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Principal vectors of second-order quantum difference equations with boundary conditions dependent on spectral parameter

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Abstract
Spectral analysis of a boundary value problem (BVP) consisting of a second-order quantum difference equation and boundary conditions depending on an eigenvalue parameter with spectral singularities was first studied by Aygar and Bohner (Appl. Math. Inf. Sci. 9(4):1725-1729, 2015). The main goal of this paper is to construct the principal vectors corresponding to the eigenvalues and the spectral singularities of this BVP. These vectors are important to get the spectral expansion formula for this BVP.

1 Introduction

Many areas including mathematical physics, engineering, economics, and quantum mechanics need the spectrum of differential and discrete operators to solve some problems. Therefore, many authors have investigated the spectral analysis of differential and discrete operators [1–10]. Because of the developments in quantum calculus, quantum difference equations became a popular topic for mathematicians. In addition to differential and discrete equations, the spectral theory of quantum difference equations has been treated in the last decade [11–13]. Hereafter, we let $q > 1$ and use the notation $q^{\mathbb{N}_0} := \{q^n : n \in \mathbb{N}_0\}$, where $\mathbb{N}_0$ denotes the set of nonnegative integers. Let us consider the BVP consisting of the second-order q-difference equation

$$qa(t)y(qt) + b(t)y(t) + a\left(\frac{t}{q}\right)y\left(\frac{t}{q}\right) = \lambda y(t), \quad t \in q^{\mathbb{N}},$$

and the boundary conditions

$$(\gamma_0 + \gamma_1\lambda)y(q) + (\beta_0 + \beta_1\lambda)y(1) = 0, \qquad \gamma_0\beta_1 - \gamma_1\beta_0 \neq 0, \qquad \gamma_1 \neq \frac{\beta_0}{a(1)},$$

where $\{a(t)\}_{t\in q^{\mathbb{N}_0}}$ and $\{b(t)\}_{t\in q^{\mathbb{N}}}$ are complex sequences, λ is a spectral parameter, $a(t) \neq 0$ for all $t \in q^{\mathbb{N}_0}$ and $\gamma_i, \beta_i \in \mathbb{C}$, $i = 0, 1$. We will introduce the Hilbert space of complex-valued functions satisfying $\langle f, f\rangle_q < \infty$, with respect to the inner product

$$\langle f, g\rangle_q := \sum_{t\in q^{\mathbb{N}}} \mu(t)f(t)\overline{g(t)}, \quad f, g : q^{\mathbb{N}} \to \mathbb{C},$$

by $\ell_2(q^{\mathbb{N}})$, where $\mu(t) = (q-1)t$ for all $t \in q^{\mathbb{N}}$. Furthermore, we will denote the q-difference operator generated in $\ell_2(q^{\mathbb{N}})$ by q-difference expression

$$(ly)(t) := qa(t)y(qt) + b(t)y(t) + a\left(\frac{t}{q}\right)y\left(\frac{t}{q}\right), \quad t \in q^{\mathbb{N}}, \tag{1.1}$$

with the boundary conditions

$$(\gamma_0 + \gamma_1\lambda)y(q) + (\beta_0 + \beta_1\lambda)y(1) = 0, \qquad \gamma_0\beta_1 - \gamma_1\beta_0 \neq 0, \qquad \gamma_1 \neq \frac{\beta_0}{a(1)}, \tag{1.2}$$

by L. In [11], it is proved that the operator L has a finite number of eigenvalues and spectral singularities with finite multiplicities under the condition

$$\sup_{t\in q^{\mathbb{N}}}\left\{\exp\left[\varepsilon\left(\frac{\ln t}{\ln q}\right)^{\delta}\right]\left(\left|1-a(t)\right| + \left|b(t)\right|\right)\right\} < \infty, \quad \varepsilon > 0, \frac{1}{2} \le \delta \le 1. \tag{1.3}$$

The set up of this paper which is an extension of [11], settled as follows: Section 2 is about the results which are proved in [11] and will be used in next section. In Section 3, we obtain principal vectors corresponding to eigenvalues and spectral singularities of L, and give some properties of them. This paper will be valuable for readers because principal vectors that we obtained corresponding to the eigenvalues and spectral singularities are important to find the spectral expansion of the operator L. It is also important to investigate the effects of spectral singularities to this expansion of L.

