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A note on the Barnes-type q-Euler polynomials

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Abstract

In this paper, we consider the Barnes-type q-Euler polynomials which are derived from the fermionic p-adic q-integrals and investigate some identities of these polynomials. Furthermore, we define the Barnes-type q-Changhee polynomials and numbers, and we derive some identities related with the Barnes-type q-Euler polynomials and the Barnes-type q-Changhee polynomials.

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1 Introduction

Let p be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will, respectively, denote the rings of p-adic integers, the fields of p-adic numbers, and the completion of the algebraic closure of $\mathbb{Q}_p$. The p-adic norm $|\cdot|_p$ is normalized as $|p|_p = \frac{1}{p}$. The space of continuous functions on $\mathbb{Z}_p$ is denoted by $C(\mathbb{Z}_p)$. Let q be an element in $\mathbb{C}_p$ with $|1-q|_p < p^{-\frac{1}{p-1}}$. The q-number of x is defined by $[x]_q = \frac{1-q^x}{1-q}$. For $f \in C(\mathbb{Z}_p)$, the fermionic p-adic integral on $\mathbb{Z}_p$ is defined by Kim,

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x)\,d\mu_{-q}(x) = \lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} f(x)(-q)^x \quad \text{(see [1–27])}, \tag{1}$$

where $[x]_{-q} = \frac{1-(-q)^x}{1+q}$. From (1), we note that

$$q^n I_{-q}(f_n) + (-1)^{n-1} I_{-q}(f) = [2]_q \sum_{l=0}^{n-1} (-1)^{n-1-l} q^l f(l), \tag{2}$$

where $f_n(x) = f(x+n)$ $(n \geq 1)$. In particular, for $n = 1$,

$$qI_{-q}(f_1) + I_{-q}(f) = [2]_q f(0). \tag{3}$$

We note that

$$\int_{\mathbb{Z}_p} e^{(x+y)t}\,d\mu_{-q}(y) = \frac{[2]_q}{qe^t+1} e^{xt}. \tag{4}$$

As is well known, the q-Euler polynomials are defined by Kim,

$$\frac{[2]_q}{qe^t+1}e^{xt}=\sum_{n=0}^{\infty}E_{n,q}(x)\frac{t^n}{n!}\quad\text{(see [2, 9, 22, 28])}. \tag{5}$$

When $x=0$, $E_{n,q}=E_{n,q}(0)$ are called the q-Euler numbers. We note that $\lim_{q\to 1}E_{n,q}(x)=E_n(x)$, where $E_n(x)$ are called the Euler polynomials which are defined by the generating function,

$$\frac{2}{e^t+1}e^{xt}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!}.$$

The Stirling number of the first kind is given by the generating function,

$$(x)_m=\sum_{l=0}^{m}S_1(m,l)x^l\quad(m\geq 0) \tag{6}$$

and the Stirling number of the second kind is defined by the generating function,

$$\left(e^t-1\right)^m=m!\sum_{l=m}^{\infty}S_2(l,m)\frac{t^l}{l!}\quad(m\geq 0)\text{ (see [7, 8, 15, 17])}. \tag{7}$$

In [21], Kim (2010) presented the generating functions related to the q-Euler polynomials of higher order and gave some interesting identities involving these polynomials. In [2], Bayad and Kim (2011) studied some relations involving values of q-Bernoulli, q-Euler, and Bernstein polynomials (see [3–6, 10, 20, 22–25, 29, 30]). Recently, Kim *et al.* studied some identities for q-analogs of the Changhee polynomials (see [11, 15]), for various degenerate Bernoulli polynomials (see [13, 16, 17, 22]), and for q-analogs of the Boole polynomials (see [10, 12]).

In recent years, a lot of people have studied various types of q-Euler polynomials and obtained many results which are interesting in number theory and combinatorics. To cite a few, in [28] one obtained eight basic identities of symmetry in three variables related to the q-Euler polynomials and a q-analog of alternating power sums. The derivation is based on the p-adic q-integrals in our case but on the p-adic integrals in [28]. In [9], some combinatorial identities involving q-Euler numbers and polynomials were obtained by adopting the ideas from [25]. It is fascinating that very recently some degenerate versions of many important polynomials were studied and some interesting results were obtained including the degenerate q-Euler polynomials. The aim of this paper is to define Barnes-type q-Euler numbers and polynomials in terms of p-adic q-integrals and to derive Witt-type formulas for them. Further, we find the connection between Barnes-type q-Euler polynomials and Barnes-type Frobenius polynomials and Barnes-type q-Changhee polynomials. This generalizes the Euler polynomials introduced in [21] by Kim.

In a forthcoming paper, we would like to give some of the applications of our results to symmetric identities involving Barnes-type q-Euler numbers and polynomials, to derivation of many identities of combinatorial nature. Also, we will investigate further properties, recurrence relations, and combinatorial identities for the Barnes-type polynomials by utilizing umbral calculus and degenerate versions of them.

The main results of this paper are some identities of the Barnes-type q-Euler polynomials. Furthermore, we define the Barnes-type q-Changhee polynomials and numbers, and we derive some identities related with the Barnes-type q-Euler polynomials and the Barnes-type q-Changhee polynomials.

