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Nonlinear Hadamard fractional differential equations with Hadamard type nonlocal non-conserved conditions

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Abstract

In this paper, we discuss the existence and uniqueness of solutions for a boundary value problem of nonlinear Hadamard fractional differential equations and nonlocal non-conserved boundary conditions in terms of Hadamard integral. Our results are new in the present configuration and are based on some classical ideas of fixed point theory. We present several examples for the illustration of main results. A companion problem has also been studied. The paper concludes with some interesting observations.

MSC: 34A60; 34A08

Keywords: Hadamard fractional derivative; integral boundary conditions; fixed point theorems

1 Introduction

In this paper, we study the following boundary value problem:

$$\begin{cases} D^\alpha x(t) = f(t, x(t)), \quad 1 < t < e, 1 < \alpha \le 2, \\ x(1) = 0, \qquad \frac{A}{\Gamma(\gamma)} \int_1^\eta (\log \frac{\eta}{s})^{\gamma-1} \frac{x(s)}{s} \, ds + Bx(e) = c, \quad \gamma > 0, 1 < \eta < e, \end{cases} \tag{1.1}$$

where D^α is the Hadamard fractional derivative of order α, $f : [1, e] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, and A, B, c are real constants.

It is well known that the conserved quantities play a key role in understanding important mathematical and physical concepts such as differential equations, laws of conservation of energy, and quantum mechanics [1–6]. The nonlocal non-conserved boundary condition in (1.1) can be interpreted as sum of a scalar multiple of the average value of the unknown function over the given interval of an arbitrary length $(1, \eta)$ (in the sense of Hadamard) and a scalar multiple of the value of the unknown function at the right end-point ($t = e$) of the given interval remains constant. In case of $B = 0$, $c = 0$, this condition reduces to

$$\frac{1}{\Gamma(\gamma)} \int_1^\eta \left(\log \frac{\eta}{s}\right)^{\gamma-1} \frac{x(s)}{s} \, ds = 0,$$

which can be conceived as a conserved nonlocal boundary condition of Hadamard type.

The recent studies on fractional differential equations indicate that a variety of interesting and important results concerning existence and uniqueness of solutions, stability properties of solutions, analytic and numerical methods of solutions of these equations have been obtained, and the surge in investigating more and more results is underway [7, 8]. The tools of fractional calculus have played a significant role in improving the modeling techniques for several real world problems. Nowadays fractional-order differential equations appear extensively in a variety of applications such as diffusion processes, chaos, thermo-elasticity, biomathematics, fractional dynamics, *etc.* [9–13]. One of the characteristics of operators of fractional-order is their nonlocal nature accounting for the hereditary properties of many phenomena and processes involved. For the recent development of the topic, we refer the reader to a series of books and papers [14–24]. However, it has been noticed that most of the work on the topic is based on Riemann-Liouville and Caputo type fractional differential equations. Another kind of fractional derivatives that appears side by side to Riemann-Liouville and Caputo derivatives in the literature is the fractional derivative due to Hadamard introduced in 1892 [25]. Details and properties of Hadamard fractional derivative and integral can be found in [15, 26–29].

The objective of this paper is to investigate a fractional integral boundary value problem involving Hadamard fractional derivative and integral. We organize the rest of the manuscript as follows. Section 2 contains some preliminary concepts and a supporting lemma to define the solution for the problem at hand, while the main results are studied in Section 3. A companion problem is discussed in Section 4.

2 Preliminaries

Definition 2.1 [15] The Hadamard derivative of fractional order q for a function $g : [1,\infty) \to \mathbb{R}$ is defined as

$$D^q g(t) = \frac{1}{\Gamma(n-q)}\left(t\frac{d}{dt}\right)^n \int_1^t \left(\log\frac{t}{s}\right)^{n-q-1}\frac{g(s)}{s}\,ds, \quad n-1<q<n, n=[q]+1,$$

where $[q]$ denotes the integer part of the real number q and $\log(\cdot) = \log_e(\cdot)$.

Definition 2.2 [15] The Hadamard fractional integral of order q for a function g is defined as

$$I^q g(t) = \frac{1}{\Gamma(q)}\int_1^t \left(\log\frac{t}{s}\right)^{q-1}\frac{g(s)}{s}\,ds, \quad q>0,$$

provided the integral exists.