2 Properties of eigenvalues and spectral singularities of *L*

Assume (1.3), then the equation $(ly)(t) := \lambda y(t)$, $t \in q^{\mathbb{N}}$ has the solution

$$e(t,z) = \alpha(t)\frac{e^{i\frac{\ln t}{\ln q}z}}{\sqrt{\mu(t)}}\left(1 + \sum_{r\in q^{\mathbb{N}}} A(t,r)e^{i\frac{\ln r}{\ln q}z}\right), \quad t \in q^{\mathbb{N}_0} \tag{2.1}$$

for $\lambda = 2\sqrt{q}\cos z$ and $z \in \overline{\mathbb{C}}_+ := \{z \in \mathbb{C} : \operatorname{Im} \ge 0\}$, where $\alpha(t)$ and $A(t,r)$ are expressed in terms of $\{a(t)\}$ and $\{b(t)\}$ as

$$\begin{aligned}
&\alpha(t) = \prod_{s\in[t,\infty)\cap q^{\mathbb{N}}} \left[a(s)\right]^{-1}, \qquad A(t,q) = -\frac{1}{\sqrt{q}}\sum_{s\in[qt,\infty)\cap q^{\mathbb{N}}} b(s), \\
&A\left(t,q^2\right) = \sum_{s\in[qt,\infty)\cap q^{\mathbb{N}}}\left\{1 - a^2(s) + \frac{b(s)}{q}\sum_{p\in[qs,\infty)\cap q^{\mathbb{N}}} b(s)\right\}, \\
&A\left(t,rq^2\right) = A(qt,r) + \sum_{s\in[qt,\infty)\cap q^{\mathbb{N}}}\left\{\left(1-a^2(s)\right)A(qs,r) - \frac{b(s)}{\sqrt{q}}A(s,qr)\right\}
\end{aligned} \tag{2.2}$$

for $r \in q^{\mathbb{N}}$ and $t \in q^{\mathbb{N}_0}$. Moreover, $A(t,r)$ satisfies

$$\left|A(t,r)\right| \le C \sum_{s\in[tq^{\lfloor\frac{\ln r}{2\ln q}\rfloor},\infty)\cap q^{\mathbb{N}}} \left(\left|1-a(s)\right| + \left|b(s)\right|\right), \tag{2.3}$$

where $\lfloor \frac{\ln r}{2\ln q} \rfloor$ is the integer part of $\frac{\ln r}{2\ln q}$ and $C > 0$ is a constant. Therefore, $e(\cdot, z)$ is analytic with respect to z in $\mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ and continuous in $\overline{\mathbb{C}}_+$. Let us define the function f using (2.1) and the boundary condition (1.2)

$$f(z) = (\gamma_0 + 2\sqrt{q}\gamma_1 \cos z)e(q, z) + (\beta_0 + 2\sqrt{q}\beta_1 \cos z)e(1, z). \tag{2.4}$$

The function f is analytic in $\mathbb{C}_+$, $f(z) = f(z + 2\pi)$, and continuous in $\overline{\mathbb{C}}_+$. If we define semi-strips $P_0 = \{z \in \mathbb{C}_+ : -\frac{\pi}{2} \leq \operatorname{Re} z \leq \frac{3\pi}{2}\}$ and $P = P_0 \cup [-\frac{\pi}{2}, \frac{3\pi}{2}]$ then we get the Green function of L as

$$G_{t,z}(z) := \begin{cases} -\frac{\phi(r,z)e(t,z)}{qa(1)f(z)}, & r = tq^{-k}, k \in \mathbb{N}_0, \\ -\frac{e(r,z)\phi(t,z)}{qa(1)f(z)}, & r = tq^{k}, k \in \mathbb{N} \end{cases}$$

for all $z \in P$ with $f(z) \neq 0$ [11], and from the definition of eigenvalues and spectral singularities [4], we have

$$\begin{aligned} \sigma_d(L) &= \{\lambda : \lambda = 2\sqrt{q}\cos z, z \in P_0, f(z) = 0\} \\ \sigma_{ss}(L) &= \left\{\lambda : \lambda = 2\sqrt{q}\cos z, z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right], f(z) = 0\right\} \setminus \{0\}, \end{aligned} \tag{2.5}$$

where $\sigma_d(L)$ and $\sigma_{ss}(L)$ denote the set of eigenvalues and spectral singularities of L, respectively. Using (2.1) and (2.4), we obtain