2 The Barnes-type q-Euler polynomials and numbers

Let $w_1, \ldots, w_r \in \mathbb{C}_p$. The Barnes-type Euler polynomials are defined by the generating function

$$\prod_{l=1}^{r}\left(\frac{2}{e^{w_l t}+1}\right)e^{xt} = \sum_{n=0}^{\infty} E_n(x|w_1,\ldots,w_r)\frac{t^n}{n!}. \tag{8}$$

When $x = 0$, $E_n(w_1,\ldots,w_r) = E_n(0|w_1,\ldots,w_r)$ are called the Barnes-type Euler numbers (see [1, 3, 5, 6, 9, 12, 14, 28–30]). By (4), we get

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} e^{(w_1x_1+\cdots+w_rx_r+x)t}\,d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r) = \prod_{l=1}^{r}\left(\frac{[2]_q}{qe^{w_l t}+1}\right)e^{xt}, \tag{9}$$

for $|t|_p < p^{-\frac{1}{p-1}}$. From (9), the Barnes-type q-Euler polynomials are defined by the generating function,

$$[2]_q^r\prod_{l=1}^{r}\left(\frac{1}{qe^{w_1 t}+1}\right)e^{xt} = \sum_{n=0}^{\infty} E_{n,q}(x|w_1,\ldots,w_r)\frac{t^n}{n!}. \tag{10}$$

When $x = 0$, $E_{n,q}(w_1,\ldots,w_r) = E_{n,q}(0|w_1,\ldots,w_r)$ are called the Barnes-type q-Euler numbers. By (9) and (10), we get

$$\begin{aligned}
&\sum_{n=0}^{\infty} E_{n,q}(x|w_1,\ldots,w_r)\frac{t^n}{n!} \\
&\quad= \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} e^{(w_1x_1+\cdots+w_rx_r+x)t}\,d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r) \\
&\quad= \sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(w_1x_1+\cdots+w_rx_r+x)^n\,d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r)\frac{t^n}{n!}.
\end{aligned} \tag{11}$$

From (11), we obtain the following theorem.

Theorem 2.1 *For $n \geq 0$, we have*

$$E_{n,q}(x|w_1,\ldots,w_r) = \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(w_1x_1+\cdots+w_rx_r+x)^n\,d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r). \tag{12}$$

From (12), we note that

$$E_{n,q}(w_1,\ldots,w_r) = \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(w_1x_1+\cdots+w_rx_r)^n\,d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r). \tag{13}$$

Now, we observe that

$$\begin{aligned}\sum_{n=0}^{\infty} E_{n,q}(x|w_1,\ldots,w_r)\frac{t^n}{n!} &= \frac{(1+q)^r}{(qe^{w_1t}+1)\cdots(qe^{w_rt}+1)}e^{xt} \\ &= \frac{(1+q^{-1})^r}{(e^{w_1t}+q^{-1})\cdots(e^{w_rt}+q^{-1})}e^{xt} \\ &= \sum_{n=0}^{\infty} H_n\bigl(x,-q^{-1}|w_1,\ldots,w_r\bigr)\frac{t^n}{n!},\end{aligned} \tag{14}$$

where $H_n(x,u|w_1,\ldots,w_r)$ are called the Barnes-type Frobenius-Euler polynomials defined by the generating function,

$$\frac{(1-u)^r}{(e^{w_1t}-u)\cdots(e^{w_rt}-u)}e^{xt} = \sum_{n=0}^{\infty} H_n(x,u|w_1,\ldots,w_r)\frac{t^n}{n!} \quad \text{(see [2, 4])}. \tag{15}$$

Therefore, by (14), we obtain the following theorem.

Theorem 2.2 *Let $n \geq 0$, we have*

$$E_{n,q}(x|w_1,\ldots,w_r) = H_n\bigl(x,-q^{-1}|w_1,\ldots,w_r\bigr). \tag{16}$$

Let $n \geq 0$ and $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$. By (2), we get

$$q^d I_{-q}(f_d) + I_{-q}(f) = [2]_q \sum_{l=0}^{d-1}(-q)^l f(l). \tag{17}$$

By (17), we get

$$\begin{aligned}&\sum_{n=0}^{\infty} E_{n,q}(x|w_1,\ldots,w_r)\frac{t^n}{n!} \\ &\quad= \left(\frac{[2]_q}{[2]_{q^d}}\right)\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} e^{(w_1x_1+\cdots+w_rx_r+x)t}\,d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r) \\ &\quad= \left(\frac{[2]_q}{[2]_{q^d}}\right)\prod_{l=1}^{r}\left(\frac{[2]_q}{q^d e^{w_l dt}+1}\right)\sum_{l_1,\ldots,l_r=0}^{d-1}(-q)^{l_1+\cdots+l_r}e^{(w_1l_1+\cdots+w_rl_r+x)t} \\ &\quad= \left(\frac{[2]_q}{[2]_{q^d}}\right)\left(\sum_{m=0}^{\infty} E_{m,q^d}(dw_1,\ldots,dw_r)\frac{t^m}{m!}\right) \\ &\qquad\times \sum_{l_1,\ldots,l_r=0}^{d-1}(-q)^{l_1+\cdots+l_r}\sum_{k=0}^{\infty}(w_1l_1+\cdots+w_rl_r+x)^k\frac{t^k}{k!} \\ &\quad= \sum_{n=0}^{\infty}\left(\frac{[2]_q}{[2]_{q^d}}\right)\left(\sum_{k=0}^{n}\binom{n}{k}\sum_{l_1,\ldots,l_r=0}^{d-1}(-q)^{l_1+\cdots+l_r}(w_1l_1+\cdots+w_rl_r+x)^k\right. \\ &\qquad\left.\times E_{n-k,q^d}(dw_1,\ldots,dw_r)\right)\frac{t^n}{n!}.\end{aligned} \tag{18}$$