Lemma 2.3 *Given $y \in C([1,e],\mathbb{R})$, the unique solution of the problem*

$$\begin{cases} D^\alpha x(t) = y(t), & 1<t<e, 1<\alpha\le 2,\\ x(1)=0, \quad AI^\gamma x(\eta) + Bx(e) = c, & 1<\eta<e, \end{cases} \tag{2.1}$$

is given by

$$x(t) = I^\alpha y(t) + (\log t)^{\alpha-1}\frac{c - AI^{\gamma+\alpha}y(\eta) - BI^\alpha y(e)}{B + \frac{A\Gamma(\alpha)}{\Gamma(\gamma+\alpha)}(\log\eta)^{\gamma+\alpha-1}}, \tag{2.2}$$

where $I^{(\cdot)}$ denotes the Hadamard fractional integral (*see Definition* 2.2).

Proof As argued in [15], the solution of Hadamard differential equation in (2.1) can be written as

$$x(t) = I^{\alpha} y(t) + c_1 (\log t)^{\alpha-1} + c_2 (\log t)^{\alpha-2}. \tag{2.3}$$

The first boundary condition gives $c_2 = 0$. Note that

$$\begin{aligned} I^{\gamma} x(\eta) &= I^{\gamma+\alpha} y(\eta) + \frac{c_1}{\Gamma(\gamma)} \int_1^{\eta} \left(\log \frac{\eta}{s}\right)^{\gamma-1} \frac{(\log s)^{\alpha-1}}{s}\, ds \\ &= I^{\gamma+\alpha} y(\eta) + c_1 \frac{\Gamma(\alpha)}{\Gamma(\gamma+\alpha)} (\log \eta)^{\gamma+\alpha-1}. \end{aligned}$$

Using the second boundary condition, we get

$$AI^{\gamma+\alpha} y(\eta) + Ac_1 \frac{\Gamma(\alpha)}{\Gamma(\gamma+\alpha)} (\log \eta)^{\gamma+\alpha-1} + BI^{\alpha} y(e) + Bc_1 = c,$$

which gives

$$c_1 = \frac{c - AI^{\gamma+\alpha} y(\eta) - BI^{\alpha} y(e)}{B + \frac{A\Gamma(\alpha)}{\Gamma(\gamma+\alpha)} (\log \eta)^{\gamma+\alpha-1}}.$$

Substituting the values of c_1 and c_2 in (2.3), we obtain (2.2). This completes the proof. □

In view of Lemma 2.3, the integral solution of problem (1.1) can be written as

$$\begin{aligned} x(t) &= \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{f(s,x(s))}{s}\, ds \\ &\quad + \frac{(\log t)^{\alpha-1}}{\Delta} \left\{ c - \frac{A}{\Gamma(\gamma+\alpha)} \int_1^{\eta} \left(\log \frac{\eta}{s}\right)^{\gamma+\alpha-1} \frac{f(s,x(s))}{s}\, ds \right. \\ &\quad \left. - \frac{B}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{f(s,x(s))}{s}\, ds \right\}, \quad t \in [1,e], \end{aligned} \tag{2.4}$$

where

$$\Delta = B + \frac{A\Gamma(\alpha)}{\Gamma(\gamma+\alpha)} (\log \eta)^{\gamma+\alpha-1}.$$

3 Main results

We define an operator $Q : C([1,e],\mathbb{R}) \to C([1,e],\mathbb{R})$ by

$$\begin{aligned} Qx(t) &= \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{f(s,x(s))}{s}\, ds \\ &\quad + \frac{(\log t)^{\alpha-1}}{\Delta} \left\{ c - \frac{A}{\Gamma(\gamma+\alpha)} \int_1^{\eta} \left(\log \frac{\eta}{s}\right)^{\gamma+\alpha-1} \frac{f(s,x(s))}{s}\, ds \right. \\ &\quad \left. - \frac{B}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{f(s,x(s))}{s}\, ds \right\}, \quad t \in [1,e]. \end{aligned} \tag{3.1}$$

Notice that problem (1.1) is equivalent to the fixed point operator equation $Qx = x$, and the existence of a fixed point of the operator Q implies the existence of a solution of problem (1.1).

