Positive solutions for a class of fractional differential equations at resonance

Yanqiang Wu[*] and Wenbin Liu

[*]Correspondence:
wyq1976819@126.com
College of Sciences, China
University of Mining and
Technology, Xuzhou, 221008, China

Abstract

In this paper, by using the Leggett-Williams norm-type theorem, we consider a m-point boundary value problem for a class of fractional differential equations at resonance. A new result on the existence of solutions for above problem is obtained.

MSC: 34A08; 34B15

Keywords: fractional differential equation; m-point boundary value problem; Leggett-Williams norm-type theorem; resonance

1 Introduction

The subject of fractional calculus has gained significant interest and been a valuable tool for both science and engineering (see [1–3]). In recent years, the fractional boundary value problems (FBVPs for short) have been considered by many authors (see [4–10] and the references therein). For example, Bai studied a FBVP at non-resonance with $1 < \alpha \leq 2$ (see [10]). FBVPs at resonance were studied by Kosmatov (see [11]) and Jiang (see [12]). But the positive solutions for FBVPs at resonance were studied very few. In [13], Yang and Wang considered the positive solutions of the following FBVP:

$$\begin{cases} -D_{0^+}^\alpha x(t) = f(t, x(t)), & t \in [0,1], \\ x(0) = 0, & x'(0) = x'(1). \end{cases}$$

In [14], Chen and Tang studied the positive solution of FBVP as follows:

$$\begin{cases} D_{0^+}^\alpha x(t) = f(t, x(t)), & t \in [0, +\infty), \\ x(0) = x'(0) = x''(0) = 0, & D_{0^+}^{\alpha-1} x(0) = \lim_{t \to +\infty} D_{0^+}^{\alpha-1} x(t). \end{cases}$$

However, to the best of our knowledge, the fractional differential equations with m-point boundary conditions at resonance have not been considered. Motivated by the papers above, we consider the existence of positive solutions for a m-point FBVP of the form

$$\begin{cases} -D_{0^+}^\alpha x(t) = f(t, x(t)), & t \in [0,1], \\ x'(0) = 0, & x(1) = \sum_{i=1}^{m-2} \beta_i x(\eta_i), \end{cases} \tag{1.1}$$

where $D_{0^+}^\alpha$ denotes the standard Caputo fractional differential operator of order α, $1 < \alpha \leq 2$, $\beta_i \in \mathbb{R}^+$, $\sum_{i=1}^{m-2} \beta_i = 1$, $0 < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < 1$, and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous. Obviously, FBVP (1.1) happens to be at resonance under the condition $\sum_{i=1}^{m-2} \beta_i = 1$.

Springer

The rest of this paper is organized as follows. Section 2 contains some necessary notations, definitions and lemmas. In Section 3, we establish a theorem on the existence of positive solutions for FBVP (1.1) under some restrictions of f, basing on the coincidence degree theory due to [15]. Finally, in Section 4, an example is given to illustrate the main result.

2 Preliminaries

For convenience of the reader, we present some definitions, notations, and preliminary statements, which can be found in [2, 16, 17].

Let X and Y be real Banach spaces, $L : \mathrm{dom}\, L \subset X \to Y$ be a Fredholm operator with index zero, where the index of a Fredholm operator L is defined by

$$\mathrm{Index}\, L := \dim \mathrm{Ker}\, L - \dim \mathrm{Coker}\, L.$$

Suppose $P : X \to X$, $Q : Y \to Y$ be continuous linear projectors such that

$$\mathrm{Im}\, P = \mathrm{Ker}\, L, \qquad \mathrm{Ker}\, Q = \mathrm{Im}\, L,$$

$$X = \mathrm{Ker}\, L \oplus \mathrm{Ker}\, P, \qquad Y = \mathrm{Im}\, L \oplus \mathrm{Im}\, Q.$$

Thus, we see that

$$L|_{\mathrm{dom}\, L \cap \mathrm{Ker}\, P} : \mathrm{dom}\, L \cap \mathrm{Ker}\, P \to \mathrm{Im}\, L$$

is invertible. We denote the inverse by K_P. Moreover, by virtue of $\dim \mathrm{Im}\, Q = \mathrm{codim}\, \mathrm{Im}\, L$, there exists an isomorphism $J : \mathrm{Im}\, Q \to \mathrm{Ker}\, L$. Then we know that the operator equation $Lx = Nx$ is equivalent to

$$x = (P + JQN)x + K_P(I - Q)Nx,$$

where $N : X \to Y$ be a nonlinear operator.

