x < b),$$

where $\Gamma(\cdot)$ is the gamma function.

Definition 2.2 ([2], p.110) The left-sided Hadamard fractional derivative of order $\alpha \in \mathbb{C}$ with $\Re(\alpha) \geq 0$ on (a,b) is defined by

$$\left({}_HD^\alpha_{a^+}\varphi\right)(x) = \delta^n\left({}_H\mathcal{J}^{n-\alpha}_{a^+}\varphi\right)(x) = \left(x\frac{d}{dx}\right)^n \frac{1}{\Gamma(n-\alpha)} \int_a^x \left(\ln\frac{x}{s}\right)^{n-\alpha-1} \varphi(s)\frac{ds}{s}$$
$$(a < x < b),$$

where $n = [\Re(\alpha)] + 1$ and differential operator $\delta = x\frac{d}{dx}$, $\delta^0 y(x) = y(x)$.

Lemma 2.3 ([2], pp.114-116) *Let $\alpha, \beta \in \mathbb{C}$ such that $\Re(\alpha) > \Re(\beta) > 0$. For $0 < a < b < \infty$, if $\varphi \in L^p(a,b)$ $(1 \leq p < \infty)$, then ${}_HD^\beta_{a^+}{}_H\mathcal{J}^\alpha_{a^+}\varphi = {}_H\mathcal{J}^{\alpha-\beta}_{a^+}\varphi$ and ${}_H\mathcal{J}^\alpha_{a^+}{}_H\mathcal{J}^\beta_{a^+}\varphi = {}_H\mathcal{J}^{\alpha+\beta}_{a^+}\varphi$.*

In [12], the left-sided Caputo-Hadamard fractional derivative is suggested and defined by

$$ {}_{C-H}D^{\alpha}_{a^+}\varphi(x) = {}_{H}D^{\alpha}_{a^+}\left[\varphi(x) - \sum_{k=0}^{n-1}\frac{\delta^k\varphi(a)}{k!}\left(\ln\frac{t}{a}\right)^k\right](x), $$

here $\Re(\alpha) \geq 0$, $n = [\Re(\alpha)] + 1$, $0 < a < b < \infty$, differential operator $\delta = x\frac{d}{dx}$, $\delta^0 y(x) = y(x)$ and

$$ \varphi(x) \in AC^n_\delta[a,b] = \left\{\varphi : [a,b] \to \mathbb{C} : \delta^{(n-1)}\varphi(x) \in AC[a,b], \delta = x\frac{d}{dx}\right\}. $$

For left-sided Caputo-Hadamard fractional derivatives, the following conclusions were given in [12].

Theorem 2.4 ([12], p.4) *Let* $\Re(\alpha) \geq 0$, $n = [\Re(\alpha)] + 1$ *and* $\varphi \in AC^n_\delta[a,b]$, $0 < a < b < \infty$. *Then* ${}_{C-H}D^{\alpha}_{a^+}\varphi(x)$ *exist everywhere on* $[a,b]$ *and*

(a) *if* $\alpha \notin \mathbb{N}_0$,

$$ {}_{C-H}D^{\alpha}_{a^+}\varphi(x) = \frac{1}{\Gamma(n-\alpha)}\int_a^x \left(\ln\frac{x}{s}\right)^{n-\alpha-1}\delta^n\varphi(s)\frac{ds}{s} = {}_{H}\mathcal{J}^{n-\alpha}_{a^+}\delta^n\varphi(x), $$

(b) *if* $\alpha = n \in \mathbb{N}_0$,

$$ {}_{C-H}D^{\alpha}_{a^+}\varphi(x) = \delta^n\varphi(x). $$

In particular, ${}_{C-H}D^{0}_{a^+}\varphi(x) = \varphi(x)$.

Lemma 2.5 ([12], p.5) *Let* $\Re(\alpha) > 0$, $n = [\Re(\alpha)] + 1$ *and* $\varphi \in C[a,b]$. *If* $\Re(\alpha) \neq 0$ *or* $\alpha \in \mathbb{N}$, *then*

$$ {}_{C-H}D^{\alpha}_{a^+}\left({}_{H}\mathcal{J}^{\alpha}_{a^+}\varphi\right)(x) = \varphi(x). $$

Lemma 2.6 ([12], p.6) *Let* $\varphi \in AC^n_\delta[a,b]$ *or* $C^n_\delta[a,b]$ *and* $\alpha \in \mathbb{C}$, *then*

$$ {}_{H}\mathcal{J}^{\alpha}_{a^+}\left({}_{C-H}D^{\alpha}_{a^+}\varphi\right)(x) = \varphi(x) - \sum_{k=0}^{n-1}\frac{\delta^k\varphi(a)}{k!}\left(\ln\frac{x}{a}\right)^k. $$

3 Main result

For system (1.1), we have

$$ \lim_{\Delta_1(u(t_1^-))\to 0,\ldots,\Delta_m(u(t_m^-))\to 0}\begin{cases} {}_{C-H}D^{q}_{a^+}u(t) = f(t,u(t)), \\ \quad t \in (a,T] \text{ and } t \neq t_k\ (k = 1,2,\ldots,m), \\ \Delta u|_{t=t_k} = u(t_k^+) - u(t_k^-) = \Delta_k(u(t_k^-)) \in \mathbb{C}, \quad k = 1,2,\ldots,m, \\ u(a^+) = u_a, \quad u_a \in \mathbb{C} \end{cases} $$

$$ \to \begin{cases} {}_{C-H}D^{q}_{a^+}u(t) = f(t,u(t)), \quad t \in (a,T], \\ u(a^+) = u_a, \quad u_a \in \mathbb{C}. \end{cases} \tag{3.1} $$

That is,

$$\begin{aligned}&\lim_{\Delta_1(u(t_1^-))\to 0,\ldots,\Delta_m(u(t_m^-))\to 0}\{\text{the solution of system (1.1)}\}\\&\quad=\{\text{the solution of system (3.1)}\}.\end{aligned}\tag{3.2}$$

Thus, the definition of solution for system (1.1) is provided.

Definition 3.1 A function $z(t):[a,T]\to\mathbb{C}$ is said to be a solution of the fractional Cauchy problem (1.1) if $z(a)=u_a$, the equation condition ${}_{C-H}D_{a^+}^q z(t)=f(t,z(t))$ for each $t\in(a,T]$ is verified, the impulsive conditions $\Delta z|_{t=t_k}=\Delta_k(z(t_k^-))$ (here $k=1,2,\ldots,m$) are satisfied, the restriction of $z(\cdot)$ to the interval $(t_k,t_{k+1}]$ (here $k=0,1,2,\ldots,m$) is continuous, and condition (3.2) holds.

A piecewise function is defined by

$$\begin{aligned}&\tilde{u}(t)=u(t_k^+)+\frac{1}{\Gamma(q)}\int_{t_k}^t\left(\ln\frac{t}{s}\right)^{q-1}f(s,u(s))\frac{ds}{s},\\&\quad\text{for } t\in(t_k,t_{k+1}]\text{ (where } k=0,1,2,\ldots,m).\end{aligned}$$