$$\begin{aligned} f(z) = {} & \alpha(1)\sqrt{\frac{q}{q-1}}\beta_1 e^{-iz} + \alpha(q)\frac{\gamma_1}{\sqrt{q-1}} + \alpha(1)\frac{\beta_0}{\sqrt{q-1}} \\ & + \left(\alpha(q)\frac{\gamma_0}{\sqrt{q(q-1)}} + \alpha(1)\sqrt{\frac{q}{q-1}}\beta_1\right)e^{iz} \\ & + \alpha(q)\frac{\gamma_1}{\sqrt{q-1}}e^{2iz} + \sum_{r \in q^{\mathbb{N}}} \alpha(1)\sqrt{\frac{q}{q-1}}\beta_1 A(1,r)e^{i(\frac{\ln r}{\ln q}-1)z} \\ & + \sum_{r \in q^{\mathbb{N}}} \left(\alpha(q)\frac{\gamma_1}{\sqrt{q-1}}A(q,r) + \alpha(1)\frac{\beta_0}{\sqrt{q-1}}A(1,r)\right)e^{i\frac{\ln r}{\ln q}z} \\ & + \sum_{r \in q^{\mathbb{N}}} \left(\alpha(q)\frac{\gamma_0}{\sqrt{q(q-1)}}A(q,r) + \alpha(1)\sqrt{\frac{q}{q-1}}\beta_1 A(1,r)\right)e^{i(\frac{\ln r}{\ln q}+1)z} \\ & + \sum_{r \in q^{\mathbb{N}}} \alpha(q)\frac{\gamma_1}{\sqrt{q-1}}A(q,r)e^{i(\frac{\ln r}{\ln q}+2)z}. \end{aligned}$$

If we define $F(z) := f(z)e^{iz}$, then the function F is also analytic in $\mathbb{C}_+$ and continuous in $\overline{\mathbb{C}}_+$, and $F(z) = F(z + 2\pi)$. It follows from (2.5) and the definition of F that

$$\begin{aligned} \sigma_d(L) &= \{\lambda : \lambda = 2\sqrt{q}\cos z, z \in P_0, F(z) = 0\}, \\ \sigma_{ss}(L) &= \left\{\lambda : \lambda = 2\sqrt{q}\cos z, z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right], F(z) = 0\right\} \setminus \{0\}. \end{aligned} \tag{2.6}$$

Definition 2.1 The multiplicity of a zero of F in P is called the multiplicity of the corresponding eigenvalue or spectral singularity of L.

It was found in [11] that under the condition (1.3), F has a finite number of zeros in P with finite multiplicities, *i.e.*, the operator L has a finite number of eigenvalues and spectral singularities with finite multiplicities.

3 Principal vectors of L

Let us define the functions $E(t,\lambda) := e(t, \arccos \frac{\lambda}{2\sqrt{q}})$ and $B(\lambda) := F(\arccos \frac{\lambda}{2\sqrt{q}})$. Using (2.1) and $\arccos \frac{\lambda}{2\sqrt{q}} = -i \ln(\frac{\lambda+\sqrt{\lambda^2-4q}}{2\sqrt{q}})$, we obtain

$$E(t,\lambda) = \frac{\alpha(t)}{\sqrt{\mu(t)}} \left(\frac{\lambda + \sqrt{\lambda^2 - 4q}}{2\sqrt{q}} \right)^{\frac{\ln t}{\ln q}} \left(1 + \sum_{r \in q^{\mathbb{N}}} A(t,r) \left(\frac{\lambda + \sqrt{\lambda^2 - 4q}}{2\sqrt{q}} \right)^{\frac{\ln r}{\ln q}} \right).$$

Since $\lambda = 2\sqrt{q} \cos z$ maps P_0 to the domain $\Lambda := \mathbb{C} \backslash [-2\sqrt{q}, 2\sqrt{q}]$, the function $E(\lambda) := \{E(t,\lambda)\}_{t \in q^{\mathbb{N}}}$ is analytic in Λ, and continuous up to the interval $[-2\sqrt{q}, 2\sqrt{q}]$, under the condition (1.3). From (2.5), we can write

$$\sigma_d(L) = \{\lambda \in \Lambda : B(\lambda) = 0\},$$
$$\sigma_{ss}(L) = \{\lambda \in [-2\sqrt{q}, 2\sqrt{q}] : B(\lambda) = 0\} \backslash \{0\}.$$

The properties of the function F in P, which were obtained in [11], give the following.

Remark 3.1 Under the condition (1.3), the function B has a finite number of zeros in Λ and in $[-2\sqrt{q}, 2\sqrt{q}]$, and each of them is of finite multiplicity.