If Ω is an open bounded subset of X such that $\mathrm{dom}\, L \cap \overline{\Omega} \neq \emptyset$, then the map $N : X \to Y$ will be called L-compact on $\overline{\Omega}$ if $QN : \overline{\Omega} \to Y$ is bounded and $K_P(I - Q)N : \overline{\Omega} \to X$ is compact.

Let C be a cone in X. Then C induces a partial order in X by

$$x \leq y \quad \text{if and only if} \quad y - x \in C.$$

Lemma 2.1 (see [15]) *Let C be a cone in X. Then for every $u \in C \setminus \{0\}$ there exists a positive number $\sigma(u)$ such that*

$$\|x + u\| \geq \sigma(u)\|x\|$$

for all $x \in C$.

Let $\gamma : X \to C$ be a retraction, that is, a continuous mapping such that $\gamma(x) = x$ for all $x \in C$. Set

$$\Psi := P + JQN + K_P(I - Q)N$$

and

$$\Psi_\gamma := \Psi \circ \gamma.$$

Lemma 2.2 (see [15]) *Let C be a cone in X and Ω_1, Ω_2 be open bounded subsets of X with $\overline{\Omega}_1 \subset \Omega_2$ and $C \cap (\overline{\Omega}_2 \setminus \Omega_1) \neq \emptyset$. Assume that the following conditions are satisfied:*

(1) *$L : \operatorname{dom} L \subset X \to Y$ be a Fredholm operator of index zero and $N : X \to Y$ be L-compact on every bounded subset of X,*

(2) *$Lx \neq \lambda Nx$ for every $(x, \lambda) \in [C \cap \partial\Omega_2 \cap \operatorname{dom} L] \times (0, 1)$,*

(3) *γ maps subsets of $\overline{\Omega}_2$ into bounded subsets of C,*

(4) *$\deg([I - (P + JQN)\gamma]|_{\operatorname{Ker} L}, \operatorname{Ker} L \cap \Omega_2, 0) \neq 0$,*

(5) *there exists $u_0 \in C \setminus \{0\}$ such that $\|x\| \leq \sigma(u_0)\|\Psi x\|$ for $x \in C(u_0) \cap \partial\Omega_1$, where $C(u_0) = \{x \in C : \mu u_0 \leq x \text{ for some } \mu > 0\}$ and $\sigma(u_0)$ is such that $\|x + u_0\| \geq \sigma(u_0)\|x\|$ for every $x \in C$,*

(6) *$(P + JQN)\gamma(\partial\Omega_2) \subset C$,*

(7) *$\Psi_\gamma(\overline{\Omega}_2 \setminus \Omega_1) \subset C$.*

Then the equation $Lx = Nx$ has at least one solution in $C \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Definition 2.3 (see [17]) The Riemann-Liouville fractional integral operator of order $\alpha > 0$ of a function x is given by

$$I_{0^+}^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} x(s) \, ds,$$

provided that the right side integral is pointwise defined on $(0, +\infty)$.

Definition 2.4 (see [17]) The Caputo fractional derivative of order $\alpha > 0$ of a continuous function x is given by

$$D_{0^+}^\alpha x(t) = \frac{1}{\Gamma(n - \alpha)} \int_0^t (t - s)^{n - \alpha - 1} x^{(n)}(s) \, ds,$$

where n is the smallest integer greater than or equal to α, provided that the right side integral is pointwise defined on $(0, +\infty)$.

Lemma 2.5 (see [18]) *For $\alpha > 0$, the general solution of the Caputo fractional differential equation*

$$D_{0^+}^\alpha x(t) = 0$$

is

$$x(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1},$$

where $c_i \in \mathbb{R}$, $i = 0, 1, \ldots, n - 1$, here n is the smallest integer greater than or equal to α.

Lemma 2.6 (see [18]) *Suppose that $D_{0^+}^\alpha x \in C[0, 1]$, $\alpha > 0$. Then*

$$I_{0^+}^\alpha D_{0^+}^\alpha x(t) = x(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1},$$

where $c_i \in \mathbb{R}$, $i = 0, 1, \ldots, n - 1$, here n is the smallest integer greater than or equal to α.

In this paper, we take $X = Y = C[0, 1]$ with the norm $\|x\|_\infty = \max_{t \in [0,1]} |x(t)|$.

