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Convergence of very weak solutions to A-Dirac equations in Clifford analysis

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Abstract

This paper is concerned with the very weak solutions to A-Dirac equations $DA(x, Du) = 0$ with Dirichlet boundary data. By means of the decomposition in a Clifford-valued function space, convergence of the very weak solutions to A-Dirac equations is obtained in Clifford analysis.

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Keywords: Dirac equations; very weak solutions; decomposition; convergence

1 Introduction

In this paper, we shall consider a nonlinear mapping $A : \Omega \times C\ell_n \to C\ell_n$ such that

(N_1) $x \to A(x, \xi)$ is measurable for all $\xi \in C\ell_n$,
(N_2) $\xi \to A(x, \xi)$ is continuous for a.e. $x \in \Omega$,
(N_3) $|A(x, \xi) - A(x, \zeta)| \le b|\xi - \zeta|^{p-1}$,
(N_4) $(A(x, \xi) - A(x, \zeta), \xi - \zeta) \ge a|\xi - \zeta|^2(|\xi| + |\zeta|)^{p-2}$,

where $0 < a \le b < \infty$.

The exponent $p > 1$ will determine the natural Sobolev class, denoted by $W^{1,p}(\Omega, C\ell_n)$, in which to consider the A-Dirac equations

$$DA(x, Du) = 0. \tag{1}$$

We call $u \in W^{1,p}_{\mathrm{loc}}(\Omega, C\ell_n)$ a weak solution to (1) if

$$\int_\Omega \overline{A(x, Du)} D\varphi \, dx = 0 \tag{2}$$

for each $\varphi \in W^{1,p}_{\mathrm{loc}}(\Omega, C\ell_n)$ with compact support.

Definition 1.1 For $s > \max\{1, p-1\}$, a Clifford-valued function $u \in W^{1,s}_{\mathrm{loc}}(\Omega, C\ell_n)$ is called a very weak solution of equation (1) if it satisfies (2) for all $\varphi \in W^{1,\frac{s}{s-p+1}}(\Omega, C\ell_n)$ with compact support.

Remark 1.2 It is clear that if $s = p$, the very weak solution is identity to the weak solution to equation (1).

It is well known that the A-harmonic equations

$$-\operatorname{div} A(x, \nabla u) = 0 \tag{3}$$

arise in the study of nonlinear elastic mechanics. More exactly, by means of qualitative theory of solutions to (3), we can study the above physics problems at equilibrium. Moreover, basic theories of (3) of degenerate condition have been studied by Iwaniec, Heinonen *et al.* systematically in [1–7]. While the regularity is not good enough, the existence of weak solutions to elliptic equations maybe not obtained in the corresponding function space, then the concept of 'very weak solution' is produced in order to study the solutions to elliptic equations in a wider space. Also, there are many researchers' works on the properties of the very weak solutions to various versions of A-harmonic equations, see [8–12]. In 1989, Gürlebeck and Sprößig studied the quaternionic analysis and elliptic boundary value problems in [13]; for more about Clifford analysis and its applications, see [14–16]. In 2010, Nolder introduced the A-Dirac equations (1) and explained how the quasi-linear elliptic equations (3) arise as components of Dirac equations (1). After that, Fu, Zhang, Bisci *et al.* studied this problem on the weighted variable exponent spaces, see [17–20]. Wang and Chen studied the relation between A-harmonic operator and A-Dirac system in [21]. In [22], Lian *et al.* studied the weak solutions to A-Dirac equations in whole. For other works in this new field, we refer readers to [23–26].

This paper is concerned with the very weak solutions to a nonlinear A-Dirac equation with Dirichlet bound data

$$\begin{cases} DA(x, Du) = 0, \\ u - u_0 \in W_0^{1,s}(\Omega, C\ell_n). \end{cases} \tag{4}$$

We study the convergence of the very weak solutions to equation (4) without the homogeneity $A(x, \lambda\xi) = |\lambda|^{p-2}\lambda A(x, \xi)$.