By Theorem 2.4, we have

$$\begin{aligned}&\left[{}_{C-H}D_{a^+}^q\tilde{u}(t)\right]_{t\in(t_k,t_{k+1}]}\\&\quad=\left\{\frac{1}{\Gamma(1-q)}\int_a^t\left(\ln\frac{t}{s}\right)^{1-q-1}\left[s\frac{d}{ds}\left(u(t_k^+)\right.\right.\right.\\&\qquad\left.\left.\left.+\frac{1}{\Gamma(q)}\int_{t_k}^s\left(\ln\frac{s}{\eta}\right)^{q-1}f(\eta,u(\eta))\frac{d\eta}{\eta}\right)\right]\frac{ds}{s}\right\}_{t\in(t_k,t_{k+1}]}\\&\quad=\left\{\frac{1}{\Gamma(1-q)\Gamma(q)}\int_{t_k}^t\left(\ln\frac{t}{s}\right)^{1-q-1}\left[s\frac{d}{ds}\left(\int_{t_k}^s\left(\ln\frac{s}{\eta}\right)^{q-1}f(\eta,u(\eta))\frac{d\eta}{\eta}\right)\right]\frac{ds}{s}\right\}_{t\in(t_k,t_{k+1}]}\\&\quad=\left\{\frac{1}{\Gamma(1-q)\Gamma(q)}\int_{t_k}^t\left(\ln\frac{t}{s}\right)^{1-q-1}\left(s\frac{-1}{q}\frac{d}{ds}\int_{t_k}^s f(\eta,u(\eta))\,d\left(\ln\frac{s}{\eta}\right)^q\right)\frac{ds}{s}\right\}_{t\in(t_k,t_{k+1}]}\\&\quad=\left\{\frac{1}{\Gamma(1-q)\Gamma(q)}\int_{t_k}^t\left(\ln\frac{t}{s}\right)^{1-q-1}\left(s\frac{-1}{q}\frac{d}{ds}\left[\left(\ln\frac{s}{\eta}\right)^q f(\eta,u(\eta))\Big|_{t_k}^s\right.\right.\right.\\&\qquad\left.\left.\left.-\int_{t_k}^s\left(\ln\frac{s}{\eta}\right)^q f'(\eta,u(\eta))\,d\eta\right]\right)\frac{ds}{s}\right\}_{t\in(t_k,t_{k+1}]}\\&\quad=f(t,u(t))|_{t\in(t_k,t_{k+1}]}.\end{aligned}$$

It shows that the piecewise function $\tilde{u}(t)$ satisfies the condition of fractional derivative in system (1.1). Thus, we assume that the piecewise function $\tilde{u}(t)$ is an approximate solution of (1.1).

Theorem 3.2 *Let* $0<\Re(q)<1$ *and* $\hbar$ *be a constant. A function* $u(t):[a,T]\to\mathbb{C}$ *is a general solution of system* (1.1) *if and only if* $u(t)$ *satisfies the fraction integral equation*

$$u(t)=\begin{cases} u_a+\frac{1}{\Gamma(q)}\int_a^t(\ln\frac{t}{s})^{q-1}f(s,u(s))\frac{ds}{s} \quad \textit{for } t\in(a,t_1],\\ u_a+\sum_{i=1}^k\Delta_i(u(t_i^-))+\frac{1}{\Gamma(q)}\int_a^t(\ln\frac{t}{s})^{q-1}f(s,u(s))\frac{ds}{s}\\ \quad+\sum_{i=1}^k\{\frac{\hbar\Delta_i(u(t_i^-))}{\Gamma(q)}[\int_a^{t_i}(\ln\frac{t_i}{s})^{q-1}f(s,u(s))\frac{ds}{s}\\ \quad+\int_{t_i}^t(\ln\frac{t}{s})^{q-1}f(s,u(s))\frac{ds}{s}-\int_a^t(\ln\frac{t}{s})^{q-1}f(s,u(s))\frac{ds}{s}]\}\\ \quad \textit{for } t\in(t_k,t_{k+1}], k=1,2,\ldots,m, \end{cases} \tag{3.3}$$

provided that the integral in (3.3) *exists.*

Proof 'Necessity', it will be verified that Eq. (3.3) satisfies the conditions of system (1.1).

Taking the Caputo-Hadamard fractional derivative to the both sides of Eq. (3.3), for $t\in(a,t_1]$, we have

$$\begin{aligned}
&{}_{C-H}D_{a^+}^q u(t)|_{t\in(a,t_1]}\\
&={}_{C-H}D_{a^+}^q\left(u_a+\frac{1}{\Gamma(q)}\int_a^t\left(\ln\frac{t}{s}\right)^{q-1}f\big(s,u(s)\big)\frac{ds}{s}\right)\\
&={}_{C-H}D_{a^+}^q\left(\frac{1}{\Gamma(q)}\int_a^t\left(\ln\frac{t}{s}\right)^{q-1}f\big(s,u(s)\big)\frac{ds}{s}\right)\\
&=\frac{1}{\Gamma(1-q)\Gamma(q)}\int_a^t\left(\ln\frac{t}{s}\right)^{1-q-1}\left[s\frac{d}{ds}\left(\int_a^s\left(\ln\frac{s}{\eta}\right)^{q-1}f\big(\eta,u(\eta)\big)\frac{d\eta}{\eta}\right)\right]\frac{ds}{s}\\
&=\frac{1}{\Gamma(1-q)\Gamma(q)}\int_a^t\left(\ln\frac{t}{s}\right)^{1-q-1}\left[s\frac{-1}{q}\frac{d}{ds}\left(\int_a^s f\big(\eta,u(\eta)\big)\,d\left(\ln\frac{s}{\eta}\right)^q\right)\right]\frac{ds}{s}\\
&=\frac{1}{\Gamma(1-q)\Gamma(q)}\int_a^t\left(\ln\frac{t}{s}\right)^{1-q-1}\left[s\frac{-1}{q}\frac{d}{ds}\left(\left(\ln\frac{s}{\eta}\right)^q f\big(\eta,u(\eta)\big)\Big|_a^s\right.\right.\\
&\quad\left.\left.-\int_a^s\left(\ln\frac{s}{\eta}\right)^q f'\big(\eta,u(\eta)\big)\,d\eta\right)\right]\frac{ds}{s}\\
&=\frac{1}{\Gamma(1-q)\Gamma(q)}\int_a^t\left(\ln\frac{t}{s}\right)^{1-q-1}\left[s\frac{-1}{q}\frac{d}{ds}\left(-\left(\ln\frac{s}{a}\right)^q f\big(a,u(a)\big)\right.\right.\\
&\quad\left.\left.-\int_a^s\left(\ln\frac{s}{\eta}\right)^q f'\big(\eta,u(\eta)\big)\,d\eta\right)\right]\frac{ds}{s}\\
&=\frac{1}{\Gamma(1-q)\Gamma(q)}\int_a^t\left(\ln\frac{t}{s}\right)^{1-q-1}\left[\left(\ln\frac{s}{a}\right)^{q-1}f\big(a,u(a)\big)\right.\\
&\quad\left.+\int_a^s\left(\ln\frac{s}{\eta}\right)^{q-1}f'\big(\eta,u(\eta)\big)\,d\eta\right]\frac{ds}{s}\\
&=f\big(a,u(a)\big)+\frac{1}{\Gamma(1-q)\Gamma(q)}\int_a^t\left(\ln\frac{t}{s}\right)^{1-q-1}\left[\int_a^s\left(\ln\frac{s}{\eta}\right)^{q-1}f'\big(\eta,u(\eta)\big)\,d\eta\right]\frac{ds}{s}\\
&=f\big(t,u(t)\big).
\end{aligned}$$

For $t\in(t_k,t_{k+1}]$ (where $k=1,2,\ldots,m$), we get

$$\begin{aligned}
&{}_{C-H}D_{a^+}^q\left(\sum_{i=1}^k\left(\frac{\hbar\Delta_i(u(t_i^-))}{\Gamma(q)}\left[\int_a^{t_i}\left(\ln\frac{t_i}{s}\right)^{q-1}f\big(s,u(s)\big)\frac{ds}{s}+\int_{t_i}^t\left(\ln\frac{t}{s}\right)^{q-1}f\big(s,u(s)\big)\frac{ds}{s}\right.\right.\right.\\
&\quad\left.\left.\left.-\int_a^t\left(\ln\frac{t}{s}\right)^{q-1}f\big(s,u(s)\big)\frac{ds}{s}\right]\right)\right)\Big|_{t\in(t_k,t_{k+1}]}
\end{aligned}$$

$$= \sum_{i=1}^{k}\left[\frac{\hbar\Delta_i(u(t_i^-))}{\Gamma(q)}{}_{C-H}D_{a^+}^q\left(\int_a^{t_i}\left(\ln\frac{t_i}{s}\right)^{q-1}f(s,u(s))\frac{ds}{s}+\int_{t_i}^t\left(\ln\frac{t}{s}\right)^{q-1}f(s,u(s))\frac{ds}{s}\right.\right.$$