In the next we give some existence and uniqueness results by using a variety of fixed point theorems.

3.1 Existence and uniqueness result via Banach's fixed point theorem

First of all, we present the existence and uniqueness result for problem (1.1). This result is based on using Banach's fixed point theorem.

For the sake of computational convenience, we set

$$\omega = \frac{1}{\Gamma(\alpha+1)} + \frac{1}{|\Delta|}\left\{\frac{|A|(\log\eta)^{\gamma+\alpha}}{\Gamma(\gamma+\alpha+1)} + \frac{|B|}{\Gamma(\alpha+1)}\right\}. \tag{3.2}$$

Theorem 3.1 *Let $f:[1,e]\times\mathbb{R}\to\mathbb{R}$ be a continuous function satisfying the following condition*:

(H_1) *There exists a constant $L_1 > 0$ such that $|f(t,x)-f(t,y)| \le L_1|x-y|$ for each $t\in[1,e]$ and $x,y\in\mathbb{R}$.*

If

$$L_1\omega < 1, \tag{3.3}$$

then the Hadamard fractional boundary value problem (1.1) *has a unique solution in* $[1,e]$.

Proof Fixing $\max_{t\in[1,e]}|f(t,0)| = M < \infty$, we define $B_r = \{x\in C([1,e],\mathbb{R}) : \|x\|\le r\}$, where $r \ge \frac{M\omega+|c|/\Delta}{1-L_1\omega}$. We show that the set B_r is invariant with respect to the operator Q, that is, $QB_r\subset B_r$.

$$\begin{aligned}
\|Qx\| \le{}& \max_{t\in[1,e]}\left\{\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{|f(s,x(s))|}{s}\,ds\right.\\
&+\frac{(\log t)^{\alpha-1}}{|\Delta|}\left[|c| + \frac{A}{\Gamma(\gamma+\alpha)}\int_1^\eta\left(\log\frac{\eta}{s}\right)^{\gamma+\alpha-1}\frac{|f(s,x(s))|}{s}\,ds\right.\\
&\left.\left.-\frac{B}{\Gamma(\alpha)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-1}\frac{|f(s,x(s))|}{s}\,ds\right]\right\}\\
\le{}& \max_{t\in[1,e]}\left\{\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{(|f(s,x(s))-f(s,0)|+|f(s,0)|)}{s}\,ds\right.\\
&+\frac{(\log t)^{\alpha-1}}{|\Delta|}\\
&\times\left[|c| + \frac{|A|}{\Gamma(\gamma+\alpha)}\int_1^\eta\left(\log\frac{\eta}{s}\right)^{\gamma+\alpha-1}\frac{(|f(s,x(s))-f(s,0)|+|f(s,0)|)}{s}\,ds\right.\\
&\left.\left.+\frac{|B|}{\Gamma(\alpha)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-1}\frac{(|f(s,x(s))-f(s,0)|+|f(s,0)|)}{s}\,ds\right]\right\}\\
\le{}& (L_1r+M)\max_{t\in[1,e]}\left\{\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{1}{s}\,ds\right.
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{(\log t)^{\alpha-1}}{|\Delta|}\left[\frac{|A|}{\Gamma(\gamma+\alpha)}\int_1^\eta \left(\log\frac{\eta}{s}\right)^{\gamma+\alpha-1}\frac{1}{s}\,ds + \frac{|B|}{\Gamma(\alpha)}\int_1^e \left(\log\frac{e}{s}\right)^{\alpha-1}\frac{1}{s}\,ds\right]\Bigg\} \\
&+ \frac{|c|(\log t)^{\alpha-1}}{|\Delta|} \\
&\le (L_1 r + M)\left[\frac{1}{\Gamma(\alpha+1)} + \frac{1}{|\Delta|}\left\{\frac{|A|(\log\eta)^{\gamma+\alpha}}{\Gamma(\gamma+\alpha+1)} + \frac{|B|}{\Gamma(\alpha+1)}\right\}\right] + \frac{|c|}{|\Delta|} \\
&= (L_1 r + M)\omega + \frac{|c|}{|\Delta|} \le r,
\end{aligned}
$$

which proves that $QB_r \subset B_r$.