2 Preliminary results

Let $e_1, e_2, \ldots, e_n$ be the standard basis of $\mathbb{R}^n$ with the relation $e_i e_j + e_j e_i = -2\delta_{ij}$. For $l = 0, 1, \ldots, n$, we denote by $C\ell_n^k = C\ell_n^k(\mathbb{R}^n)$ the linear space of all k-vectors, spanned by the reduced products $e_I = e_{i_1} e_{i_2} \cdots e_{i_k}$, corresponding to all ordered k-tuples $I = (i_1, i_2, \ldots, i_k)$, $1 \le i_1 < i_2 < \cdots < i_k \le n$. Thus, Clifford algebra $C\ell_n = \oplus C\ell_n^k$ is a graded algebra and $C\ell_n^0 = \mathbb{R}$ and $C\ell_n^1 = \mathbb{R}^n$. $\mathbb{R} \subset \mathbb{C} \subset \mathbb{H} \subset C\ell_n^3 \subset \cdots$ is an increasing chain. For $u \in C\ell_n$, u can be written as

$$u = \sum_I u_I e_I = \sum_{1 \le i_1 < \cdots < i_k \le n} u_{i_1,\ldots,i_k} e_{i_1} \cdots e_{i_k},$$

where $1 \le k \le n$.

The norm of $u \in C\ell_n$ is given by $|u| = (\sum_I u_I^2)^{1/2}$. Clifford conjugation $\overline{e_{\alpha_1} \cdots e_{\alpha_k}} = (-1)^k e_{\alpha_k} \cdots e_{\alpha_1}$. For each $I = (i_1, i_2, \ldots, i_k)$, we have

$$e_I \overline{e_I} = \overline{e_I} e_I = 1. \tag{5}$$

For $u = \sum_I u_I e_I \in C\ell_n$, $v = \sum_J v_J e_J \in C\ell_n$,

$$\langle u, v\rangle = \left\langle \sum_I u_I e_I, \sum_J v_J e_J \right\rangle = \sum_I u_I v_I$$

defines the corresponding inner product on $C\ell_n$. For $u \in C\ell_n$, $\mathrm{Sc}(u)$ denotes the scalar part of u, that is, the coefficient of the element u. Also we have $\langle u, v\rangle = \mathrm{Sc}(\overline{u}v)$.

The Dirac operator is given by

$$D = \sum_{j=1}^{n} e_j \frac{\partial}{\partial x_j}.$$

u is called a monogenic function if $Du = 0$. Also $D^2 = -\triangle$, where $\triangle$ is the Laplace operator which operates only on coefficients.

Throughout the paper, Ω is a bounded domain. $C_0^\infty(\Omega, C\ell_n)$ is the space of Clifford-valued functions in Ω whose coefficients belong to $C_0^\infty(\Omega)$. For $s > 0$, denote by $L^s(\Omega, C\ell_n)$ the space of Clifford-valued functions in Ω whose coefficients belong to the usual $L^s(\Omega)$ space. Denoted by $\nabla u = (\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \dots, \frac{\partial u}{\partial x_n})$, then $W^{1,s}(\Omega, C\ell_n)$ is the space of Clifford-valued functions in Ω whose coefficients as well as their first distributional derivatives are in $L^s(\Omega)$. We similarly write $W^{1,s}_{\mathrm{loc}}(\Omega, C\ell)$ and $W_0^{1,s}(\Omega, C\ell_n)$.

Let $G(x) = \frac{1}{\omega_n} \frac{\overline{x}}{|x|^n}$, the Teodorescu operator here is given by

$$Tf = \int_\Omega G(x-y) f(y)\, \mathrm{d}y.$$

Next, we introduce the Borel-Prompieu result for a Clifford-valued function.

Theorem 2.1 ([14]) *If Ω is a domain in $\mathbb{R}^n$, then for each $f \in C_0^\infty(\Omega, C\ell_n)$, we have*

$$f(z) = \int_{\partial\Omega} G(x-z)\, \mathrm{d}\sigma(x) f(x) - \int_\Omega G(x-z) Df(x)\, \mathrm{d}x, \tag{6}$$

where $z \in \Omega$.

According to Theorem 2.1, Lian proved the following theorem.