Let $\lambda_1, \lambda_2, \ldots, \lambda_s$ and $\lambda_{s+1}, \lambda_{s+2}, \ldots, \lambda_\nu$ denote the zeros of B in Λ (which are eigenvalues of L), and in $[-2\sqrt{q}, 2\sqrt{q}]$ (which are spectral singularities of L) with multiplicities $m_1, m_2, \ldots, m_s$ and $m_{s+1}, m_{s+2}, \ldots, m_\nu$, respectively.

Definition 3.2 Let λ_0 be an eigenvalue of L. If the vectors $y^{(k)} = \{y^{(k)}(t)\}_{t \in q^{\mathbb{N}}}$ for $k = 1, 2, \ldots, n$ satisfy

$$\begin{cases} (ly^{(0)})(t) - \lambda_0 y^{(0)}(t) = 0, \\ (ly^{(k)})(t) - \lambda_0 y^{(k)}(t) - y^{(k-1)}(t) = 0, \quad k = 1, 2, \ldots, n; n \in q^{\mathbb{N}}, \end{cases} \tag{3.1}$$

then the vector $y^{(0)}$ is called the eigenvector corresponding to the eigenvalue λ_0 of L. The vectors $y^{(1)}, y^{(2)}, \ldots, y^{(n)}$ are called the associated vectors corresponding to $\lambda = \lambda_0$. The eigenvector and the associated vectors corresponding to λ_0 are called the principal vectors of the eigenvalue $\lambda = \lambda_0$. The principal vectors of the spectral singularities of L are defined similarly.

Now, we define the vectors for $\lambda = 2\sqrt{q} \cos z$, $z \in P$,

$$V^{(k)}(t,\lambda_j) = \frac{1}{k!} \left\{ \frac{\mathrm{d}^k}{\mathrm{d}\lambda^k} E(t,\lambda) \right\}_{\lambda=\lambda_j}, \quad k = 0, 1, \ldots, m_j - 1; j = 1, 2, \ldots, s$$

and

$$V^{(k)}(t,\lambda_j) = \frac{1}{k!} \left\{ \frac{\mathrm{d}^k}{\mathrm{d}\lambda^k} E(t,\lambda) \right\}_{\lambda=\lambda_j}, \quad k = 0, 1, \ldots, m_j - 1; j = s+1, s+2, \ldots, \nu.$$

Furthermore, if $y(\lambda) = \{y(t,\lambda)\}_{t\in q^{\mathbb{N}}}$ is a solution $(ly)(t) = \lambda y(t)$, then we get

$$\begin{aligned} &qa(t)\frac{\mathrm{d}^k}{\mathrm{d}\lambda^k}y(qt,\lambda) + b(t)\frac{\mathrm{d}^k}{\mathrm{d}\lambda^k}y(t,\lambda) + a\left(\frac{t}{q}\right)\frac{\mathrm{d}^k}{\mathrm{d}\lambda^k}y\left(\frac{t}{q},\lambda\right) \\ &\quad = k\frac{\mathrm{d}^{k-1}}{\mathrm{d}\lambda^{k-1}}y(t,\lambda) + \lambda\frac{\mathrm{d}^k}{\mathrm{d}\lambda^k}y(t,\lambda). \end{aligned} \tag{3.2}$$

Using (3.1) and (3.2), we have, for $k = 0,1,\dots,m_j - 1$ and $j = 1,2,\dots,\nu$,

$$\begin{cases} (lV^{(0)}(\lambda_j))(t) - \lambda_j V^{(0)}(\lambda_j)(t) = 0, \\ (lV^{(k)}(\lambda_j))(t) - \lambda_j V^{(k)}(\lambda_j)(t) - V^{(k-1)}(\lambda_j)(t) = 0. \end{cases} \tag{3.3}$$

So, the vectors $V^{(k)}(t,\lambda_j)$, $k = 0,1,\dots,m_j - 1$; $j = 1,2,\dots,s$, and $V^{(k)}(t,\lambda_j)$, $k = 0,1,\dots,m_j - 1$; $j = s+1, s+2,\dots,\nu$ are the principal vectors of eigenvalues and spectral singularities of L, respectively.