Now let $x, y \in C([1,e],\mathbb{R})$. Then, for $t \in [1,e]$, we have

$$
\begin{aligned}
&\big|(Qx)(t) - (Qy)(t)\big| \\
&\le \max_{t\in[1,e]}\Bigg\{\frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{|f(s,x(s)) - f(s,y(s))|}{s}\,ds \\
&\quad + \frac{(\log t)^{\alpha-1}}{|\Delta|}\left[\frac{|A|}{\Gamma(\gamma+\alpha)}\int_1^\eta \left(\log\frac{\eta}{s}\right)^{\gamma+\alpha-1}\frac{|f(s,x(s)) - f(s,y(s))|}{s}\,ds \right.\\
&\quad \left. + \frac{|B|}{\Gamma(\alpha)}\int_1^e \left(\log\frac{e}{s}\right)^{\alpha-1}\frac{|f(s,x(s)) - f(s,y(s))|}{s}\,ds\right]\Bigg\} \\
&\le L_1\|x-y\|\max_{t\in[1,e]}\Bigg\{\frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{1}{s}\,ds \\
&\quad + \frac{(\log t)^{\alpha-1}}{|\Delta|}\left[\frac{|A|}{\Gamma(\gamma+\alpha)}\int_1^\eta \left(\log\frac{\eta}{s}\right)^{\gamma+\alpha-1}\frac{1}{s}\,ds + \frac{|B|}{\Gamma(\alpha)}\int_1^e \left(\log\frac{e}{s}\right)^{\alpha-1}\frac{1}{s}\,ds\right]\Bigg\} \\
&= L_1\omega\|x-y\|.
\end{aligned}
$$

Therefore,

$$
\|Qx - Qy\| \le L_1\omega\|x-y\|.
$$

It follows from assumption (3.3) that Q is a contraction. In consequence, it follows by Banach's fixed point theorem that the operator Q has a fixed point which corresponds to the unique solution of problem (1.1). This completes the proof. □

3.2 Existence result via Krasnoselskii's fixed point theorem

Lemma 3.2 (Krasnoselskii's fixed point theorem [30]) *Let M be a closed, bounded, convex, and nonempty subset of a Banach space X. Let A, B be the operators such that* (a) $Ax + By \in M$ *whenever* $x, y \in M$; (b) *A is compact and continuous*; (c) *B is a contraction mapping. Then there exists $z \in M$ such that $z = Az + Bz$.*

Theorem 3.3 *Let $f : [1,e] \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying* (H_1). *In addition, we assume that*

(H_2) $|f(t,x)| \le \mu(t)$, $\forall (t,u) \in [1,e] \times \mathbb{R}$, *and* $\mu \in C([1,e],\mathbb{R}^+)$.

Then problem (1.1) *has at least one solution on* $[1,e]$ *if*

$$\frac{1}{|\Delta|}\left\{\frac{|A|(\log\eta)^{\gamma+\alpha}}{\Gamma(\gamma+\alpha+1)}+\frac{|B|}{\Gamma(\alpha+1)}\right\}<1. \tag{3.4}$$

Proof We define $\sup_{t\in[1,e]}|\mu(t)| = \|\mu\|$ and choose a suitable constant $\bar{r}$ as

$$\bar{r}\geq\|\mu\||\omega|+\frac{|c|}{\Delta},$$

where ω is defined by (3.2). We define the operators $\mathcal{P}$ and $\mathcal{Q}$ on $B_{\bar{r}} = \{x\in C([1,e],\mathbb{R}) : \|x\|\leq\bar{r}\}$ as

$$\begin{aligned}(\mathcal{P}x)(t) &= \frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{f(s,x(s))}{s}\,ds,\\ (\mathcal{Q}x)(t) &= \frac{(\log t)^{\alpha-1}}{\Delta}\left[c-\frac{A}{\Gamma(\gamma+\alpha)}\int_1^\eta\left(\log\frac{\eta}{s}\right)^{\gamma+\alpha-1}\frac{f(s,x(s))}{s}\,ds\right.\\ &\quad\left.-\frac{1}{\Gamma(\alpha)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-1}\frac{f(s,x(s))}{s}\,ds\right].\end{aligned}$$