$$\left.\left.-\int_a^t\left(\ln\frac{t}{s}\right)^{q-1}f(s,u(s))\frac{ds}{s}\right)\right]\Big|_{t\in(t_k,t_{k+1}]}$$

$$= \sum_{i=1}^{k}\left[\frac{\hbar\Delta_i(u(t_i^-))}{\Gamma(1-q)\Gamma(q)}\int_{t_0}^t\left(\ln\frac{t}{s}\right)^{1-q-1}\left[s\frac{d}{ds}\left(\int_{t_i}^s\left(\ln\frac{s}{\eta}\right)^{q-1}f(\eta,u(\eta))\frac{d\eta}{\eta}\right.\right.\right.$$

$$\left.\left.\left.-\int_a^s\left(\ln\frac{s}{\eta}\right)^{q-1}f(\eta,u(\eta))\frac{d\eta}{\eta}\right)\right]\frac{ds}{s}\right]\Big|_{t\in(t_k,t_{k+1}]}$$

$$= \sum_{i=1}^{k}\left[\frac{\hbar\Delta_i(u(t_i^-))}{\Gamma(1-q)\Gamma(q)}\left(\int_{t_i}^t\left(\ln\frac{t}{s}\right)^{1-q-1}s\frac{d}{ds}\left(\int_{t_i}^s\left(\ln\frac{s}{\eta}\right)^{q-1}f(\eta,u(\eta))\frac{d\eta}{\eta}\right)\frac{ds}{s}\right.\right.$$

$$\left.\left.-\int_a^t\left(\ln\frac{t}{s}\right)^{1-q-1}s\frac{d}{ds}\left(\int_a^s\left(\ln\frac{s}{\eta}\right)^{q-1}f(\eta,u(\eta))\frac{d\eta}{\eta}\right)\frac{ds}{s}\right)\right]\Big|_{t\in(t_k,t_{k+1}]}$$

$$= \sum_{i=1}^{k}\left[\hbar\Delta_i(u(t_i^-))\left(f(t,u(t))|_{t\geq t_i}-f(t,u(t))|_{t\geq a}\right)\right]|_{t\in(t_k,t_{k+1}]} = 0.$$

Thus,

$$\begin{aligned}
&{}_{C-H}D_{a^+}^q u(t)|_{t\in(t_k,t_{k+1}]}\\
&\quad= {}_{C-H}D_{a^+}^q\left\{u_a+\sum_{i=1}^k\Delta_i(u(t_i^-))+\frac{1}{\Gamma(q)}\int_a^t\left(\ln\frac{t}{s}\right)^{q-1}f(s,u(s))\frac{ds}{s}\right.\\
&\qquad+\sum_{i=1}^k\left(\frac{\hbar\Delta_i(u(t_i^-))}{\Gamma(q)}\left[\int_a^{t_i}\left(\ln\frac{t_i}{s}\right)^{q-1}f(s,u(s))\frac{ds}{s}\right.\right.\\
&\qquad\left.\left.\left.+\int_{t_i}^t\left(\ln\frac{t}{s}\right)^{q-1}f(s,u(s))\frac{ds}{s}-\int_a^t\left(\ln\frac{t}{s}\right)^{q-1}f(s,u(s))\frac{ds}{s}\right]\right)\right\}_{t\in(t_k,t_{k+1}]}\\
&\quad=\left\{{}_{C-H}D_{a^+}^q\left(\frac{1}{\Gamma(q)}\int_a^t\left(\ln\frac{t}{s}\right)^{q-1}f(s,u(s))\frac{ds}{s}\right)\right\}_{t\in(t_k,t_{k+1}]}\\
&\quad=\left\{f(t,u(t))|_{t\geq a}\right\}_{t\in(t_k,t_{k+1}]}=f(t,u(t))_{t\in(t_k,t_{k+1}]}.
\end{aligned}$$

So, Eq. (3.3) satisfies the condition of fractional derivative in system (1.1).

Next, by (3.3), for each t_k (here $k = 1,2,\dots,m$), we have

$$\begin{aligned}
u(t_k^+)-u(t_k^-) &= \lim_{t\to t_k^+}u(t)-u(t_k)\\
&= u_a+\sum_{i=1}^k\Delta_i(u(t_i^-))+\frac{1}{\Gamma(q)}\int_a^{t_k}\left(\ln\frac{t_k}{s}\right)^{q-1}f(s,u(s))\frac{ds}{s}\\
&\quad+\sum_{i=1}^k\left(\frac{\hbar\Delta_i(u(t_i^-))}{\Gamma(q)}\left[\int_a^{t_i}\left(\ln\frac{t_i}{s}\right)^{q-1}f(s,u(s))\frac{ds}{s}\right.\right.\\
&\quad\left.\left.+\int_{t_i}^{t_k}\left(\ln\frac{t_k}{s}\right)^{q-1}f(s,u(s))\frac{ds}{s}-\int_a^{t_k}\left(\ln\frac{t_k}{s}\right)^{q-1}f(s,u(s))\frac{ds}{s}\right]\right)
\end{aligned}$$

$$
\begin{aligned}
&- u_a - \sum_{i=1}^{k-1} \Delta_i\bigl(u\bigl(t_i^-\bigr)\bigr) + \frac{1}{\Gamma(q)} \int_a^{t_k} \left(\ln \frac{t_k}{s}\right)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s} \\
&- \sum_{i=1}^{k-1} \left(\frac{\hbar \Delta_i(u(t_i^-))}{\Gamma(q)} \left[\int_a^{t_i} \left(\ln \frac{t_i}{s}\right)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s} \right.\right. \\
&\left.\left. + \int_{t_i}^{t_k} \left(\ln \frac{t_k}{s}\right)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s} - \int_a^{t_k} \left(\ln \frac{t_k}{s}\right)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s} \right] \right) \\
={}& \Delta_k\bigl(u\bigl(t_k^-\bigr)\bigr).
\end{aligned}
$$

Therefore, Eq. (3.3) satisfies the impulsive condition of (1.1).

Finally, it can be easily verified that Eq. (3.3) satisfies condition (3.2).

'Sufficiency', we will prove that the solutions of system (1.1) satisfy Eq. (3.3) by using the inductive method. For $t \in (a, t_1]$, we obtain the following equation by Lemma 2.6:

$$
u(t) = u_a + \frac{1}{\Gamma(q)} \int_a^{t} \left(\ln \frac{t}{s}\right)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s} \quad \text{for } t \in (a, t_1]. \tag{3.4}
$$

Using (3.4), we have

$$
\begin{aligned}
u\bigl(t_1^+\bigr) &= u\bigl(t_1^-\bigr) + \Delta_1\bigl(u\bigl(t_1^-\bigr)\bigr) \\
&= u_a + \Delta_1\bigl(u\bigl(t_1^-\bigr)\bigr) + \frac{1}{\Gamma(q)} \int_a^{t_1} \left(\ln \frac{t_1}{s}\right)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s}.
\end{aligned}
$$

Then the approximate solution is given by

$$
\begin{aligned}
\tilde{u}(t) &= u\bigl(t_1^+\bigr) + \frac{1}{\Gamma(q)} \int_{t_1}^{t} \left(\ln \frac{t}{s}\right)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s} \\
&= u_a + \Delta_1\bigl(u\bigl(t_1^-\bigr)\bigr) + \frac{1}{\Gamma(q)} \left[\int_a^{t_1} \left(\ln \frac{t_1}{s}\right)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s} \right. \\
&\quad \left. + \int_{t_1}^{t} \left(\ln \frac{t}{s}\right)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s} \right] \quad \text{for } t \in (t_1, t_2].
\end{aligned} \tag{3.5}
$$

Let $e_1(t) = u(t) - \tilde{u}(t)$, for $t \in (t_1, t_2]$. Due to

$$
\lim_{\Delta_1(u(t_1^-)) \to 0} u(t) = u_a + \frac{1}{\Gamma(q)} \int_a^{t} \left(\ln \frac{t}{s}\right)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s}
$$

for $t \in (t_1, t_2]$, we get

$$
\begin{aligned}
\lim_{\Delta_1(u(t_1^-)) \to 0} e_1(t) &= \lim_{\Delta_1(u(t_1^-)) \to 0} \bigl\{u(t) - \tilde{u}(t)\bigr\} \\
&= \frac{1}{\Gamma(q)} \left[\int_a^{t} \left(\ln \frac{t}{s}\right)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s} - \int_a^{t_1} \left(\ln \frac{t_1}{s}\right)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s} \right. \\
&\quad \left. - \int_{t_1}^{t} \left(\ln \frac{t}{s}\right)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s} \right].
\end{aligned}
$$