Theorem 3.3 *Under the condition* (1.3), $V^{(k)}(t,\lambda_j) \in \ell_2(q^{\mathbb{N}})$ *for* $k = 0,1,\dots,m_j - 1$; $j = 1,2,\dots,s$, *but* $V^{(k)}(t,\lambda_j) \notin \ell_2(q^{\mathbb{N}})$ *for* $k = 0,1,\dots,m_j - 1$; $j = s+1,s+2,\dots,\nu$.

Proof By using $E(t,\lambda) = e(t, \arccos\frac{\lambda}{2\sqrt{q}})$, we find

$$\left\{\frac{\mathrm{d}^k}{\mathrm{d}\lambda^k}E(t,\lambda)\right\}_{\lambda=\lambda_j} = \sum_{m=0}^{k} C_m \left\{\frac{\mathrm{d}^m}{\mathrm{d}\lambda^m}e(t,z)\right\}_{z=z_j}, \tag{3.4}$$

where $\lambda_j = 2\sqrt{q}\cos z_j$, $z_j \in P$, $j = 1,2,\dots,\nu$, and C_m is a constant depending on λ_j. From (2.1), we get

$$\left\{\frac{\mathrm{d}^m}{\mathrm{d}\lambda^m}e(t,z)\right\}_{z=z_j} = \frac{\alpha(t)}{\sqrt{\mu(t)}}e^{i\frac{\ln t}{\ln q}z_j}\left\{\left(i\frac{\ln t}{\ln q}\right)^m + \sum_{r\in q^{\mathbb{N}}}\left(i\frac{\ln(tr)}{\ln q}\right)^m A(t,r)e^{i\frac{\ln r}{\ln q}z_j}\right\} \tag{3.5}$$

for all $t \in q^{\mathbb{N}}$ and $j = 1,2,\dots,\nu$. For the principal vectors $V^{(k)}(t,\lambda_j)$, $k = 0,1,\dots,m_j - 1$; $j = 1,2,\dots,s$, corresponding to the eigenvalues $\lambda_j = 2\sqrt{q}\cos z_j$ of L, we obtain

$$\left\{\frac{\mathrm{d}^k}{\mathrm{d}\lambda^k}E(t,\lambda)\right\}_{\lambda=\lambda_j} = \sum_{m=0}^{k} C_m \frac{\alpha(t)}{\sqrt{\mu(t)}}e^{i\frac{\ln t}{\ln q}z_j}\left\{\left(i\frac{\ln t}{\ln q}\right)^m + \sum_{r\in q^{\mathbb{N}}}\left(i\frac{\ln(tr)}{\ln q}\right)^m A(t,r)e^{i\frac{\ln r}{\ln q}z_j}\right\};$$

then

$$V^{(k)}(t,\lambda_j) = \frac{1}{k!}\sum_{m=0}^{k} C_m \frac{\alpha(t)}{\sqrt{\mu(t)}}e^{i\frac{\ln t}{\ln q}z_j}\left\{\left(i\frac{\ln t}{\ln q}\right)^m + \sum_{r\in q^{\mathbb{N}}}\left(i\frac{\ln(tr)}{\ln q}\right)^m A(t,r)e^{i\frac{\ln r}{\ln q}z_j}\right\} \tag{3.6}$$

for $k = 0,1,\dots,m_j - 1$ and $j = 1,2,\dots,s$. Now define the functions

$$g_1(t,z) := \frac{1}{k!}\sum_{m=0}^{k} C_m \frac{\alpha(t)}{\sqrt{\mu(t)}}e^{i\frac{\ln t}{\ln q}z_j}\left(i\frac{\ln t}{\ln q}\right)^m$$

and

$$g_2(t,z) := \frac{1}{k!}\sum_{m=0}^{k} C_m \frac{\alpha(t)}{\sqrt{\mu(t)}} e^{i\frac{\ln t}{\ln q}z_j} \sum_{r\in q^{\mathbb{N}}} \left(i\frac{\ln(tr)}{\ln q}\right)^m A(t,r) e^{i\frac{\ln r}{\ln q}z_j} \tag{3.7}$$

for $j = 1,2,\ldots,s$. Since $\operatorname{Im} z_j > 0$ for the eigenvalues $\lambda_j = 2\sqrt{q}\cos z_j$, $j = 1,2,\ldots,s$ of L, we get