For $x,y\in B_{\bar{r}}$, we find that

$$\begin{aligned}&\|\mathcal{P}x+\mathcal{Q}y\|\\ &\quad\leq\|\mu\|\left\{\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{1}{s}\,ds+\frac{(\log t)^{\alpha-1}}{|\Delta|}\left[\frac{|A|}{\Gamma(\gamma+\alpha)}\int_1^\eta\left(\log\frac{\eta}{s}\right)^{\gamma+\alpha-1}\frac{1}{s}\,ds\right.\right.\\ &\qquad\left.\left.+\frac{|B|}{\Gamma(\alpha)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-1}\frac{1}{s}\,ds\right]\right\}+\frac{(\log t)^{\alpha-1}|c|}{|\Delta|}\\ &\quad\leq\|\mu\|\omega+\frac{|c|}{|\Delta|}\\ &\quad\leq\bar{r}.\end{aligned}$$

Thus, $\mathcal{P}x+\mathcal{Q}y\in B_{\bar{r}}$. It follows from assumption (H_1) together with (3.4) that $\mathcal{Q}$ is a contraction mapping. Continuity of f implies that the operator $\mathcal{P}$ is continuous. Also, $\mathcal{P}$ is uniformly bounded on $B_{\bar{r}}$ as

$$\|\mathcal{P}x\|\leq\frac{\|\mu\|}{\Gamma(\alpha+1)}.$$

Now we prove the compactness of the operator $\mathcal{P}$.

We define $\sup_{(t,x)\in[1,e]\times B_{\bar{r}}}|f(t,x)| = \bar{f}<\infty$, $\tau_1,\tau_2\in[1,e]$ with $\tau_1<\tau_2$, and consequently we have

$$\begin{aligned}\left|(\mathcal{P}x)(\tau_2)-(\mathcal{P}x)(\tau_1)\right| &\leq\frac{\bar{f}}{\Gamma(\alpha)}\left|\int_1^{\tau_1}\left(\log\frac{\tau_1}{s}\right)^{\alpha-1}\frac{1}{s}\,ds-\int_1^{\tau_2}\left(\log\frac{\tau_2}{s}\right)^{\alpha-1}\frac{1}{s}\,ds\right|\\ &\leq\frac{\bar{f}}{\Gamma(\alpha)}\left|\int_1^{\tau_1}\left[\left(\log\frac{\tau_1}{s}\right)^{\alpha-1}-\left(\log\frac{\tau_2}{s}\right)^{\alpha-1}\right]\frac{1}{s}\,ds\right|\\ &\quad+\frac{\bar{f}}{\Gamma(\alpha)}\left|\int_{\tau_1}^{\tau_2}\left(\log\frac{\tau_2}{s}\right)^{\alpha-1}\frac{1}{s}\,ds\right|\end{aligned}$$

$$= \frac{\bar{f}}{\Gamma(\alpha+1)}\Big[\big|(\log\tau_1)^\alpha + (\log\tau_2-\log\tau_1)^\alpha - (\log\tau_2)^\alpha\big| + \big|(\log\tau_2-\log\tau_1)^\alpha\big|\Big],$$

which is independent of x and tends to zero as $\tau_2-\tau_1\to 0$. Thus, $\mathcal{P}$ is equicontinuous. So $\mathcal{P}$ is relatively compact on $B_{\bar{r}}$. Hence, by the Arzelá-Ascoli theorem, $\mathcal{P}$ is compact on $B_{\bar{r}}$. Thus all the assumptions of Lemma 3.2 are satisfied. So the conclusion of Lemma 3.2 implies that the fractional boundary value problem (1.1) has at least one solution on $[1,e]$. The proof is completed. □

3.3 Existence result via Leray-Schauder's nonlinear alternative

Lemma 3.4 (Nonlinear alternative for single-valued maps [31]) *Let E be a Banach space, C be a closed, convex subset of E, U be an open subset of C and $0\in U$. Suppose that $F:\overline{U}\to C$ is a continuous, compact (that is, $F(\overline{U})$ is a relatively compact subset of C) map. Then either*