Then we assume

$$\begin{aligned} e_1(t) &= \sigma\big(\Delta_1\big(u\big(t_1^-\big)\big)\big) \lim_{\Delta_1(u(t_1^-))\to 0} e_1(t) \\ &= \frac{\sigma(\Delta_1(u(t_1^-)))}{\Gamma(q)}\left[\int_a^t \left(\ln\frac{t}{s}\right)^{q-1} f\big(s,u(s)\big)\frac{ds}{s} - \int_a^{t_1}\left(\ln\frac{t_1}{s}\right)^{q-1} f\big(s,u(s)\big)\frac{ds}{s} \right. \\ &\quad \left. - \int_{t_1}^t \left(\ln\frac{t}{s}\right)^{q-1} f\big(s,u(s)\big)\frac{ds}{s}\right], \end{aligned}$$

where the function $\sigma(\cdot)$ is an undetermined function with $\sigma(0) = 1$. Therefore,

$$\begin{aligned} u(t) &= \tilde{u}(t) + e_1(t) = u_a + \Delta_1\big(u\big(t_1^-\big)\big) + \frac{1}{\Gamma(q)}\int_a^t \left(\ln\frac{t}{s}\right)^{q-1} f\big(s,u(s)\big)\frac{ds}{s} \\ &\quad + \frac{1-\sigma(\Delta_1(u(t_1^-)))}{\Gamma(q)}\left[\int_a^{t_1}\left(\ln\frac{t_1}{s}\right)^{q-1} f\big(s,u(s)\big)\frac{ds}{s} + \int_{t_1}^t \left(\ln\frac{t}{s}\right)^{q-1} f\big(s,u(s)\big)\frac{ds}{s} \right. \\ &\quad \left. - \int_a^t \left(\ln\frac{t}{s}\right)^{q-1} f\big(s,u(s)\big)\frac{ds}{s}\right] \quad \text{for } t \in (t_1, t_2]. \end{aligned} \tag{3.6}$$

By (3.6), we get

$$\begin{aligned} u\big(t_2^+\big) &= u\big(t_2^-\big) + \Delta_2\big(u\big(t_2^-\big)\big) \\ &= u_a + \Delta_1\big(u\big(t_1^-\big)\big) + \Delta_2\big(u\big(t_2^-\big)\big) + \frac{1}{\Gamma(q)}\int_a^{t_2}\left(\ln\frac{t_2}{s}\right)^{q-1} f\big(s,u(s)\big)\frac{ds}{s} \\ &\quad + \frac{1-\sigma(\Delta_1(u(t_1^-)))}{\Gamma(q)}\left[\int_a^{t_1}\left(\ln\frac{t_1}{s}\right)^{q-1} f\big(s,u(s)\big)\frac{ds}{s} + \int_{t_1}^{t_2}\left(\ln\frac{t_2}{s}\right)^{q-1} f\big(s,u(s)\big)\frac{ds}{s} \right. \\ &\quad \left. - \int_a^{t_2}\left(\ln\frac{t_2}{s}\right)^{q-1} f\big(s,u(s)\big)\frac{ds}{s}\right]. \end{aligned}$$

Therefore, the approximate solution is provided by

$$\begin{aligned} \tilde{u}(t) &= u\big(t_2^+\big) + \frac{1}{\Gamma(q)}\int_{t_2}^t \left(\ln\frac{t}{s}\right)^{q-1} f\big(s,u(s)\big)\frac{ds}{s} \\ &= u_a + \Delta_1\big(u\big(t_1^-\big)\big) + \Delta_2\big(u\big(t_2^-\big)\big) + \frac{1}{\Gamma(q)}\left[\int_a^{t_2}\left(\ln\frac{t_2}{s}\right)^{q-1} f\big(s,u(s)\big)\frac{ds}{s} \right. \\ &\quad \left. + \int_{t_2}^t \left(\ln\frac{t}{s}\right)^{q-1} f\big(s,u(s)\big)\frac{ds}{s}\right] \\ &\quad + \frac{1-\sigma(\Delta_1(u(t_1^-)))}{\Gamma(q)}\left[\int_a^{t_1}\left(\ln\frac{t_1}{s}\right)^{q-1} f\big(s,u(s)\big)\frac{ds}{s} + \int_{t_1}^{t_2}\left(\ln\frac{t_2}{s}\right)^{q-1} f\big(s,u(s)\big)\frac{ds}{s} \right. \\ &\quad \left. - \int_a^{t_2}\left(\ln\frac{t_2}{s}\right)^{q-1} f\big(s,u(s)\big)\frac{ds}{s}\right] \quad \text{for } t \in (t_2, t_3]. \end{aligned} \tag{3.7}$$

Let $e_2(t) = u(t) - \tilde{u}(t)$, for $t \in (t_2, t_3]$. For the exact solution $u(t)$ of system (1.1), we have

$$\begin{aligned} &\lim_{\Delta_1(u(t_1^-))\to 0, \Delta_2(u(t_2^-))\to 0} u(t) = u_a + \frac{1}{\Gamma(q)}\int_a^t \left(\ln\frac{t}{s}\right)^{q-1} f\big(s,u(s)\big)\frac{ds}{s} \quad \text{for } t \in (t_2, t_3], \\ &\lim_{\Delta_1(u(t_1^-))\to 0} u(t) = u_a + \Delta_2\big(u\big(t_2^-\big)\big) + \frac{1}{\Gamma(q)}\int_a^t \left(\ln\frac{t}{s}\right)^{q-1} f\big(s,u(s)\big)\frac{ds}{s} \end{aligned}$$

$$
\begin{aligned}
&+\frac{1-\sigma(\Delta_2(u(t_2^-)))}{\Gamma(q)}\left[\int_a^{t_2}\left(\ln\frac{t_2}{s}\right)^{q-1}f(s,u(s))\frac{ds}{s}\right.\\
&\left.+\int_{t_2}^{t}\left(\ln\frac{t}{s}\right)^{q-1}f(s,u(s))\frac{ds}{s}-\int_a^{t}\left(\ln\frac{t}{s}\right)^{q-1}f(s,u(s))\frac{ds}{s}\right]\\
&\text{for } t\in(t_2,t_3],
\end{aligned}
$$

$$
\begin{aligned}
\lim_{\Delta_2(u(t_2^-))\to 0}u(t)={}&u_a+\Delta_1(u(t_1^-))+\frac{1}{\Gamma(q)}\int_a^{t}\left(\ln\frac{t}{s}\right)^{q-1}f(s,u(s))\frac{ds}{s}\\
&+\frac{1-\sigma(\Delta_1(u(t_1^-)))}{\Gamma(q)}\left[\int_a^{t_1}\left(\ln\frac{t_1}{s}\right)^{q-1}f(s,u(s))\frac{ds}{s}\right.\\
&\left.+\int_{t_1}^{t}\left(\ln\frac{t}{s}\right)^{q-1}f(s,u(s))\frac{ds}{s}-\int_a^{t}\left(\ln\frac{t}{s}\right)^{q-1}f(s,u(s))\frac{ds}{s}\right]\\
&\text{for } t\in(t_2,t_3].
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\lim_{\substack{\Delta_1(u(t_1^-))\to 0,\\ \Delta_2(u(t_2^-))\to 0}}e_2(t)&=\lim_{\substack{\Delta_1(u(t_1^-))\to 0,\\ \Delta_2(u(t_2^-))\to 0}}\{u(t)-\tilde{u}(t)\}\\
&=\frac{1}{\Gamma(q)}\left[\int_a^{t}\left(\ln\frac{t}{s}\right)^{q-1}f(s,u(s))\frac{ds}{s}-\int_a^{t_2}\left(\ln\frac{t_2}{s}\right)^{q-1}f(s,u(s))\frac{ds}{s}\right.\\
&\quad\left.-\int_{t_2}^{t}\left(\ln\frac{t}{s}\right)^{q-1}f(s,u(s))\frac{ds}{s}\right], \qquad (3.8)
\end{aligned}
$$