$$\begin{aligned}
\sum_{t\in q^{\mathbb{N}}} \mu(t)\left|g_1(t,z)\right|^2 &\leq \frac{1}{(k!)^2}\left\{\sum_{t\in q^{\mathbb{N}}}\sum_{m=0}^{k} |C_m|\left|\alpha(t)\right| e^{-\frac{\ln t}{\ln q}\operatorname{Im} z_j}\left(\frac{\ln t}{\ln q}\right)^m\right\}^2 \\
&\leq H\left\{\sum_{t\in q^{\mathbb{N}}} e^{-\frac{\ln t}{\ln q}\operatorname{Im} z_j}\left[1+\frac{\ln t}{\ln q}+\cdots+\left(\frac{\ln t}{\ln q}\right)^k\right]\right\}^2 \\
&\leq H(k+1)^2\left[\sum_{t\in q^{\mathbb{N}}} e^{-\frac{\ln t}{\ln q}\operatorname{Im} z_j}\left(\frac{\ln t}{\ln q}\right)^k\right]^2 < \infty,
\end{aligned} \tag{3.8}$$

where H is a constant. Using (3.7), we also have

$$\begin{aligned}
\left|g_2(t,z)\right| &\leq \sum_{m=0}^{k} \frac{|C_m||\alpha(t)|}{\sqrt{\mu(t)}} e^{-\frac{\ln t}{\ln q}\operatorname{Im} z_j} \sum_{r\in q^{\mathbb{N}}} \left|\frac{\ln(tr)}{\ln q}\right|^m \left|A(t,r)\right| e^{-\frac{\ln r}{\ln q}\operatorname{Im} z_j} \\
&\leq |C_0|\frac{|\alpha(t)|}{\sqrt{\mu(t)}} e^{-\frac{\ln t}{\ln q}\operatorname{Im} z_j} \sum_{r\in q^{\mathbb{N}}} \left|A(t,r)\right| e^{-\frac{\ln r}{\ln q}\operatorname{Im} z_j} \\
&\quad + |C_1|\frac{|\alpha(t)|}{\sqrt{\mu(t)}} e^{-\frac{\ln t}{\ln q}\operatorname{Im} z_j} \sum_{r\in q^{\mathbb{N}}} \frac{\ln(tr)}{\ln q}\left|A(t,r)\right| e^{-\frac{\ln r}{\ln q}\operatorname{Im} z_j} \\
&\quad + \cdots + |C_k|\frac{|\alpha(t)|}{\sqrt{\mu(t)}} e^{-\frac{\ln t}{\ln q}\operatorname{Im} z_j} \sum_{r\in q^{\mathbb{N}}} \left(\frac{\ln(tr)}{\ln q}\right)^k \left|A(t,r)\right| e^{-\frac{\ln r}{\ln q}\operatorname{Im} z_j} \\
&\leq \widetilde{C} e^{-\frac{\ln t}{\ln q}\operatorname{Im} z_j},
\end{aligned}$$

where $\widetilde{C} = \max\{|C_0|, |C_1|, \ldots, |C_k|\}\frac{|\alpha(t)|}{\sqrt{\mu(t)}}\sum_{r\in q^{\mathbb{N}}}\sum_{m=0}^{k} |A(t,r)|(\frac{\ln(tr)}{\ln q})^k e^{-\frac{\ln r}{\ln q}\operatorname{Im} z_j}$. Then we get, for $j = 1,2,\ldots,s$,

$$\sum_{t\in q^{\mathbb{N}}} \mu(t)\left|g_2(t,z)\right|^2 \leq \sum_{t\in q^{\mathbb{N}}} \widetilde{C}^2 e^{-2\frac{\ln t}{\ln q}\operatorname{Im} z_j} < \infty. \tag{3.9}$$

It follows from (3.8) and (3.9) that

$$V^{(k)}(t,\lambda_j) \in \ell_2\left(q^{\mathbb{N}}\right)$$

for $k = 0,1,\ldots,m_j - 1$ and $j = 1,2,\ldots,s$. Now, we will use (3.6) for the principal vectors corresponding to the spectral singularities of L for $\lambda_j = 2\sqrt{q}\cos z_j$ and $j = s+1, s+2, \ldots, \nu$. Then we have

$$V^{(k)}(t,\lambda_j) = \frac{1}{k!}\sum_{m=0}^{k} C_m \frac{\alpha(t)}{\sqrt{\mu(t)}} e^{i\frac{\ln t}{\ln q}z_j}\left\{\left(i\frac{\ln t}{\ln q}\right)^m + \sum_{r\in q^{\mathbb{N}}}\left(i\frac{\ln(tr)}{\ln q}\right)^m A(t,r) e^{i\frac{\ln r}{\ln q}z_j}\right\} \tag{3.10}$$