(i) *F has a fixed point in $\overline{U}$, or*

(ii) *there is $u\in\partial U$ (the boundary of U in C) and $\lambda\in(0,1)$ with $u=\lambda F(u)$.*

Theorem 3.5 *Let $f:[1,e]\times\mathbb{R}\to\mathbb{R}$ be a continuous function. Assume that:*

(H_3) *there exist a continuous nondecreasing function $\psi:[0,\infty)\to(0,\infty)$ and a function $p\in C([1,e],\mathbb{R}^+)$ such that*

$$\big|f(t,u)\big|\le p(t)\psi\big(\|u\|\big)\quad \textit{for each } (t,u)\in[1,e]\times\mathbb{R};$$

(H_4) *there exists a constant $M>0$ such that*

$$\frac{M}{\psi(M)\|p\|\omega+\frac{|c|}{|\Delta|}}>1.$$

Then the fractional boundary value problem (1.1) *has at least one solution on* $[1,e]$.

Proof We complete the proof in several steps. We begin by showing that *Q maps bounded sets (balls) into bounded sets in $C([1,e],\mathbb{R})$*. For a positive number r, let $B_r=\{u\in C([1,e],\mathbb{R}):\|u\|\le r\}$ be a bounded ball in $C([1,e],\mathbb{R})$. Then, for $t\in[1,e]$, we have

$$\begin{aligned}\big|Qx(t)\big| &\le \max_{t\in[1,e]}\Bigg\{\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{|f(s,x(s))|}{s}\,ds \\ &\quad+\frac{(\log t)^{\alpha-1}}{|\Delta|}\Bigg[|c|+\frac{|A|}{\Gamma(\gamma+\alpha)}\int_1^\eta\left(\log\frac{\eta}{s}\right)^{\gamma+\alpha-1}\frac{|f(s,x(s))|}{s}\,ds \\ &\quad+\frac{|B|}{\Gamma(\alpha)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-1}\frac{|f(s,x(s))|}{s}\,ds\Bigg]\Bigg\} \\ &\le \max_{t\in[1,e]}\psi\big(\|x\|\big)\|p\|\Bigg\{\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{1}{s}\,ds \\ &\quad+\frac{(\log t)^{\alpha-1}}{|\Delta|}\Bigg[\frac{|A|}{\Gamma(\gamma+\alpha)}\int_1^\eta\left(\log\frac{\eta}{s}\right)^{\gamma+\alpha-1}\frac{1}{s}\,ds\end{aligned}$$

$$+ \frac{|B|}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{1}{s}\, ds\bigg]\bigg\} + \frac{(\log t)^{\alpha-1}|c|}{|\Delta|}$$
$$\leq \psi\big(\|x\|\big)\|p\|\omega + \frac{|c|}{|\Delta|}.$$

Consequently,

$$\|Qx\| \leq \psi(r)\|p\|\omega + \frac{|c|}{|\Delta|}.$$

Next we show that Q *maps bounded sets into equicontinuous sets of* $C([1,e],\mathbb{R})$. Let $\tau_1, \tau_2 \in [1,e]$ with $\tau_1 < \tau_2$ and $x \in B_r$. Then we have