Thus, by (18), we obtain the following theorem.

Theorem 2.3 *Let $n \geq 0$. Then, for positive integer d with $d \equiv 1 \pmod 2$,*

$$\begin{aligned} &E_{n,q}(x|w_1,\dots,w_r) \\ &= \left(\frac{[2]_q}{[2]_{q^d}}\right) \sum_{k=0}^{n} \binom{n}{k} \sum_{l_1,\dots,l_r=0}^{d-1} (-q)^{l_1+\cdots+l_r} (w_1 l_1 + \cdots + w_1 l_1 + x)^k \\ &\quad \times E_{n-k,q^d}(dw_1,\dots,dw_r). \end{aligned} \tag{19}$$

We note that in [6], the authors considered the q-extensions of Changhee polynomials which are derived from the fermionic p-adic q-integral on $\mathbb{Z}_p$, and they gave some identities for these polynomials. Finally, we consider the Barnes-type q-Changhee polynomials. By (3), we note that, for $l = 1,\dots,r$,

$$\int_{\mathbb{Z}_p} (1+t)^{w_l x}\, d\mu_{-q}(x) = \frac{[2]_q}{q(1+t)^{w_l}+1}, \tag{20}$$

where $|t|_p < p^{-\frac{1}{p-1}}$. By (20), we get

$$\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+t)^{w_1x_1+\cdots+w_rx_r+x}\, d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r) = \prod_{l=1}^{r} \frac{[2]_q}{q(1+t)^{w_l}+1}(1+t)^x. \tag{21}$$

From (21), the Barnes-type q-Changhee polynomials are defined by the generating function,

$$\prod_{l=1}^{r} \frac{[2]_q}{q(1+t)^{w_l}+1}(1+t)^x = \sum_{n=0}^{\infty} Ch_{n,q}(x|w_1,\dots,w_r)\frac{t^n}{n!}. \tag{22}$$

When $x = 0$, $Ch_{n,q}(w_1,\dots,w_r) = Ch_{n,q}(0|w_1,\dots,w_r)$ are called the Barnes-type q-Changhee numbers (see [7, 11, 13, 15]). By (21) and (22), we have

$$\begin{aligned} &\sum_{n=0}^{\infty} Ch_{n,q}(x|w_1,\dots,w_r)\frac{t^n}{n!} \\ &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+t)^{w_1x_1+\cdots+w_rx_r+x}\, d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r) \\ &= \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \binom{w_1x_1+\cdots+w_rx_r+x}{n} t^n\, d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r) \\ &= \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (w_1x_1+\cdots+w_rx_r+x)_n\, d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \sum_{l=0}^{n} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} S_1(n,l)(w_1x_1+\cdots+w_rx_r+x)^l\, d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \sum_{l=0}^{n} S_1(n,l) E_{l,q}(x|w_1,\dots,w_r)\frac{t^n}{n!}. \end{aligned} \tag{23}$$

By (23), we obtain the following theorem.

Theorem 2.4 *Let $n \geq 0$. Then we have*

$$Ch_{n,q}(x|w_1,\ldots,w_r) = \sum_{l=0}^{n} S_1(n,l)E_{l,q}(x|w_1,\ldots,w_r). \tag{24}$$

By replacing t by $e^t - 1$, we have

$$\begin{aligned} \prod_{l=1}^{r} \frac{[2]_q}{qe^{w_l t}+1} e^{xt} &= \sum_{m=0}^{\infty} Ch_{m,q}(x|w_1,\ldots,w_r)\frac{(e^t-1)^m}{m!} \\ &= \sum_{m=0}^{\infty} Ch_{m,q}(x|w_1,\ldots,w_r)\frac{1}{m!}m!\sum_{n=m}^{\infty} S_2(n,m)\frac{t^m}{m!} \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{n} S_2(n,m)Ch_{m,q}(x|w_1,\ldots,w_r)\frac{t^n}{n!}. \end{aligned} \tag{25}$$

By (25) we obtain the following theorem.

Theorem 2.5 *Let $n \geq 0$. Then we have*

$$E_{n,q}(x|w_1,\ldots,w_r) = \sum_{m=0}^{n} S_2(n,m)Ch_{n,q}(x|w_1,\ldots,w_r). \tag{26}$$

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to this work. All authors read and approved the final manuscript.

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