Define the operator $L : \operatorname{dom} L \subset X \to Y$ by

$$Lx = -D_{0^+}^\alpha x, \tag{2.1}$$

where

$$\operatorname{dom} L = \left\{ x \in X : D_{0^+}^\alpha x \in Y, x'(0) = 0, x(1) = \sum_{i=1}^{m-2} \beta_i x(\eta_i) \right\}.$$

Let $N : X \to Y$ be the Nemytskii operator

$$Nx(t) = f\big(t, x(t)\big), \quad \forall t \in [0, 1].$$

Then FBVP (1.1) is equivalent to the operator equation

$$Lx = Nx, \quad x \in \operatorname{dom} L.$$

3 Main result

In this section, a theorem on the existence of positive solutions for FBVP (1.1) will be given.

For simplicity of notation, we set

$$l_i(s) = \begin{cases} (1-s)^{\alpha-1} - (\eta_i - s)^{\alpha-1}, & 0 \le s \le \eta_i \le 1, \\ (1-s)^{\alpha-1}, & 0 \le \eta_i \le s \le 1, \end{cases}$$

and

$$G(t, s) = \begin{cases} \dfrac{1}{\Gamma(\alpha+1)}(1-s)^\alpha \\ \qquad + \dfrac{\alpha(\Gamma(\alpha+2)-1+(\alpha+1)t^\alpha)}{\Gamma(\alpha+2)\sum_{i=1}^{m-2}\beta_i(1-\eta_i^\alpha)} \sum_{i=1}^{m-2} \beta_i l_i(s), & 0 \le t \le s \le 1, \\ \dfrac{1}{\Gamma(\alpha+1)}(1-s)^\alpha - \dfrac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1} \\ \qquad + \dfrac{\alpha(\Gamma(\alpha+2)-1+(\alpha+1)t^\alpha)}{\Gamma(\alpha+2)\sum_{i=1}^{m-2}\beta_i(1-\eta_i^\alpha)} \sum_{i=1}^{m-2} \beta_i l_i(s), & 0 \le s \le t \le 1. \end{cases}$$

Obviously, $\max_{0 \le s \le 1} \sum_{i=1}^{m-2} \beta_i l_i(s) \le 1$. We denote

$$\kappa := \min \left\{ 1, \frac{\sum_{i=1}^{m-2} \beta_i(1-\eta_i^\alpha)}{\alpha}, \frac{1}{\max_{t,s \in [0,1]} G(t,s)} \right\}.$$

Thus, one has

$$1 - \frac{\kappa\alpha \sum_{i=1}^{m-2} \beta_i l_i(s)}{\sum_{i=1}^{m-2} \beta_i(1-\eta_i^\alpha)} \ge 0, \quad 1 - \kappa G(t,s) \ge 0. \tag{3.1}$$

Theorem 3.1 *Let $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ be continuous. Suppose that*:

(H$_1$) *there exist nonnegative functions $a, b \in X$ with $\frac{\Gamma(\alpha+1)}{2} b_1 < 1$ such that*

$$\big|f(t, u)\big| \le a(t) + b(t)|u|, \quad \forall t \in [0, 1], u \in \mathbb{R},$$

where $b_1 = \|b\|_\infty$,

(H_2) *there exists a constant $B > 0$ such that*

$$uf(t, u) < 0, \quad \forall t \in [0, 1], |u| > B,$$

(H_3) $f(t, u) > -\kappa u$, *for all* $(t, u) \in [0, 1] \times [0, \infty)$,

(H_4) *there exist* $r \in (0, +\infty)$, $t_0 \in [0, 1]$, $M \in (0, 1)$ *and continuous function* $h : (0, r] \to [0, \infty)$ *such that* $f(t, u) \geq h(u)$ *for all* $t \in [0, 1]$, $u \in (0, r]$, *and* $\frac{h(u)}{u}$ *is non-increasing on* $(0, r]$ *with*

$$\frac{h(r)}{r} \int_0^1 G(t_0, s)\,ds \geq \frac{1 - M}{M}.$$

Then FBVP (1.1) *has at least one solution in* X.

Now, we begin with some lemmas that are useful in what follows.