$$\begin{aligned}
&\big|(Qx)(\tau_2) - (Qx)(\tau_1)\big| \\
&\leq \frac{\psi(r)\|p\|}{\Gamma(\alpha)}\left|\int_1^{\tau_1}\left(\log\frac{\tau_1}{s}\right)^{\alpha-1}\frac{1}{s}\,ds - \int_1^{\tau_2}\left(\log\frac{\tau_2}{s}\right)^{\alpha-1}\frac{1}{s}\,ds\right| \\
&\quad + \frac{\psi(r)\|p\||(\log\tau_2)^{\alpha-1} - (\log\tau_1)^{\alpha-1}|}{|\Delta|}\left[\frac{|A|}{\Gamma(\gamma+\alpha)}\int_1^{\eta}\left(\log\frac{\eta}{s}\right)^{\gamma+\alpha-1}\frac{1}{s}\,ds\right. \\
&\quad \left. + \frac{|B|}{\Gamma(\alpha)}\int_1^{e}\left(\log\frac{e}{s}\right)^{\alpha-1}\frac{1}{s}\,ds\right] + \frac{|(\log\tau_2)^{\alpha-1} - (\log\tau_1)^{\alpha-1}||c|}{|\Delta|} \\
&\leq \frac{\psi(r)\|p\|}{\Gamma(\alpha)}\left|\int_1^{\tau_1}\left[\left(\log\frac{\tau_1}{s}\right)^{\alpha-1} - \left(\log\frac{\tau_2}{s}\right)^{\alpha-1}\right]\frac{1}{s}\,ds\right| \\
&\quad + \frac{\psi(r)\|p\|}{\Gamma(\alpha)}\left|\int_{\tau_1}^{\tau_2}\left(\log\frac{\tau_2}{s}\right)^{\alpha-1}\frac{1}{s}\,ds\right| \\
&\quad + \frac{\psi(r)\|p\||(\log\tau_2)^{\alpha-1} - (\log\tau_1)^{\alpha-1}|}{|\Delta|}\left[\frac{|A|}{\Gamma(\gamma+\alpha)}\int_1^{\eta}\left(\log\frac{\eta}{s}\right)^{\gamma+\alpha-1}\frac{1}{s}\,ds\right. \\
&\quad \left. + \frac{|B|}{\Gamma(\alpha)}\int_1^{e}\left(\log\frac{e}{s}\right)^{\alpha-1}\frac{1}{s}\,ds\right] + \frac{|(\log\tau_2)^{\alpha-1} - (\log\tau_1)^{\alpha-1}||c|}{|\Delta|}.
\end{aligned}$$

Obviously the right-hand side of the above inequality tends to zero independently of $x \in B_r$ as $\tau_2 - \tau_1 \to 0$. As Q satisfies the above assumptions, therefore it follows by the Arzelá-Ascoli theorem that $Q : C([1,e],\mathbb{R}) \to C([1,e],\mathbb{R})$ is completely continuous.

Let x be a solution. Then, for $t \in [1,e]$, as in the first step, we have

$$\|x\| \leq \psi\big(\|x\|\big)\|p\|\omega + \frac{|c|}{|\Delta|},$$

which implies that

$$\frac{\|x\|}{\psi(\|x\|)\|p\|\omega + \frac{|c|}{|\Delta|}} \leq 1.$$

In view of (H_4), there exists M such that $\|u\| \neq M$. Let us set

$$U = \big\{u \in C\big([1,e],\mathbb{R}\big) : \|u\| < M\big\}.$$

Note that the operator $Q : \overline{U} \to C([1,e],\mathbb{R})$ is continuous and completely continuous. From the choice of U, there is no $u \in \partial U$ such that $u = \lambda Qu$ for some $\lambda \in (0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type (Lemma 3.4), we deduce that Q has a fixed point $u \in \overline{U}$ which is a solution of problem (1.1). This completes the proof. □

3.4 Existence results via Leray-Schauder degree

Theorem 3.6 *Let $f : [1,e] \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that*

(H_5) *there exist constants $0 \le \kappa < \omega^{-1}$ and $M_1 > 0$ such that*

$$\left|f(t,x)\right| \le \kappa|x| + M_1 \quad \textit{for all } (t,x) \in [1,e] \times \mathbb{R}.$$

Then the fractional boundary value problem (1.1) *has at least one solution on* $[1,e]$.

Proof We define an operator $Q : C([1,e],\mathbb{R}) \to C([1,e],\mathbb{R})$ as in (3.1) and consider the fixed point problem

$$u = Qu. \tag{3.5}$$

We are going to prove that there exists a fixed point $u \in C([1,e],\mathbb{R})$ satisfying (3.5). It is sufficient to show that $Q : \overline{B}_R \to C([1,e],\mathbb{R})$ satisfies

$$x \neq \lambda Qx, \quad \forall x \in \partial B_R, \forall \lambda \in [0,1], \tag{3.6}$$

where $B_R = \{x \in C([1,e],\mathbb{R}) : \max_{t\in[1,e]} |x(t)| < R, R > 0\}$. We define

$$H(\lambda,x) = \lambda Qx, \quad u \in C\bigl([1,e],\mathbb{R}\bigr), \lambda \in [0,1].$$