Theorem 2.2 ([22]) (Poincaré inequality) *For every $u \in W_0^{1,s}(\Omega, C\ell_n)$, $1 < s < \infty$, there exists a constant c such that*

$$\int_\Omega |u|^s\, \mathrm{d}x \le c|\Omega|^{\frac{1}{n}} \int_\Omega |Du|^s\, \mathrm{d}x. \tag{7}$$

For $s > 1$, we can find that $f = TDf$ when $f \in W_0^{1,s}(\Omega, C\ell_n)$. Also, we have $f = DTf$ when $f \in L^s(\Omega, C\ell_n)$, see [27].

Lemma 2.3 ([18]) *Suppose Clifford-valued function $u \in C_0^\infty(\Omega, C\ell_n)$, $1 < p < \infty$, then there exists a constant c such that*

$$\int_\Omega \left|\nabla(Tu)\right|^p \mathrm{d}x \le c(n, p, \Omega) \int_\Omega |u|^p\, \mathrm{d}x.$$

Then we can easily get the following lemma.

Lemma 2.4 *Let u be a Clifford-valued function in $W_0^{1,s}(\Omega, C\ell_n)$, $1 < s < \infty$. Then there exists a constant c such that*

$$\int_\Omega |\nabla u|^s \,\mathrm{d}x = \int_\Omega |\nabla TDu|^s \,\mathrm{d}x \leq c \int_\Omega |Du|^s \,\mathrm{d}x. \tag{8}$$

Remark 2.5 From Theorem 2.2 and Lemma 2.4, we know that, for Clifford-valued function $u \in W_0^{1,p}(\Omega, C\ell_n)$, we have

$$\int_\Omega \left(|u|^p + |\nabla u|^p\right) \mathrm{d}x \leq c(n,p,\Omega) \int_\Omega |Du|^p \,\mathrm{d}x.$$

3 Decomposition in Clifford-valued function space

In this section, we mainly discuss the properties of decomposition of Clifford-valued functions, these properties play an important role in studying the solutions of A-Dirac equations.

In [27], Kähler gave the following decomposition for Clifford-valued function space $L^s(\Omega, C\ell_n)$:

$$L^s(\Omega, C\ell_n) = \left[\ker D \cap L^s(\Omega, C\ell_n)\right] \oplus DW_0^{1,s}(\Omega, C\ell_n). \tag{9}$$

This means that for $\omega \in L^s(\Omega, C\ell_n)$, there exist the uniqueness $\alpha \in L^s(\Omega, C\ell_n) \cap \ker D$, $\beta \in W_0^{1,s}(\Omega, C\ell_n)$ such that $\omega = \alpha + D\beta$.

Let (X, μ) be a measure space and let E be a separable complex Hilbert space. Consider a bounded linear operator $F : L^s(X,E) \to L^s(X,E)$ for all $r \in [s_1, s_2]$, where $1 \leq s_1 < s_2 \leq \infty$. Denote its norm by $\|F\|_s$.

Lemma 3.1 ([11]) *Suppose that $\frac{s}{s_2} \leq 1 + \varepsilon \leq \frac{s}{s_1}$. Then*

$$\left\|F\left(|f|^\varepsilon\right)f\right\|_{\frac{s}{1+\varepsilon}} \leq K|\varepsilon|\|f\|_r^{1+\varepsilon} \tag{10}$$

for each $f \in L^s(X,E) \cap G$, where

$$K = \frac{2s(s_2 - s_1)}{(s - s_1)(s - s_2)}\left(\|F\|_{s_1} + \|F\|_{s_2}\right).$$

Remark 3.2 For Clifford-valued function $\omega = \alpha + D\beta \in W^{1,s}(\Omega, C\ell_n)$, let $F\omega = \alpha$, then we have $F(D\omega) = 0$.

Proof From (9), for $D\omega \in L^s(\Omega, C\ell_n)$, there exist

$$\alpha_1 \in \ker D \cap L^s(\Omega, C\ell_n), \qquad \beta_1 \in DW_0^{1,s}(\Omega, C\ell_n)$$

such that $D\omega = \alpha_1 + D\beta_1$, then $F(D\omega) = \alpha_1$. So we have $DD\omega = D\alpha_1 + DD\beta_1$, this means that

$$\begin{cases} \triangle(\omega - \beta_1) = 0, \\ \omega - \beta_1 \in W_0^{1,s}(\Omega, C\ell_n). \end{cases} \tag{11}$$