$$
\begin{aligned}
\lim_{\Delta_2(u(t_2^-))\to 0}e_2(t)&=\lim_{\Delta_2(u(t_2^-))\to 0}\{u(t)-\tilde{u}(t)\}=\frac{\sigma(\Delta_1(u(t_1^-)))}{\Gamma(q)}\left[\int_a^{t}\left(\ln\frac{t}{s}\right)^{q-1}f(s,u(s))\frac{ds}{s}\right.\\
&\quad\left.-\int_a^{t_2}\left(\ln\frac{t_2}{s}\right)^{q-1}f(s,u(s))\frac{ds}{s}-\int_{t_2}^{t}\left(\ln\frac{t}{s}\right)^{q-1}f(s,u(s))\frac{ds}{s}\right]\\
&\quad+\frac{1-\sigma(\Delta_1(u(t_1^-)))}{\Gamma(q)}\left[\int_{t_1}^{t}\left(\ln\frac{t}{s}\right)^{q-1}f(s,u(s))\frac{ds}{s}\right.\\
&\quad\left.-\int_{t_1}^{t_2}\left(\ln\frac{t_2}{s}\right)^{q-1}f(s,u(s))\frac{ds}{s}-\int_{t_2}^{t}\left(\ln\frac{t}{s}\right)^{q-1}f(s,u(s))\frac{ds}{s}\right], \qquad (3.9)
\end{aligned}
$$

$$
\begin{aligned}
\lim_{\Delta_1(u(t_1^-))\to 0}e_2(t)&=\lim_{\Delta_1(u(t_1^-))\to 0}\{u(t)-\tilde{u}(t)\}\\
&=\frac{\sigma(\Delta_2(u(t_2^-)))}{\Gamma(q)}\left[\int_a^{t}\left(\ln\frac{t}{s}\right)^{q-1}f(s,u(s))\frac{ds}{s}\right.\\
&\quad\left.-\int_a^{t_2}\left(\ln\frac{t_2}{s}\right)^{q-1}f(s,u(s))\frac{ds}{s}-\int_{t_2}^{t}\left(\ln\frac{t}{s}\right)^{q-1}f(s,u(s))\frac{ds}{s}\right]. \qquad (3.10)
\end{aligned}
$$

By (3.8)-(3.10), we obtain

$$
\begin{aligned}
e_2(t)={}&\frac{\sigma(\Delta_1(u(t_1^-)))+\sigma(\Delta_2(u(t_2^-)))-1}{\Gamma(q)}\left[\int_a^{t}\left(\ln\frac{t}{s}\right)^{q-1}f(s,u(s))\frac{ds}{s}\right.\\
&\left.-\int_a^{t_2}\left(\ln\frac{t_2}{s}\right)^{q-1}f(s,u(s))\frac{ds}{s}-\int_{t_2}^{t}\left(\ln\frac{t}{s}\right)^{q-1}f(s,u(s))\frac{ds}{s}\right]
\end{aligned}
$$

$$+\frac{1-\sigma(\Delta_1(u(t_1^-)))}{\Gamma(q)}\left[\int_{t_1}^{t}\left(\ln\frac{t}{s}\right)^{q-1}f\big(s,u(s)\big)\frac{ds}{s}-\int_{t_1}^{t_2}\left(\ln\frac{t_2}{s}\right)^{q-1}f\big(s,u(s)\big)\frac{ds}{s}\right.$$
$$\left.-\int_{t_2}^{t}\left(\ln\frac{t}{s}\right)^{q-1}f\big(s,u(s)\big)\frac{ds}{s}\right]. \tag{3.11}$$

Therefore,

$$\begin{aligned}
u(t) &= \tilde{u}(t)+e_2(t)\\
&= u_a+\Delta_1\big(u\big(t_1^-\big)\big)+\Delta_2\big(u\big(t_2^-\big)\big)+\frac{1}{\Gamma(q)}\int_a^t\left(\ln\frac{t}{s}\right)^{q-1}f\big(s,u(s)\big)\frac{ds}{s}\\
&\quad+\frac{1-\sigma(\Delta_1(u(t_1^-)))}{\Gamma(q)}\left[\int_a^{t_1}\left(\ln\frac{t_1}{s}\right)^{q-1}f\big(s,u(s)\big)\frac{ds}{s}+\int_{t_1}^{t}\left(\ln\frac{t}{s}\right)^{q-1}f\big(s,u(s)\big)\frac{ds}{s}\right.\\
&\quad\left.-\int_a^{t}\left(\ln\frac{t}{s}\right)^{q-1}f\big(s,u(s)\big)\frac{ds}{s}\right]\\
&\quad+\frac{1-\sigma(\Delta_2(u(t_2^-)))}{\Gamma(q)}\left[\int_a^{t_2}\left(\ln\frac{t_2}{s}\right)^{q-1}f\big(s,u(s)\big)\frac{ds}{s}+\int_{t_2}^{t}\left(\ln\frac{t}{s}\right)^{q-1}f\big(s,u(s)\big)\frac{ds}{s}\right.\\
&\quad\left.-\int_a^{t}\left(\ln\frac{t}{s}\right)^{q-1}f\big(s,u(s)\big)\frac{ds}{s}\right].
\end{aligned} \tag{3.12}$$

Letting $t_2\to t_1$, we have

$$\lim_{t_2\to t_1}\begin{cases}{}_{C-H}D_{a^+}^q u(t)=f(t,u(t)), & t\in(a,t_3]\text{ and }t\neq t_1\text{ and }t\neq t_2,\\ \Delta u|_{t=t_k}=u(t_k^+)-u(t_k^-)=\Delta_k(u(t_k^-))\in\mathbb{C}, & k=1,2,\\ u(a^+)=u_a, & u_a\in\mathbb{C}\end{cases} \tag{3.13}$$

$$\to\begin{cases}{}_{C-H}D_{a^+}^q u(t)=f(t,u(t)), & t\in(a,t_3]\text{ and }t\neq t_1,\\ \Delta u|_{t=t_1}=u(t_1^+)-u(t_1^-)+u(t_2^+)-u(t_2^-)=\Delta_1(u(t_1^-))+\Delta_2(u(t_2^-)),\\ u(a^+)=u_a, \quad u_a\in\mathbb{C}.\end{cases} \tag{3.14}$$

Using (3.6) and (3.12) to systems (3.14) and (3.13), respectively, we get

$$\begin{aligned}
&1-\sigma\big(\Delta_1\big(u\big(t_1^-\big)\big)+\Delta_2\big(u\big(t_2^-\big)\big)\big)=1-\sigma\big(\Delta_1\big(u\big(t_1^-\big)\big)\big)+1-\sigma\big(\Delta_2\big(u\big(t_2^-\big)\big)\big),\\
&\quad\forall\Delta_1\big(u\big(t_1^-\big)\big),\Delta_2\big(u\big(t_2^-\big)\big)\in\mathbb{C}.
\end{aligned}$$