Lemma 3.2 *Let L be defined by* (2.1), *then*

$$\operatorname{Ker} L = \left\{ x \in X : x(t) = c, \forall t \in [0, 1], c \in \mathbb{R} \right\}, \tag{3.2}$$

$$\operatorname{Im} L = \left\{ y \in Y : \sum_{i=1}^{m-2} \beta_i \int_0^1 l_i(s) y(s)\,ds = 0 \right\}. \tag{3.3}$$

Proof By Lemma 2.5, $D_{0^+}^\alpha x(t) = 0$ has solution

$$x(t) = c_0 + c_1 t, \quad c_0, c_1 \in \mathbb{R}.$$

Combining with the boundary conditions of FBVP (1.1), one sees that (3.2) holds.

For $y \in \operatorname{Im} L$, there exists $x \in \operatorname{dom} L$ such that $y = Lx \in Y$. By Lemma 2.6, we have

$$x(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} y(s)\,ds + c_0 + c_1 t, \quad c_0, c_1 \in \mathbb{R}.$$

Then we get

$$x'(t) = -\frac{1}{\Gamma(\alpha - 1)} \int_0^t (t - s)^{\alpha - 2} y(s)\,ds + c_1.$$

By the boundary conditions of FBVP (1.1), we see that y satisfies

$$\int_0^1 (1 - s)^{\alpha - 1} y(s)\,ds = \sum_{i=1}^{m-2} \beta_i \int_0^{\eta_i} (\eta_i - s)^{\alpha - 1} y(s)\,ds.$$

That is,

$$\sum_{i=1}^{m-2} \beta_i \int_0^1 l_i(s) y(s)\,ds = 0. \tag{3.4}$$

On the other hand, suppose $y \in Y$ and satisfies (3.4). Let $x(t) = -I_{0^+}^\alpha y(t) + x(0)$, then $x \in \operatorname{dom} L$ and $D_{0^+}^\alpha x(t) = -y(t)$. Thus, $y \in \operatorname{Im} L$. Hence (3.3) holds. The proof is complete. □

Lemma 3.3 *Let L be defined by* (2.1), *then L is a Fredholm operator of index zero, and the linear continuous projector operators* $P : X \to X$ *and* $Q : Y \to Y$ *can be defined as*

$$Px(t) = \int_0^1 x(s)\,ds, \quad \forall t \in [0,1],$$

$$Qy(t) = \frac{\alpha}{\sum_{i=1}^{m-2} \beta_i(1 - \eta_i^\alpha)} \sum_{i=1}^{m-2} \beta_i \int_0^1 l_i(s)y(s)\,ds, \quad \forall t \in [0,1].$$

Furthermore, the operator $K_P : \operatorname{Im} L \to \operatorname{dom} L \cap \operatorname{Ker} P$ *can be written by*

$$K_P y(t) = \int_0^1 k(t,s)y(s)\,ds, \quad \forall t \in [0,1],$$

where

$$k(t,s) = \begin{cases} \frac{1}{\Gamma(\alpha+1)}(1-s)^\alpha, & 0 \le t \le s \le 1, \\ \frac{1}{\Gamma(\alpha+1)}(1-s)^\alpha - \frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1}, & 0 \le s \le t \le 1. \end{cases} \tag{3.5}$$

Proof Obviously, $\operatorname{Im} P = \operatorname{Ker} L$ and $P^2 x = Px$. It follows from $x = (x - Px) + Px$ that $X = \operatorname{Ker} P + \operatorname{Ker} L$. By a simple calculation, one obtain $\operatorname{Ker} L \cap \operatorname{Ker} P = \{0\}$. Thus, we get

$$X = \operatorname{Ker} L \oplus \operatorname{Ker} P.$$

For $y \in Y$, we have

$$Q^2 y = Q(Qy) = Qy \cdot \frac{\alpha}{\sum_{i=1}^{m-2} \beta_i(1 - \eta_i^\alpha)} \sum_{i=1}^{m-2} \beta_i \int_0^1 l_i(s)\,ds = Qy.$$

Let $y = (y - Qy) + Qy$, where $y - Qy \in \operatorname{Ker} Q$, $Qy \in \operatorname{Im} Q$. It follows from $\operatorname{Ker} Q = \operatorname{Im} L$ and $Q^2 y = Qy$ that $\operatorname{Im} Q \cap \operatorname{Im} L = \{0\}$. Then one has

$$Y = \operatorname{Im} L \oplus \operatorname{Im} Q.$$

Thus, we obtain

$$\dim \operatorname{Ker} L = \dim \operatorname{Im} Q = \dim \operatorname{Coker} L = 1.$$

That is, L is a Fredholm operator of index zero.