Hence we get $\omega = \beta_1$, then $D\omega = D\beta_1$. Then we get the final result directly. □

Lemma 3.3 *For each* $\omega \in W^{1,s}(\Omega, C\ell_n)$, $\max\{1, p-1\} \leq s < p$, *there exist* $\mu \in W_0^{1,\frac{s}{1+\varepsilon}}(\Omega, C\ell_n)$, $\pi \in L^{\frac{s}{1+\varepsilon}}(\Omega, C\ell_n)$ *such that*

$$|D\omega|^{\varepsilon} D\omega = D\mu + \pi, \tag{12}$$

and also

$$\begin{aligned} \|\pi\|_{\frac{s}{1+\varepsilon}} &= \big\|F\big(|D\omega|^{\varepsilon} D\omega\big)\big\|_{\frac{s}{1+\varepsilon}} \leq k|\varepsilon|\|D\omega\|_s^{1+\varepsilon}, \\ \|D\mu\|_{\frac{s}{1+\varepsilon}} &= \big\||D\omega|^{\varepsilon} D\omega - \pi\big\|_{\frac{s}{1+\varepsilon}} \leq C\|D\omega\|_s^{1+\varepsilon}. \end{aligned} \tag{13}$$

Proof We can get (12) from (9) immediately, so it is only needed to prove (13). It follows by the definition of the operator F that $F(D\omega) = 0$ and

$$F : W^{1,s}(\Omega, C\ell_n) \to W^{1,s}(\Omega, C\ell_n)$$

is a bounded linear mapping. And according to Lemma 3.1, we have

$$\|\pi\|_{\frac{s}{1+\varepsilon}} = \big\|F\big(|D\omega|^{\varepsilon} D\omega\big)\big\|_{\frac{s}{1+\varepsilon}} \leq k|\varepsilon|\|D\omega\|_s^{1+\varepsilon}.$$

Then by Minkowski's inequality, we get

$$\begin{aligned} \|D\mu\|_{\frac{s}{1+\varepsilon}} &= \big\||D\omega|^{\varepsilon} D\omega - \pi\big\|_{\frac{s}{1+\varepsilon}} \\ &\leq \big\||D\omega|^{\varepsilon} D\omega\big\|_{\frac{s}{1+\varepsilon}} + \|\pi\|_{\frac{s}{1+\varepsilon}} \\ &\leq c\|D\omega\|_s^{1+\varepsilon}, \end{aligned}$$

this completes the proof. □

4 Main results

In this section, we will show the convergence of very weak solutions of (4). Suppose that $\{u_{0,j}\}$, $j = 1, 2, \ldots$, is the sequence converging to u_0 in $W^{1,s}(\Omega, C\ell_n)$, and let $u_j \in W^{1,s}_{\mathrm{loc}}(\Omega, C\ell_n)$, $j = 1, 2, \ldots$, be the very weak solutions of the boundary value problem

$$\begin{cases} DA(x, Du_j) = 0, \\ u_j - u_{0,j} \in W_0^{1,s}(\Omega, C\ell_n), \end{cases} \tag{14}$$

where $\max\{1, p-1\} \leq s < p$.

The main result of this section is the following theorem.

Theorem 4.1 *Under the hypotheses above, for* $\max\{1, p-1\} \leq s \leq p$, *there exists* $u \in W^{1,s}(\Omega, C\ell_n)$ *such that* u_j *converges to* u *in* $W^{1,s}_{\mathrm{loc}}(\Omega, C\ell_n)$ *and* u *is a very weak solution to equation* (4) *satisfying* $u - u_0 \in W^{1,s}(\Omega, C\ell_n)$.

We first discuss the non-homogeneous A-Dirac equations

$$DA(x, g + D\omega) = 0, \tag{15}$$

where $g \in L^s(\Omega, C\ell_n)$.

Definition 4.2 The Clifford-valued function $\omega \in W^{1,s}(\Omega, C\ell_n)$ is called a very weak solution to (15), $\max\{1, p-1\} \leq s < p$, if

$$\int_\Omega \overline{A(x, g + D\omega)} D\varphi \, \mathrm{d}x = 0 \tag{16}$$

holds for all $\varphi \in W^{1, \frac{s}{s-p+1}}(\Omega, C\ell_n)$ with compact support.