Letting $\rho(z)=1-\sigma(z)$ ($\forall z\in\mathbb{C}$), we have $\rho(z+w)=\rho(z)+\rho(w)$ ($\forall z,w\in\mathbb{C}$). Therefore $\rho(z)=\hbar z$, where $\hbar$ is a constant. So, we obtain the following two equations:

$$\begin{aligned}
u(t) &= u_a+\Delta_1\big(u\big(t_1^-\big)\big)+\frac{1}{\Gamma(q)}\int_a^t\left(\ln\frac{t}{s}\right)^{q-1}f\big(s,u(s)\big)\frac{ds}{s}\\
&\quad+\frac{\hbar\Delta_1(u(t_1^-))}{\Gamma(q)}\left[\int_a^{t_1}\left(\ln\frac{t_1}{s}\right)^{q-1}f\big(s,u(s)\big)\frac{ds}{s}+\int_{t_1}^{t}\left(\ln\frac{t}{s}\right)^{q-1}f\big(s,u(s)\big)\frac{ds}{s}\right.\\
&\quad\left.-\int_a^{t}\left(\ln\frac{t}{s}\right)^{q-1}f\big(s,u(s)\big)\frac{ds}{s}\right]\quad\text{for }t\in(t_1,t_2]
\end{aligned} \tag{3.15}$$

and

$$\begin{aligned}u(t) &= u_a + \Delta_1\big(u(t_1^-)\big) + \Delta_2\big(u(t_2^-)\big) + \frac{1}{\Gamma(q)}\int_a^t\left(\ln\frac{t}{s}\right)^{q-1} f\big(s,u(s)\big)\frac{ds}{s} \\ &\quad + \frac{\hbar\Delta_1(u(t_1^-))}{\Gamma(q)}\left[\int_a^{t_1}\left(\ln\frac{t_1}{s}\right)^{q-1} f\big(s,u(s)\big)\frac{ds}{s} + \int_{t_1}^t\left(\ln\frac{t}{s}\right)^{q-1} f\big(s,u(s)\big)\frac{ds}{s}\right. \\ &\quad \left. - \int_a^t\left(\ln\frac{t}{s}\right)^{q-1} f\big(s,u(s)\big)\frac{ds}{s}\right] \\ &\quad + \frac{\hbar\Delta_2(u(t_2^-))}{\Gamma(q)}\left[\int_a^{t_2}\left(\ln\frac{t_2}{s}\right)^{q-1} f\big(s,u(s)\big)\frac{ds}{s} + \int_{t_2}^t\left(\ln\frac{t}{s}\right)^{q-1} f\big(s,u(s)\big)\frac{ds}{s}\right. \\ &\quad \left. - \int_a^t\left(\ln\frac{t}{s}\right)^{q-1} f\big(s,u(s)\big)\frac{ds}{s}\right] \quad \text{for } t \in (t_2,t_3].\end{aligned} \tag{3.16}$$

For $t \in (t_n, t_{n+1}]$, suppose

$$\begin{aligned}u(t) &= u_a + \sum_{i=1}^n \Delta_i\big(u(t_i^-)\big) + \frac{1}{\Gamma(q)}\int_a^t\left(\ln\frac{t}{s}\right)^{q-1} f\big(s,u(s)\big)\frac{ds}{s} \\ &\quad + \sum_{i=1}^n\left(\frac{\hbar\Delta_i(u(t_i^-))}{\Gamma(q)}\left[\int_a^{t_i}\left(\ln\frac{t_i}{s}\right)^{q-1} f\big(s,u(s)\big)\frac{ds}{s} + \int_{t_i}^t\left(\ln\frac{t}{s}\right)^{q-1} f\big(s,u(s)\big)\frac{ds}{s}\right.\right. \\ &\quad \left.\left. - \int_a^t\left(\ln\frac{t}{s}\right)^{q-1} f\big(s,u(s)\big)\frac{ds}{s}\right]\right) \quad \text{for } t \in (t_n,t_{n+1}].\end{aligned} \tag{3.17}$$

By (3.17), we have

$$\begin{aligned}u\big(t_{n+1}^+\big) &= u\big(t_{n+1}^-\big) + \Delta_{n+1}\big(u\big(t_{n+1}^-\big)\big) \\ &= u_a + \sum_{i=1}^{n+1} \Delta_i\big(u(t_i^-)\big) + \frac{1}{\Gamma(q)}\int_a^{t_{n+1}}\left(\ln\frac{t_{n+1}}{s}\right)^{q-1} f\big(s,u(s)\big)\frac{ds}{s} \\ &\quad + \sum_{i=1}^n\left(\frac{\hbar\Delta_i(u(t_i^-))}{\Gamma(q)}\left[\int_a^{t_i}\left(\ln\frac{t_i}{s}\right)^{q-1} f\big(s,u(s)\big)\frac{ds}{s}\right.\right. \\ &\quad \left.\left. + \int_{t_i}^{t_{n+1}}\left(\ln\frac{t_{n+1}}{s}\right)^{q-1} f\big(s,u(s)\big)\frac{ds}{s} - \int_a^{t_{n+1}}\left(\ln\frac{t_{n+1}}{s}\right)^{q-1} f\big(s,u(s)\big)\frac{ds}{s}\right]\right).\end{aligned}$$

So, the approximate solution is provided by

$$\begin{aligned}\tilde{u}(t) &= u\big(t_{n+1}^+\big) + \frac{1}{\Gamma(q)}\int_{t_{n+1}}^t\left(\ln\frac{t}{s}\right)^{q-1} f\big(s,u(s)\big)\frac{ds}{s} \\ &= u_a + \sum_{i=1}^{n+1} \Delta_i\big(u(t_i^-)\big) + \frac{1}{\Gamma(q)}\left[\int_a^{t_{n+1}}\left(\ln\frac{t_{n+1}}{s}\right)^{q-1} f\big(s,u(s)\big)\frac{ds}{s}\right. \\ &\quad \left. + \int_{t_{n+1}}^t\left(\ln\frac{t}{s}\right)^{q-1} f\big(s,u(s)\big)\frac{ds}{s}\right] + \sum_{i=1}^n\left(\frac{\hbar\Delta_i(u(t_i^-))}{\Gamma(q)}\left[\int_a^{t_i}\left(\ln\frac{t_i}{s}\right)^{q-1} f\big(s,u(s)\big)\frac{ds}{s}\right.\right. \\ &\quad \left.\left. + \int_{t_i}^{t_{n+1}}\left(\ln\frac{t_{n+1}}{s}\right)^{q-1} f\big(s,u(s)\big)\frac{ds}{s} - \int_a^{t_{n+1}}\left(\ln\frac{t_{n+1}}{s}\right)^{q-1} f\big(s,u(s)\big)\frac{ds}{s}\right]\right) \\ &\quad \text{for } t \in (t_{n+1}, t_{n+2}].\end{aligned} \tag{3.18}$$

Let $e_{n+1}(t) = u(t) - \tilde{u}(t)$, for $t \in (t_{n+1}, t_{n+2}]$. For the exact solution $u(t)$ of system (1.1), we have

$$\lim_{\Delta_1(u(t_1^-))\to 0,\ldots,\Delta_{n+1}(u(t_{n+1}^-))\to 0} u(t) = u_a + \frac{1}{\Gamma(q)}\int_a^t \left(\ln\frac{t}{s}\right)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s} \quad \text{for } t \in (t_{n+1}, t_{n+2}],$$

$$\begin{aligned}
\lim_{\Delta_j(u(t_j^-))\to 0} u(t) &= u_a + \sum_{\substack{1\le i\le n+1\\ \text{and } i\neq j}} \Delta_i\bigl(u(t_i^-)\bigr) + \frac{1}{\Gamma(q)}\int_a^t \left(\ln\frac{t}{s}\right)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s} \\
&\quad + \sum_{\substack{1\le i\le n+1\\ \text{and } i\neq j}} \left(\frac{\hbar\Delta_i(u(t_i^-))}{\Gamma(q)}\left[\int_a^{t_i}\left(\ln\frac{t_i}{s}\right)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s}\right.\right. \\
&\quad \left.\left. + \int_{t_i}^t \left(\ln\frac{t}{s}\right)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s} - \int_a^t \left(\ln\frac{t}{s}\right)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s}\right]\right), \\
&\quad \text{here } 1 \le j \le n+1.
\end{aligned}$$