Now, we will prove that K_P is the inverse of $L|_{\operatorname{dom} L \cap \operatorname{Ker} P}$. In fact, for $y \in \operatorname{Im} L$, we have

$$K_P y(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\,ds + c_0, \tag{3.6}$$

where

$$c_0 = \frac{1}{\Gamma(\alpha+1)} \int_0^1 (1-s)^\alpha y(s)\,ds.$$

It is easy to see that $LK_P y = y$. Moreover, for $x \in \operatorname{dom} L \cap \operatorname{Ker} P$, we get $x'(0) = 0$ and

$$
\begin{aligned}
K_P L x(t) &= I_{0^+}^\alpha D_{0^+}^\alpha x(t) - I_{0^+}^{\alpha+1} D_{0^+}^\alpha x(t)|_{t=1} \\
&= x(t) - x(0) - \int_0^1 \big(x(s) - x(0)\big)\, ds \\
&= x - Px.
\end{aligned}
\tag{3.7}
$$

Combining (3.6) with (3.7), we know that $K_P = (L|_{\operatorname{dom} L \cap \operatorname{Ker} P})^{-1}$. The proof is complete. □

Lemma 3.4 *Assume $\Omega \subset X$ is an open bounded subset such that $\operatorname{dom} L \cap \overline{\Omega} \neq \emptyset$, then N is L-compact on $\overline{\Omega}$.*

Proof By the continuity of f, we see that $QN(\overline{\Omega})$ and $K_P(I - Q)N(\overline{\Omega})$ are bounded. That is, there exist constants $A, B > 0$ such that $|(I - Q)Nx| \leq A$ and $|K_P(I - Q)Nx| \leq B$, $\forall x \in \overline{\Omega}$, $t \in [0,1]$. Thus, one need only prove that $K_P(I - Q)N(\overline{\Omega}) \subset X$ is equicontinuous.

Let $K_{P,Q} = K_P(I - Q)N$, for $0 \leq t_1 < t_2 \leq 1$, $x \in \overline{\Omega}$, we get

$$
\begin{aligned}
&\big|(K_{P,Q}x)(t_2) - (K_{P,Q}x)(t_1)\big| \\
&\leq \frac{1}{\Gamma(\alpha)} \left| \int_0^{t_2} (t_2 - s)^{\alpha-1} (I - Q)Nx(s)\, ds - \int_0^{t_1} (t_1 - s)^{\alpha-1} (I - Q)Nx(s)\, ds \right| \\
&\leq \frac{A}{\Gamma(\alpha)} \left[\int_0^{t_1} (t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}\, ds + \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1}\, ds \right] \\
&= \frac{A}{\Gamma(\alpha+1)} \big(t_2^\alpha - t_1^\alpha\big).
\end{aligned}
$$

Since t^α is uniformly continuous on $[0,1]$, we see that $K_{P,Q}N(\overline{\Omega}) \subset X$ is equicontinuous. Thus, we see that $K_{P,Q}N : \overline{\Omega} \to X$ is compact. The proof is completed. □

Lemma 3.5 *Suppose (H_1) and (H_2) hold, then the set*

$$
\Omega_0 = \big\{ x \in \operatorname{dom} L : Lx = \lambda Nx, \lambda \in (0,1) \big\}
$$

is bounded.

Proof Take $x \in \Omega_0$, then $Nx \in \operatorname{Im} L$. By (3.2), we have

$$
\sum_{i=1}^{m-2} \beta_i \int_0^1 l_i(s) f\big(s, x(s)\big)\, ds = 0.
$$

Then, by the integral mean value theorem, there exists a constant $\xi \in (0,1)$ such that $f(\xi, x(\xi)) = 0$. So, from (H_2), we get $|x(\xi)| \leq B$. By Lemma 2.6, one has

$$
x(t) = x(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} D_{0^+}^\alpha x(s)\, ds,
$$

$$
x(\xi) = x(0) + \frac{1}{\Gamma(\alpha)} \int_0^\xi (\xi - s)^{\alpha-1} D_{0^+}^\alpha x(s)\, ds.
$$