Theorem 4.3 *Let $\omega \in W^{1,s}(\Omega, C\ell_n)$ be a very weak solution to* (15), *then there exists a constant c such that*

$$\int_\Omega |D\omega|^s \, \mathrm{d}x \leq c \int_\Omega |g|^s \, \mathrm{d}x. \tag{17}$$

Proof From Lemma 3.3, we have the following decomposition:

$$|D\omega|^s D\omega = D\varphi + f, \tag{18}$$

where $\varphi \in W_0^{1, \frac{p}{s-p+1}}(\Omega, C\ell_n)$, so φ can be as a test function in (16). Then we have

$$\int_\Omega \mathrm{Sc}\big(\overline{A(x, g + D\omega)} D\varphi\big) \, \mathrm{d}x = \int_\Omega \big\langle A(x, g + D\omega), D\varphi \big\rangle \, \mathrm{d}x = 0.$$

Combining with (18), it follows

$$\int_\Omega \big\langle A(x, D\omega), |D\omega|^{s-p} D\omega - f \big\rangle \, \mathrm{d}x = 0,$$

i.e.,

$$\begin{aligned} &\int_\Omega \big\langle A\big(x, |D\omega|^{s-p} D\omega\big)\big\rangle \, \mathrm{d}x \\ &\quad = \int_\Omega \big\langle A(x, D\omega) - A(x, g + D\omega), |D\omega|^{s-p} D\omega \big\rangle \, \mathrm{d}x \\ &\qquad + \int_\Omega \big\langle A(x, g + D\omega, f)\big\rangle \, \mathrm{d}x. \end{aligned}$$

Using the structure condition (N_3), (N_4), we get

$$a \int_\Omega |D\omega|^s \, \mathrm{d}x \leq b \int_\Omega |g|^{p-1} |D\omega|^{s-p+1} \, \mathrm{d}x + \int_\Omega |g + D\omega|^{p-1} |f| \, \mathrm{d}x.$$

Then, by Hölder, it yields

$$\begin{aligned} a \int_\Omega |D\omega|^s \, \mathrm{d}x &\leq b \left(\int_\Omega |g|^s \, \mathrm{d}x \right)^{\frac{p-1}{s}} \left(\int_\Omega |D\omega|^s \, \mathrm{d}x \right)^{\frac{s-p+1}{s}} \\ &\quad + \left(\int_\Omega |g + D\omega|^s \, \mathrm{d}x \right)^{\frac{p-1}{s}} \left(\int_\Omega |f|^{\frac{s}{s-p+1}} \, \mathrm{d}x \right)^{\frac{s-p+1}{s}}. \end{aligned}$$

According to Lemma 3.3, there is

$$\left(\int_\Omega |f|^{\frac{s}{s-p+1}}\,\mathrm{d}x\right)^{\frac{s-p+1}{s}} \le k|s-p|\left(\int_\Omega |D\omega|^s\,\mathrm{d}x\right)^{\frac{s-p+1}{s}}.$$

And then, combining with Young's inequality, we get

$$\begin{aligned} a\int_\Omega |D\omega|^s\,\mathrm{d}x &\le c\tau_1\int_\Omega |D\omega|^s\,\mathrm{d}x + c(\tau_1)\int_\Omega |g|^s\,\mathrm{d}x + c\tau_2\int_\Omega |D\omega|^s\,\mathrm{d}x \\ &\quad + c(\tau_2)\int_\Omega |D\omega|^s\,\mathrm{d}x + c(\tau_2)|s-p|\int_\Omega |D\omega|^s\,\mathrm{d}x \\ &\le \bigl(c\tau_1 + c\tau_2 + c(\tau_2)|s-p|\bigr)\int_\Omega |D\omega|^s\,\mathrm{d}x \\ &\quad + \bigl(c(\tau_1) + c(\tau_2)\bigr)\int_\Omega |g|^s\,\mathrm{d}x. \end{aligned}$$