Then

$$\begin{aligned}
&\lim_{\Delta_1(u(t_1^-))\to 0,\ldots,\Delta_{n+1}(u(t_{n+1}^-))\to 0} e_{n+1}(t) \\
&\quad = \lim_{\Delta_1(u(t_1^-))\to 0,\ldots,\Delta_{n+1}(u(t_{n+1}^-))\to 0} \bigl\{u(t) - \tilde{u}(t)\bigr\} \\
&\quad = \frac{1}{\Gamma(q)}\left[\int_a^t \left(\ln\frac{t}{s}\right)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s} - \int_a^{t_{n+1}} \left(\ln\frac{t_{n+1}}{s}\right)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s}\right. \\
&\qquad \left. - \int_{t_{n+1}}^t \left(\ln\frac{t}{s}\right)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s}\right],
\end{aligned} \tag{3.19}$$

$$\begin{aligned}
\lim_{\Delta_j(u(t_j^-))\to 0} e_{n+1}(t) &= \lim_{\Delta_j(u(t_j^-))\to 0} \bigl\{u(t) - \tilde{u}(t)\bigr\} \\
&= \frac{1 - \hbar\sum_{\substack{1\le i\le n+1\\ \text{and } i\neq j}} \Delta_i(u(t_i^-))}{\Gamma(q)}\left[\int_a^t \left(\ln\frac{t}{s}\right)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s}\right. \\
&\quad \left. - \int_a^{t_{n+1}} \left(\ln\frac{t_{n+1}}{s}\right)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s} - \int_{t_{n+1}}^t \left(\ln\frac{t}{s}\right)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s}\right] \\
&\quad + \sum_{\substack{1\le i\le n\\ \text{and } i\neq j}} \left(\frac{\hbar\Delta_i(u(t_i^-))}{\Gamma(q)}\left[\int_{t_i}^t \left(\ln\frac{t}{s}\right)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s}\right.\right. \\
&\quad \left.\left. - \int_{t_i}^{t_{n+1}} \left(\ln\frac{t_{n+1}}{s}\right)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s} - \int_{t_{n+1}}^t \left(\ln\frac{t}{s}\right)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s}\right]\right), \\
&\quad \text{here } 1 \le j \le n+1.
\end{aligned} \tag{3.20}$$

So, by (3.19) and (3.20), we obtain

$$\begin{aligned}
e_{n+1}(t) &= \frac{1 - \hbar\sum_{1\le i\le n+1} \Delta_i(u(t_i^-))}{\Gamma(q)}\left[\int_a^t \left(\ln\frac{t}{s}\right)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s}\right. \\
&\quad \left. - \int_a^{t_{n+1}} \left(\ln\frac{t_{n+1}}{s}\right)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s} - \int_{t_{n+1}}^t \left(\ln\frac{t}{s}\right)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s}\right]
\end{aligned}$$

$$+ \sum_{1\le i\le n} \left(\frac{\hbar\Delta_i(u(t_i^-))}{\Gamma(q)} \left[\int_{t_i}^{t} \left(\ln\frac{t}{s}\right)^{q-1} f\big(s,u(s)\big)\frac{ds}{s} - \int_{t_i}^{t_{n+1}} \left(\ln\frac{t_{n+1}}{s}\right)^{q-1} f\big(s,u(s)\big)\frac{ds}{s} - \int_{t_{n+1}}^{t} \left(\ln\frac{t}{s}\right)^{q-1} f\big(s,u(s)\big)\frac{ds}{s} \right] \right).$$

Thus,

$$\begin{aligned} u(t) &= \tilde{u}(t) + e_{n+1}(t) \\ &= u_a + \sum_{i=1}^{n+1} \Delta_i\big(u(t_i^-)\big) + \frac{1}{\Gamma(q)} \int_a^t \left(\ln\frac{t}{s}\right)^{q-1} f\big(s,u(s)\big)\frac{ds}{s} \\ &\quad + \sum_{i=1}^{n+1} \left(\frac{\hbar\Delta_i(u(t_i^-))}{\Gamma(q)} \left[\int_a^{t_i} \left(\ln\frac{t_i}{s}\right)^{q-1} f\big(s,u(s)\big)\frac{ds}{s} + \int_{t_i}^{t} \left(\ln\frac{t}{s}\right)^{q-1} f\big(s,u(s)\big)\frac{ds}{s} \right.\right. \\ &\quad \left.\left. - \int_a^{t} \left(\ln\frac{t}{s}\right)^{q-1} f\big(s,u(s)\big)\frac{ds}{s} \right] \right) \quad \text{for } t \in (t_{n+1}, t_{n+2}]. \end{aligned} \tag{3.21}$$

So, the solution of system (1.1) satisfies Eq. (3.3).

By the proof of Sufficiency and Necessity, it is shown that system (1.1) is equivalent to the integral equation (3.3). The proof is now completed. □

Remark 3.1 Due to uncertainty of the constant $\hbar$, the deviation caused by impulse for the fractional order nonlinear systems is undetermined.

Next, we provide an analysis of the connection between impulsive Caputo-Hadamard fractional differential equations and impulsive first-order differential equations. Letting $q \to 1^-$ for system (1.1), we have

$$\begin{aligned} &\lim_{q\to 1^-} \begin{cases} {}_{C-H}D_{a^+}^q u(t) = f(t,u(t)), & t \in (a,T] \text{ and } t \neq t_k\ (k = 1,2,\ldots,m), \\ \Delta u|_{t=t_k} = u(t_k^+) - u(t_k^-) = \Delta_k(u(t_k^-)) \in \mathbb{C}, & k = 1,2,\ldots,m, \\ u(a) = u_a, \quad u_a \in \mathbb{C} \end{cases} \\ &\to \begin{cases} t\frac{dx(t)}{dt} = f(t,x(t)), & t \in (a,T] \text{ and } t \neq t_k\ (k = 1,2,\ldots,m), \\ \Delta u|_{t=t_k} = u(t_k^+) - u(t_k^-) = \Delta_k(u(t_k^-)) \in \mathbb{C}, & k = 1,2,\ldots,m, \\ u(a) = u_a, \quad u_a \in \mathbb{C}. \end{cases} \end{aligned} \tag{3.22}$$

On the other hand, by Theorem 3.2, the general solution of system (1.1) satisfies

$$\begin{aligned} \lim_{q\to 1^-} u(t) &= \begin{cases} \lim_{q\to 1^-} [u_a + \frac{1}{\Gamma(q)} \int_a^t (\ln\frac{t}{s})^{q-1} f(s,u(s))\frac{ds}{s}] \quad \text{for } t \in (a,t_1], \\ \lim_{q\to 1^-} \{u_a + \sum_{i=1}^k \Delta_i(u(t_i^-)) + \frac{1}{\Gamma(q)} \int_a^t (\ln\frac{t}{s})^{q-1} f(s,u(s))\frac{ds}{s} \\ \quad + \sum_{i=1}^k (\frac{\hbar\Delta_i(u(t_i^-))}{\Gamma(q)} [\int_a^{t_i} (\ln\frac{t_i}{s})^{q-1} f(s,u(s))\frac{ds}{s} + \int_{t_i}^t (\ln\frac{t}{s})^{q-1} f(s,u(s))\frac{ds}{s} \\ \quad - \int_a^t (\ln\frac{t}{s})^{q-1} f(s,u(s))\frac{ds}{s}])\} \quad \text{for } t \in (t_k,t_{k+1}], k = 1,2,\ldots,m \end{cases} \\ &= \begin{cases} u_a + \int_a^t \frac{f(s,u(s))}{s}\,ds \quad \text{for } t \in (a,t_1], \\ u_a + \sum_{i=1}^k \Delta_i(u(t_i^-)) + \int_a^t \frac{f(s,u(s))}{s}\,ds \\ \quad \text{for } t \in (t_k,t_{k+1}], k = 1,2,\ldots,m. \end{cases} \end{aligned} \tag{3.23}$$

Moreover, it is true that Eq. (3.23) is the solution of impulsive system (3.22) and indirectly verifies the formula of general solution for nonlinear systems with Caputo-Hadamard fractional derivatives and impulsive effect.

4 Example

In this section, an example is given to illustrate the usefulness of the result in this paper.