for $k = 0, 1, \ldots, m_j - 1$ and $j = s+1, s+2, \ldots, \nu$. Since $\operatorname{Im} z_j = 0$ for the spectral singularities $\lambda_j = 2\sqrt{q}\cos z_j$, $j = s+1, s+2, \ldots, \nu$ of L, we find that

$$\frac{1}{k!}\sum_{t\in q^{\mathbb{N}}}\mu(t)\left|\sum_{m=0}^{k} C_m \frac{\alpha(t)}{\sqrt{\mu(t)}} e^{i\frac{\ln t}{\ln q}z_j}\left(i\frac{\ln t}{\ln q}\right)^m\right|^2 = \infty. \tag{3.11}$$

If we define the function h as

$$h(t,z) := \sum_{m=0}^{k}\sum_{r\in q^{\mathbb{N}}}\left(i\frac{\ln(tr)}{\ln q}\right)^m A(t,r) e^{i\frac{\ln r}{\ln q}z_j},$$

then using (1.3) and (2.3), we obtain

$$\begin{aligned}
\left|h(t,z)\right| &\leq \sum_{m=0}^{k}\sum_{r\in q^{\mathbb{N}}}\left(\frac{\ln(tr)}{\ln q}\right)^m \left|A(t,r)\right| \\
&\leq \sum_{m=0}^{k}\sum_{r\in q^{\mathbb{N}}}\left(\frac{\ln(tr)}{\ln q}\right)^m C \sum_{s\in[tq^{\lfloor\frac{\ln r}{2\ln q}\rfloor},\infty)\cap q^{\mathbb{N}}} Q(s) \\
&\leq C\sum_{m=0}^{k}\sum_{r\in q^{\mathbb{N}}}\left(\frac{\ln(tr)}{\ln q}\right)^m \sum_{s\in[tq^{\lfloor\frac{\ln r}{2\ln q}\rfloor},\infty)\cap q^{\mathbb{N}}} \exp\left[-\varepsilon\left(\frac{\ln s}{\ln q}\right)^\delta\right]\exp\left[\varepsilon\left(\frac{\ln s}{\ln q}\right)^\delta\right] Q(s) \\
&\leq C\sum_{m=0}^{k}\sum_{r\in q^{\mathbb{N}}}\left(\frac{\ln(tr)}{\ln q}\right)^m \exp\left\{-\varepsilon\left[\left(\frac{\ln t}{\ln q}\right)^\delta + \left(\frac{\ln r}{\ln q}\right)^\delta\right]\right\} P(s) \\
&\leq C_1\sum_{m=0}^{k}\sum_{r\in q^{\mathbb{N}}}\left(\frac{\ln(tr)}{\ln q}\right)^m \exp\left\{\frac{-\varepsilon}{4}\left[\left(\frac{\ln t}{\ln q}\right)^\delta + \left(\frac{\ln r}{\ln q}\right)^\delta\right]\right\} \\
&= C_1 \exp\left[\frac{-\varepsilon}{4}\left(\frac{\ln t}{\ln q}\right)^\delta\right]\sum_{m=0}^{k}\sum_{r\in q^{\mathbb{N}}}\left(\frac{\ln(tr)}{\ln q}\right)^m \exp\left[\frac{-\varepsilon}{4}\left(\frac{\ln r}{\ln q}\right)^\delta\right] \\
&= C_1 \exp\left[\frac{-\varepsilon}{4}\left(\frac{\ln t}{\ln q}\right)^\delta\right]\sum_{r\in q^{\mathbb{N}}}\sum_{m=0}^{k}\left(\frac{\ln(tr)}{\ln q}\right)^m \exp\left[\frac{-\varepsilon}{4}\left(\frac{\ln r}{\ln q}\right)^\delta\right] \\
&= D\exp\left[\frac{-\varepsilon}{4}\left(\frac{\ln t}{\ln q}\right)^\delta\right]
\end{aligned}$$

and

$$\frac{1}{k!}\sum_{t\in q^{\mathbb{N}}}\left|\frac{\alpha(t)}{\sqrt{\mu(t)}} e^{i\frac{\ln t}{\ln q}z_j} h(t,z)\right|^2 \mu(t) \leq \frac{1}{k!}\sum_{t\in q^{\mathbb{N}}}\alpha^2(t) D^2 e^{-\frac{\varepsilon}{2}(\frac{\ln t}{\ln q})^\delta} < \infty, \tag{3.12}$$