Thus, we get

$$x(t) - x(\xi) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} D_{0^+}^\alpha x(s)\, ds$$

$$- \frac{1}{\Gamma(\alpha)} \int_0^\xi (\xi-s)^{\alpha-1} D_{0^+}^\alpha x(s)\, ds,$$

which together with (H$_1$) implies that

$$\left| x(t) \right| \le \left| x(\xi) \right| + \frac{1}{\Gamma(\alpha)} \left\| D_{0^+}^\alpha x \right\|_\infty \cdot \frac{1}{\alpha} \left(t^\beta + \xi^\beta \right)$$

$$\le B + \frac{2}{\Gamma(\alpha+1)} \left\| D_{0^+}^\alpha x \right\|_\infty$$

$$\le B + \frac{2}{\Gamma(\alpha+1)} \cdot \max_{t \in [0,1]} \left| f\left(t, x(t)\right) \right|$$

$$\le B + \frac{2}{\Gamma(\alpha+1)} \left(\|a\|_\infty + b_1 \|x\|_\infty \right), \quad \forall t \in [0,1].$$

That is,

$$\|x\|_\infty \le B + \frac{2}{\Gamma(\alpha+1)} \left(\|a\|_\infty + b_1 \|x\|_\infty \right).$$

In view of $\frac{2}{\Gamma(\alpha+1)} b_1 < 1$, there exists a constant $D_2 > 0$ such that

$$\|x\|_\infty \le D_2.$$

Hence, Ω_0 is bounded. The proof is complete. $\qquad\square$

Proof of Theorem 3.1 Set $C = \{x \in X : x(t) \ge 0, t \in [0,1]\}$, $\Omega_1 = \{x \in X : r > |x(t)| > M\|x\|_\infty, t \in [0,1]\}$, and $\Omega_2 = \{x \in X : \|x\|_\infty < R\}$, where $R = \max\{B, D_2\}$. Clearly, Ω_1, Ω_2 are open bounded subsets of X and

$$\overline{\Omega}_1 = \left\{ x \in X : r \ge \left| x(t) \right| \ge M\|x\|_\infty, t \in [0,1] \right\} \subset \Omega_2.$$

From Lemma 3.3, Lemma 3.4, and Lemma 3.5, we see that the conditions (1) and (2) of Lemma 2.2 are satisfied.

Let $\gamma x(t) = |x(t)|$ for $x \in X$ and $J = I$. One can see that γ is a retraction and maps subsets of $\overline{\Omega}_2$ into bounded subsets of C, which means that the condition (3) of Lemma 2.2 holds.

For $x \in \operatorname{Ker} L \cap \Omega_2$, we have $x(t) \equiv c$. Let

$$H(c, \lambda) = c - \lambda|c| - \frac{\lambda\alpha}{\sum_{i=1}^{m-2} \beta_i(1-\eta_i^\alpha)} \sum_{i=1}^{m-2} \beta_i \int_0^1 l_i(s) f\left(s, |c|\right) ds.$$

From $H(c, \lambda) = 0$, one has $c \ge 0$. Moreover, if $H(R, \lambda) = 0$, we get

$$0 \le R(1-\lambda) = \frac{\lambda\alpha}{\sum_{i=1}^{m-2} \beta_i(1-\eta_i^\alpha)} \sum_{i=1}^{m-2} \beta_i \int_0^1 l_i(s) f(s, R)\, ds,$$

which contradicts (H_2). Thus $H(c, \lambda) \neq 0$ for $x \in \partial\Omega_2$, $\lambda \in [0, 1]$. Hence

$$
\deg\big(\big[I - (P + JQN)\gamma\big]|_{\mathrm{Ker}\,L}, \mathrm{Ker}\,L \cap \Omega_2, 0\big)
$$

$$
= \deg\big(H(c, 1), \mathrm{Ker}\,L \cap \Omega_2, 0\big)
$$

$$
= \deg\big(H(c, 0), \mathrm{Ker}\,L \cap \Omega_2, 0\big)
$$

$$
= \deg(I, \mathrm{Ker}\,L \cap \Omega_2, 0)
$$

$$
= 1.
$$

So, the condition (4) of Lemma 2.2 holds.