Example 1 Let us consider the general solution of the impulsive fractional system

$$\begin{cases} {}_{C-H}D_{1^+}^{\frac{1}{2}}u(t) = \ln t, \quad t \in (1,3] \text{ and } t \neq 2, \\ \Delta u| = u(2^+) - u(2^-) = \Delta, \\ u(1^+) = u_1. \end{cases} \tag{4.1}$$

By Theorem 3.2, after some elementary computation, the general solution is obtained as follows:

$$u(t) = \begin{cases} u_1 + \frac{1}{\Gamma(\frac{1}{2})}\int_1^t (\ln\frac{t}{s})^{\frac{1}{2}-1}\ln s\frac{ds}{s} \quad \text{for } t \in (1,2], \\ u_1 + \Delta + \frac{1}{\Gamma(\frac{1}{2})}\int_1^t (\ln\frac{t}{s})^{\frac{1}{2}-1}\ln s\frac{ds}{s} + \frac{\hbar\Delta}{\Gamma(\frac{1}{2})}[\int_1^2 (\ln\frac{2}{s})^{\frac{1}{2}-1}\ln s\frac{ds}{s} \\ \quad + \int_2^t (\ln\frac{t}{s})^{\frac{1}{2}-1}\ln s\frac{ds}{s} - \int_1^t (\ln\frac{t}{s})^{\frac{1}{2}-1}\ln s\frac{ds}{s}] \quad \text{for } t \in (2,3]. \end{cases} \tag{4.2}$$

That is,

$$u(t) = \begin{cases} u_1 + \frac{4}{3}\frac{1}{\Gamma(\frac{1}{2})}(\ln t)^{\frac{3}{2}} \quad \text{for } t \in (1,2], \\ u_1 + \Delta + \frac{4}{3}\frac{1}{\Gamma(\frac{1}{2})}(\ln t)^{\frac{3}{2}}|_{t\geq 1} + \frac{\hbar\Delta}{\Gamma(\frac{1}{2})}[\frac{4}{3}(\ln 2)^{\frac{3}{2}} + (\frac{4}{3}(\ln\frac{t}{2})^{\frac{3}{2}} \\ \quad + 2(\ln\frac{t}{2})^{\frac{1}{2}}\ln 2)|_{t\geq 2} - \frac{4}{3}(\ln t)^{\frac{3}{2}}|_{t\geq 1}] \quad \text{for } t \in (2,3]. \end{cases} \tag{4.2a}$$

Next, it is verified that Eq. (4.2a) satisfies the condition of system (4.1).

Taking the Caputo-Hadamard fractional derivative to the both sides of Eq. (4.2a), after some elementary computation, we have

(i) for $t \in (1,2]$,

$$\begin{aligned} {}_{C-H}D_{1^+}^{\frac{1}{2}}u(t)|_{t\in(1,2]} &= \left\{\frac{1}{\Gamma(1-\frac{1}{2})}\int_1^t \left(\ln\frac{t}{s}\right)^{1-\frac{1}{2}-1}\left(s\frac{d}{ds}\right)\left[u_1 + \frac{4}{3}\frac{1}{\Gamma(\frac{1}{2})}(\ln s)^{\frac{3}{2}}\right]\frac{ds}{s}\right\}_{t\in(1,2]} \\ &= \left\{\frac{1}{\Gamma(1-\frac{1}{2})\Gamma(\frac{1}{2})}\int_1^t \left(\ln\frac{t}{s}\right)^{1-\frac{1}{2}-1}\left[2(\ln s)^{\frac{1}{2}}\right]\frac{ds}{s}\right\}_{t\in(1,2]} \\ &= \ln t|_{t\in(1,2]}, \end{aligned}$$

(ii) for $t \in (2,3]$,

$$\begin{aligned} &{}_{C-H}D_{1^+}^{\frac{1}{2}}u(t)|_{t\in(2,3]} \\ &= \frac{1}{\Gamma(1-\frac{1}{2})}\int_1^t \left(\ln\frac{t}{s}\right)^{1-\frac{1}{2}-1}\left(s\frac{d}{ds}\right)\left\{u_1 + \Delta + \frac{4}{3}\frac{1}{\Gamma(\frac{1}{2})}(\ln s)^{\frac{3}{2}}\Big|_{t\geq 1}\right. \\ &\quad \left. + \frac{\hbar\Delta}{\Gamma(\frac{1}{2})}\left[\frac{4}{3}(\ln 2)^{\frac{3}{2}} + \left(\frac{4}{3}\left(\ln\frac{s}{2}\right)^{\frac{3}{2}} + 2\left(\ln\frac{s}{2}\right)^{\frac{1}{2}}\ln 2\right)\Big|_{t\geq 2} - \frac{4}{3}(\ln s)^{\frac{3}{2}}\Big|_{t\geq 1}\right]\right\}\frac{ds}{s}\Big|_{t\in(2,3]} \end{aligned}$$

$$= \left\{ \frac{1}{\Gamma(1-\frac{1}{2})\Gamma(\frac{1}{2})} \int_1^t \left(\ln\frac{t}{s}\right)^{1-\frac{1}{2}-1} \left[2(\ln s)^{\frac{1}{2}} - 2\hbar\Delta(\ln s)^{\frac{1}{2}}\right]\frac{ds}{s} \right.$$
$$\left. + \frac{\hbar\Delta}{\Gamma(1-\frac{1}{2})\Gamma(\frac{1}{2})} \int_2^t \left(\ln\frac{t}{s}\right)^{1-\frac{1}{2}-1} \left(2\left(\ln\frac{s}{2}\right)^{\frac{1}{2}} + \ln 2\left(\ln\frac{s}{2}\right)^{-\frac{1}{2}}\right)\frac{ds}{s} \right\}_{t\in(2,3]}$$
$$= \left\{ \ln t|_{t\geq 1} - \hbar\Delta \ln t|_{t\geq 1} + \hbar\Delta\left(\left(\ln\frac{t}{2}\right) + \ln 2\right)\Big|_{t\geq 2} \right\}_{t\in(2,3]}$$
$$= \ln t|_{t\in(2,3]}.$$

So, Eq. (4.2a) satisfies the Hadamard fractional derivative condition of system (4.1).

By Eq. (4.2a), we have

$$u(2^+) - u(2^-) = \lim_{t\to 2^+}\left\{ u_1 + \Delta + \frac{4}{3}\frac{1}{\Gamma(\frac{1}{2})}(\ln t)^{\frac{3}{2}}\Big|_{t\geq 1} + \frac{\hbar\Delta}{\Gamma(\frac{1}{2})}\left[\frac{4}{3}(\ln 2)^{\frac{3}{2}} + \left(\frac{4}{3}\left(\ln\frac{t}{2}\right)^{\frac{3}{2}} \right.\right.\right.$$
$$\left.\left.\left. + 2\left(\ln\frac{t}{2}\right)^{\frac{1}{2}}\ln 2\right)\Big|_{t\geq 2} - \frac{4}{3}(\ln t)^{\frac{3}{2}}\Big|_{t\geq 1}\right]\right\} - u_1 - \frac{4}{3}\frac{1}{\Gamma(\frac{1}{2})}(\ln 2)^{\frac{3}{2}}$$
$$= \Delta.$$

That is, Eq. (4.2a) satisfies the impulsive condition in system (4.1).

Finally, it is obvious that Eq. (4.2a) satisfies the following limit case:

$$\lim_{\Delta\to 0}\begin{cases} {}_{C-H}D_{1^+}^{\frac{1}{2}}u(t) = \ln t, & t\in(1,3] \text{ and } t\neq 2,\\ \Delta u|_{t=2} = u(2^+) - u(2^-) = \Delta\in\mathbb{R},\\ u(1^+) = u_1\in\mathbb{R} \end{cases}$$
$$\to \begin{cases} {}_{C-H}D_{1^+}^{\frac{1}{2}}u(t) = \ln t, & t\in(1,3],\\ u(1^+) = u_1\in\mathbb{R}. \end{cases} \tag{4.3}$$

So, Eq. (4.2a) is the general solution of system (4.1).

5 Conclusion

For the first-order impulsive differential equations, the solution is determined by initial value. However, Caputo-Hadamard fractional differential equations with impulsive effect have a general solution and need more conditions to decide the constant $\hbar$ than the first-order impulsive ones. Moreover, due to the non-uniqueness of solution for the impulsive fractional-order nonlinear system, it means that there appear new problems on impulsive control of the fractional-order nonlinear systems.

Competing interests
The author declares that he has no competing interests.

Author's contributions
The author wrote the first version of the manuscript and approved the final manuscript by himself.

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