where

$$Q(s) = \left(\left|1-a(s)\right| + \left|b(s)\right|\right), \qquad P(s) = \sum_{s\in[tq^{\lfloor\frac{\ln r}{2\ln q}\rfloor},\infty)\cap q^{\mathbb{N}}} \exp\left[\varepsilon\left(\frac{\ln s}{\ln q}\right)^\delta\right] Q(s),$$

$$D = C_1\sum_{r\in q^{\mathbb{N}}}\sum_{m=0}^{k}\left(\frac{\ln(tr)}{\ln q}\right)^m \exp\left[\frac{-\varepsilon}{4}\left(\frac{\ln r}{\ln q}\right)^\delta\right],$$

and $\frac{1}{2} \le \delta \le 1$. It follows from (3.10), (3.11), and (3.12) that

$$V^{(k)}(t,\lambda_j) \notin \ell_2(q^{\mathbb{N}})$$

for $k = 0,1,\dots,m_j - 1$ and $j = s+1, s+2,\dots,\nu$. □

Let us introduce Hilbert spaces,

$$H_k(q^{\mathbb{N}}) := \{y = y(t)_{t\in q^{\mathbb{N}}} : \|y\|_k < \infty\}$$

and

$$H_{-k}(q^{\mathbb{N}}) := \{u = u(t)_{t\in q^{\mathbb{N}}} : \|u\|_{-k} < \infty\}$$

for $k \in \mathbb{N}_0$ with the norms

$$\|y\|_k^2 = \sum_{t\in q^{\mathbb{N}}} \mu(t)\left(1 + \frac{\ln t}{\ln q}\right)^{2k} \left|y(t)\right|^2$$

and

$$\|u\|_{-k}^2 = \sum_{t\in q^{\mathbb{N}}} \mu(t)\left(1 + \frac{\ln t}{\ln q}\right)^{-2k} \left|u(t)\right|^2,$$

respectively. It is obvious that $H_0(q^{\mathbb{N}}) = \ell_2(q^{\mathbb{N}})$ and

$$H_{k+1}(q^{\mathbb{N}}) \subsetneqq H_k(q^{\mathbb{N}}) \subsetneqq \ell_2(q^{\mathbb{N}}) \subsetneqq H_{-k}(q^{\mathbb{N}}) \subsetneqq H_{-(k+1)}(q^{\mathbb{N}}), \quad k = 0,1,2,\dots.$$

Theorem 3.4 $V^{(k)}(t,\lambda_j) \in H_{k+1}(q^{\mathbb{N}})$ *for* $k = 0,1,\dots,m_j - 1$ *and* $j = s+1, s+2,\dots,\nu$.

Proof Using (3.10), we get

$$\sum_{t\in q^{\mathbb{N}}} \mu(t)\left(1 + \frac{\ln t}{\ln q}\right)^{-2(k+1)} \left|\frac{1}{k!}\sum_{m=0}^{k} C_m \frac{\alpha(t)}{\sqrt{\mu(t)}}\left(i\frac{\ln t}{\ln q}\right)^m e^{i\frac{\ln t}{\ln q}z_j}\right|^2 < \infty$$

and

$$\sum_{t\in q^{\mathbb{N}}} \mu(t)\left(1 + \frac{\ln t}{\ln q}\right)^{-2(k+1)} \left|\frac{1}{k!}\sum_{m=0}^{k} C_m \frac{\alpha(t)}{\sqrt{\mu(t)}} e^{i\frac{\ln t}{\ln q}z_j} \sum_{r\in q^{\mathbb{N}}} \left(\frac{\ln(tr)}{\ln q}\right)^m A(t,r) e^{i\frac{\ln r}{\ln q}z_j}\right|^2 < \infty$$

for $k = 0,1,\dots,m_j - 1$ and $j = s+1, s+2,\dots,\nu$. This completes the proof. □

Let us choose $m_0 = \max\{m_{s+1}, m_{s+2},\dots,m_\nu\}$. Now, we can give the following theorem as a result of Theorem 3.4.

Theorem 3.5 $V^{(k)}(t,\lambda_j) \in H_{m_0}(q^{\mathbb{N}})$ *for* $k = 0,1,\dots,m_j - 1$ *and* $j = s+1, s+2,\dots,\nu$.

Competing interests
The author declares to have no competing interests.

Author's contributions
The author performed all tasks of this research: drafting, thinking of the study, writing, and revision of paper.

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