Let $x \in \overline{\Omega}_2 \setminus \Omega_1$, $t \in [0, 1]$, we have

$$
\Psi_\gamma x(t) = \int_0^1 |x(s)|\, ds + \frac{\alpha}{\sum_{i=1}^{m-2} \beta_i(1 - \eta_i^\alpha)} \sum_{i=1}^{m-2} \beta_i \int_0^1 l_i(s) f\big(s, |x(s)|\big)\, ds
$$

$$
+ \int_0^1 k(t, s)\left[f\big(s, |x(s)|\big) - \frac{\alpha}{\sum_{i=1}^{m-2} \beta_i(1 - \eta_i^\alpha)} \sum_{i=1}^{m-2} \beta_i \int_0^1 l_i(\tau) f\big(\tau, |x(\tau)|\big)\, d\tau\right] ds
$$

$$
= \int_0^1 |x(s)|\, ds + \int_0^1 G(t, s) f\big(s, |x(s)|\big)\, ds,
$$

which together with (H_3) and (3.1) yields

$$
\Psi_\gamma x(t) \geq \int_0^1 |x(s)|\, ds - \kappa \int_0^1 G(t, s)|x(s)|\, ds = \int_0^1 \big(1 - \kappa G(t, s)\big)|x(s)|\, ds \geq 0.
$$

Thus, the condition (7) of Lemma 2.2 holds. In addition, we can prove the condition (6) of Lemma 2.2 holds too by a similar process.

Finally, we will show that the condition (5) of Lemma 2.2 is satisfied. Let $u_0(t) \equiv 1$, $t \in [0, 1]$, then $u_0 \in C \setminus \{0\}$, $C(u_0) = \{x \in C : x(t) > 0, t \in [0, 1]\}$ and we can take $\sigma(u_0) = 1$. For $x \in C(u_0) \cap \partial\Omega_1$, we have $x(t) > 0$, $t \in [0, 1]$, $0 < \|x\|_\infty \leq r$, and $x(t) \geq M\|x\|_\infty$, $t \in [0, 1]$. So, from (H_4), we obtain

$$
\Psi x(t_0) = \int_0^1 x(s)\, ds + \int_0^1 G(t_0, s) f\big(s, x(s)\big)\, ds
$$

$$
\geq M\|x\| + \int_0^1 G(t_0, s) h\big(x(s)\big)\, ds
$$

$$
= M\|x\| + \int_0^1 G(t_0, s) \frac{h(x(s))}{x(s)} x(s)\, ds
$$

$$
\geq M\|x\| + \frac{h(r)}{r} \int_0^1 G(t_0, s) x(s)\, ds
$$

$$
\geq M\|x\| + \frac{h(r)}{r} \int_0^1 G(t_0, s) M\|x\|\, ds
$$

$$
\geq M\|x\| + (1 - M)\|x\|
$$

$$
= \|x\|.
$$

Then the condition (5) of Lemma 2.2 holds.

Consequently, by Lemma 2.2, the equation $Lx = Nx$ has at least one solution $x^* \in C \cap (\overline{\Omega}_2 \setminus \Omega_1)$. Namely, FBVP (1.1) has at least one positive solution in X. The proof is complete. □

4 Example

We consider the following FBVP:

$$\begin{cases} -D_{0^+}^{\frac{3}{2}} x(t) = \sin t - \frac{1}{10} x(t) + 10 + \sin x(t), & t \in [0,1], \\ x'(0) = 0, \qquad x(1) = x(\frac{1}{2}). \end{cases} \tag{4.1}$$

Thus, we have

$$l(s) = \begin{cases} \sqrt{1-s} - \sqrt{\frac{1}{2}-s}, & 0 \le s \le \frac{1}{2}, \\ \sqrt{1-s}, & \frac{1}{2} \le s \le 1, \end{cases}$$

$$G(t,s) = \begin{cases} \frac{1}{\Gamma(\frac{5}{2})}(1-s)^{\frac{3}{2}} + \frac{3(\Gamma(\frac{7}{2})-1+(\frac{5}{2})t^{\frac{3}{2}})}{2\Gamma(\frac{7}{2})(1-(\frac{1}{2})^{\frac{3}{2}})}l(s), & 0 \le t \le s \le 1, \\ \frac{1}{\Gamma(\frac{5}{2})}(1-s)^{\frac{3}{2}} - \frac{1}{\Gamma(\frac{3}{2})}(t-s)^{\frac{1}{2}} + \frac{3(\Gamma(\frac{7}{2})-1+(\frac{5}{2})t^{\frac{3}{2}})}{2\Gamma(\frac{7}{2})(1-(\frac{1}{2})^{\frac{3}{2}})}l(s), & 0 \le s \le t \le 1. \end{cases}$$