We now determine τ_1, τ_2, ε to ensure that

$$\int_\Omega c\tau_1 + c\tau_2 + c(\tau_2)|s-p| \le \frac{a}{2}.$$

Thus

$$\int_\Omega |D\omega|^s\,\mathrm{d}x \le c\int_\Omega |g|^s\,\mathrm{d}x.$$

This proof is completed. □

Corollary 4.4 *Suppose that $u_0 \in W^{1,s}(\Omega, C\ell_n)$, $\max\{1, p-1\} \le s < p$, $u \in W^{1,s}(\Omega, C\ell_n)$ is a very weak solution to* (4) *with $u - u_0 \in W^{1,s}(\Omega, C\ell_n)$. Then*

$$\int_\Omega |Du|^s\,\mathrm{d}x \le c\int_\Omega |Du_0|^s\,\mathrm{d}x. \tag{19}$$

Proof Let $\omega = u - u_0$, we have $Du = D\omega + Du_0$. Since u is a very weak solution to (4), ω is a very weak solution to $DA(x, D\omega + Du_0) = 0$. From Theorem 4.3, we get

$$\int_\Omega |D\omega|^s\,\mathrm{d}x \le c\int_\Omega |Du_0|^s\,\mathrm{d}x.$$

By means of Minkowski's inequality, we obtain

$$\int_\Omega |Du|^s\,\mathrm{d}x \le c\int_\Omega |D\omega|^s\,\mathrm{d}x + c\int_\Omega |Du_0|^s\,\mathrm{d}x \le c\int_\Omega |Du_0|^s\,\mathrm{d}x.$$

This completes the proof. □

Proof of Theorem 4.1 By (19), we have the uniform bounds for $|Du_j|^s$,

$$\int_\Omega |Du_j|^s\,\mathrm{d}x \le c\int_\Omega |Du_{0,j}|^s\,\mathrm{d}x \le 2c\int_\Omega |Du_0|^s\,\mathrm{d}x,$$

when j is sufficiently large. Using Lemma 2.4, we have

$$\begin{aligned}\int_\Omega |\nabla u_j|^s \,\mathrm{d}x &\le c\int_\Omega \big(\big|\nabla(u_j-u_{0,j})\big|^s + |\nabla u_{0,j}|^s\big)\,\mathrm{d}x\\ &\le c\int_\Omega \big(\big|D(u_j-u_{0,j})\big|^s + |\nabla u_{0,j}|^s\big)\,\mathrm{d}x\\ &\le c'\left(\int_\Omega |Du_0|^s\,\mathrm{d}x + \int_\Omega |\nabla u_0|^s\,\mathrm{d}x\right).\end{aligned}$$

Write $u_j = \sum_I u_j^I e_I$, we have $u_j^I \in W^{1,s}(\Omega)$, and $\|u_j^I\|_{W^{1,s}(\Omega)} \le C$. Then there exists a subsequence, still denoted by $\{u_j^I\}$ and $u_I \in W^{1,s}(\Omega)$, such that

$$\begin{cases} u_j^I \rightharpoonup u_I, & \text{in } W^{1,s}(\Omega),\\ u_j^I \to u_I, & \text{in } L^s(\Omega),\\ u_j^I \to u_I, & \text{pointwise a.e. in } \Omega.\end{cases} \tag{20}$$

Let $u = \sum_I u_I e_I$, then $u_j \to u$ in $L^s(\Omega, C\ell_n)$, $u_j \to u$ pointwise a.e. Ω. Since $\nabla u_j^I \rightharpoonup u_I$ in $L^s(\Omega)$ for each $j = 1, 2, \ldots$, we have

$$\int_\Omega y_\alpha \frac{\partial u_j^I}{\partial x_j}\,\mathrm{d}x \to \int_\Omega y_\alpha \frac{\partial u_I}{\partial x_j}\,\mathrm{d}x,$$

whenever $y_\alpha \in L^{\frac{s}{s-1}}(\Omega)$. Then

$$\int_\Omega y Du_i\,\mathrm{d}x \to \int_\Omega y Du\,\mathrm{d}x$$

for each $y \in L^{\frac{s}{s-1}}(\Omega, C\ell_n)$, which implies that $Du_j \rightharpoonup Du$ in $L^s(\Omega)$.