Moreover, $f(t,u) \ge 8 - \frac{1}{10}u \ge -\frac{1}{4}u$ for all $u \ge 0$, and $l(s) \le 1$, $G(t,s) \le 4$, $\kappa = -\frac{1}{4}$. So, we can find that (H$_1$), (H$_2$), (H$_3$) hold. Next, we take $t_0 = 0$, $h(x) = x$, and $M = \frac{2}{3}$, thus $G(0,s) = \frac{1}{\Gamma(\frac{5}{2})}(1-s)^{\frac{3}{2}} + \frac{3(\Gamma(\frac{7}{2})-1)}{2\Gamma(\frac{7}{2})(1-(\frac{1}{2})^{\frac{3}{2}})}l(s)$, $0 \le s \le 1$, and $\int_0^1 G(0,s)\,ds = 1$. Then (H$_4$) is satisfied. According to the above points, by Theorem 3.1, we can conclude that FBVP (4.1) has at least one positive solution.

References

1. Glockle, WG, Nonnenmacher, TF: A fractional calculus approach of self-similar protein dynamics. Biophys. J. **68**, 46-53 (1995)
2. Oldham, KB, Spanier, J: The Fractional Calculus. Academic Press, New York (1974)
3. Diethelm, K, Freed, AD: On the solution of nonlinear fractional order differential equations used in the modeling of viscoplasticity. In: Keil, F, Mackens, W, Voss, H, Werther, J (eds.) Scientific Computing in Chemical Engineering II. Computational Fluid Dynamics, Reaction Engineering and Molecular Properties, pp. 217-224. Springer, Heidelberg (1999)
4. Agarwal, RP, O'Regan, D, Stanek, S: Positive solutions for Dirichlet problems of singular nonlinear fractional differential equations. J. Math. Anal. Appl. **371**, 57-68 (2010)
5. Bai, Z, Lü, H: Positive solutions for boundary value problem of nonlinear fractional differential equation. J. Math. Anal. Appl. **311**, 495-505 (2005)
6. Kaufmann, ER, Mboumi, E: Positive solutions of a boundary value problem for a nonlinear fractional differential equation. Electron. J. Qual. Theory Differ. Equ. **2008**, 3 (2008)
7. Zhang, S: Positive solutions for boundary-value problems of nonlinear fractional differential equations. Electron. J. Differ. Equ. **2006**, 36 (2006)
8. Jafari, H, Gejji, VD: Positive solutions of nonlinear fractional boundary value problems using Adomian decomposition method. Appl. Math. Comput. **180**, 700-706 (2006)

9. Liang, S, Zhang, J: Positive solutions for boundary value problems of nonlinear fractional differential equation. Nonlinear Anal. **71**, 5545-5550 (2009)
10. Bai, Z, Zhang, Y: Solvability of fractional three-point boundary value problems with nonlinear growth. Appl. Math. Comput. **218**, 1719-1725 (2011)
11. Kosmatov, N: A boundary value problem of fractional order at resonance. Electron. J. Differ. Equ. **2010**, 135 (2010)
12. Jiang, W: The existence of solutions to boundary value problems of fractional differential equations at resonance. Nonlinear Anal. **74**, 1987-1994 (2011)
13. Yang, A, Wang, H: Positive solutions of two-point boundary value problems of nonlinear fractional differential equation at resonance. Electron. J. Qual. Theory Differ. Equ. **2011**, 71 (2011)
14. Chen, Y, Tang, X: Positive solutions of fractional differential equations at resonance on the half-line. Bound. Value Probl. **2012**, 64 (2012)
15. O'Regan, D, Zima, M: Leggett-Williams norm-type theorems for coincidences. Arch. Math. **87**, 222-244 (2006)
16. Mainardi, F: Fractional calculus: some basic problems in continuum and statistical mechanics. In: Carpinteri, A, Mainardi, F (eds.) Fractals and Fractional Calculus in Continuum Mechanics, pp. 291-348. Springer, Wien (1997)
17. Metzler, F, Schick, W, Kilian, HG, Nonnenmacher, TF: Relaxation in filled polymers: a fractional calculus approach. J. Chem. Phys. **103**, 7180-7186 (1995)
18. Lakshmikantham, V, Leela, S, Vasundhara Devi, J: Theory of Fractional Dynamic Systems. Cambridge Academic Publishers, Cambridge (2009)