The next stage is to extract a further subsequence, so that $Du_j \to Du$ pointwise a.e. in Ω. Through

$$\begin{aligned}\int_\Omega \big|\big|A(x,Du_j) - A(x,Du)\big||Du_j - Du|^{s-p}\big|^{\frac{s}{s-1}}\,\mathrm{d}x &\le b\int_\Omega |Du_j - Du|^s\,\mathrm{d}x\\ &\le c\int_\Omega \big(|Du|^s + |Du_j|^s\big)\,\mathrm{d}x < \infty,\end{aligned}$$

we know that $|A(x,Du_j) - A(x,Du)||Du_j - Du|^{s-p} \in L^{\frac{s}{s-1}}(\Omega)$, together with $Du_j \rightharpoonup Du$ in $L^s(\Omega)$ yields

$$\int_\Omega \big\langle A(x,Du_j) - A(x,Du), |Du_j - Du|^{s-p}(Du_j - Du)\big\rangle\,\mathrm{d}x \to 0 \quad (j\to\infty).$$

Then it follows from the structure condition (N_3) that

$$\begin{aligned}&a\int_\Omega |Du_j - Du|^s\,\mathrm{d}x\\ &\quad\le \int_\Omega \big\langle A(x,Du_j) - A(x,Du), |Du_j - Du|^{s-p}(Du_j - Du)\big\rangle\,\mathrm{d}x \to 0,\end{aligned}$$

that is to say $Du_j \to Du$ a.e. in Ω. Since $\xi \to A(x,\xi)$ is continuous, for each $\varphi \in W^{1,\frac{s}{s-p+1}}(\Omega, C\ell_n)$, we get

$$\int_\Omega \langle A(x,Du), D\varphi\rangle \, dx = \lim_{j\to\infty} \int_\Omega \langle A(x,Du_j), D\varphi\rangle \, dx = 0. \tag{21}$$

Next, we show that

$$\int_\Omega \overline{(A(x,Du))} D\varphi \, dx = 0. \tag{22}$$

Write $\overline{(A(x,Du))}D\varphi = \sum_J \nu_J e_J$, then $\langle A(x,Du), D\varphi\rangle = \mathrm{Sc}\, \overline{(A(x,Du))}D\varphi = \nu_0$. So (21) yields $\int_\Omega \nu_0 \, dx = 0$. Now, for each J, let $\varphi' = \varphi\overline{e_J}$, we find that $D\varphi' = (D\varphi)\overline{e_J}$ still in $W^{1,\frac{s}{s-p+1}}(\Omega, C\ell_n)$. Then φ' can be as a test function, so we obtain

$$0 = \int_\Omega \langle A(x,Du), D\varphi'\rangle \, dx = \int_\Omega \nu_J \, dx. \tag{23}$$

Thus, for each J, $\int_\Omega \nu_J \, dx = 0$, this implies

$$\int_\Omega \overline{(A(x,Du))} D\varphi \, dx = \left(\int_\Omega \nu_J \, dx\right) e_J = 0,$$

i.e., (22) holds for each $\varphi \in W^{1,\frac{s}{s-p+1}}(\Omega, C\ell_n)$ with compact support.

At last, we show that $u - u_0 \in W_0^{1,s}(\Omega, C\ell_n)$. Let $u_{0,j} = \sum_I u_{0,j}^I e_I$, $u_0 = \sum_I u_0^I e_I$. Since $u_{0,j} \to u_0$ in $W^{1,s}(\Omega, C\ell_n)$, we have $u_{0,j}^I \to u_0^I$ in $W^{1,s}(\Omega)$. On the other hand, $u_j^I \rightharpoonup u_I$ in $W^{1,s}(\Omega)$, this yields $u_j^I - u_{0,j}^I \rightharpoonup u_I - u_0^I$ in $W^{1,s}(\Omega)$. Also,

$$\int_\Omega \left|\nabla(u_j - u_{0,j})\right|^s dx \le c\int_\Omega |Du_j - Du_{0,j}|^s \, dx \le c\int_\Omega |Du_0|^s \, dx,$$

i.e., $u_j^I - u_{0,j}^I$ is bounded in $W_0^{1,s}(\Omega)$, then $u_I - u_0^I \in W_0^{1,s}(\Omega)$, which implies that $u - u_0 \in W_0^{1,s}(\Omega, C\ell_n)$. So we obtain that $u \in W^{1,s}(\Omega, C\ell_n)$ is the very weak solution to equation (4), the theorem follows. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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