As shown in Theorem 3.5, we have that the operator Q is continuous, uniformly bounded, and equicontinuous. Then, by the Arzelá-Ascoli theorem, a continuous map h_λ defined by $h_\lambda(x) = u - H(\lambda,x) = x - \lambda Qx$ is completely continuous. If (3.6) is true, then the following Leray-Schauder degrees are well defined, and by the homotopy invariance of topological degree it follows that

$$\begin{aligned} \deg(h_\lambda, B_R, 0) &= \deg(I - \lambda Q, B_R, 0) = \deg(h_1, B_R, 0) \\ &= \deg(h_0, B_R, 0) = \deg(I, B_R, 0) = 1 \neq 0, \quad 0 \in B_R, \end{aligned} \tag{3.7}$$

where I denotes the identity operator. By the nonzero property of Leray-Schauder degree, $h_1(x) = x - Qx = 0$ for at least one $x \in B_R$. In order to prove (3.6), we assume that $x = \lambda Qx$ for some $\lambda \in [0,1]$ and for all $t \in [1,e]$. Then, with $\|x\| = \sup_{t\in[1,e]} |x(t)|$, we have

$$\begin{aligned} \bigl|Qx(t)\bigr| \le \max_{t\in[1,e]} \biggl\{ &\frac{1}{\Gamma(\alpha)} \int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} \frac{|f(s,x(s))|}{s}\,ds \\ &+ \frac{(\log t)^{\alpha-1}}{|\Delta|} \biggl[|c| + \frac{|A|}{\Gamma(\gamma+\alpha)} \int_1^\eta \left(\log\frac{\eta}{s}\right)^{\gamma+\alpha-1} \frac{|f(s,x(s))|}{s}\,ds \\ &+ \frac{|B|}{\Gamma(\alpha)} \int_1^e \left(\log\frac{e}{s}\right)^{\alpha-1} \frac{|f(s,x(s))|}{s}\,ds \biggr] \biggr\} \end{aligned}$$

$$
\begin{aligned}
&\leq \bigl(\kappa \|x\| + M\bigr)\left\{ \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} \frac{1}{s}\, ds \right.\\
&\quad + \frac{(\log t)^{\alpha-1}}{|\Delta|} \left[\frac{|A|}{\Gamma(\gamma+\alpha)} \int_1^\eta \left(\log\frac{\eta}{s}\right)^{\gamma+\alpha-1} \frac{1}{s}\, ds \right.\\
&\quad \left.\left. + \frac{|B|}{\Gamma(\alpha)} \int_1^e \left(\log\frac{e}{s}\right)^{\alpha-1} \frac{1}{s}\, ds \right]\right\} + \frac{|c|}{|\Delta|}\\
&\leq \bigl(\kappa \|x\| + M_1\bigr)\omega + \frac{|c|}{|\Delta|},
\end{aligned}
$$

which, on solving for $\|x\|$, yields

$$\|x\| \leq \frac{M_1\omega + (|c|/|\Delta|)}{1-\kappa\omega}.$$

If $R = \frac{M_1\omega + (|c|/|\Delta|)}{1-\kappa\omega} + 1$, inequality (3.6) holds. This completes the proof. □

3.5 Examples

Example 3.7 Consider the problem

$$
\begin{cases}
D^{3/2}x(t) = \frac{L}{2}\left(\sin x + \frac{|x|^3}{1+|x|^3}\right) + \frac{\sqrt{t}+1}{e}, & 1 < t < e,\\
x(1) = 0, \qquad I^{1/2}x(2) + x(e) = 4.
\end{cases}
\tag{3.8}
$$

Here, $\alpha = 3/2$, $\gamma = 1/2$, $\eta = 2$, $A = 1$, $B = 1$, $c = 4$, and $f(t,x) = \frac{L}{2}(\sin x + \frac{|x|^3}{1+|x|^3}) + \frac{\sqrt{t}+1}{e}$. With the given values, we find that

$$
\begin{aligned}
&\Delta = B + \frac{A\Gamma(\alpha)}{\Gamma(\gamma+\alpha)}(\log\eta)^{\gamma+\alpha-1} \simeq 1 + \sqrt{\pi}\log\sqrt{2} \approx 2.228571,\\
&\omega = \frac{1}{\Gamma(\alpha+1)} + \frac{1}{\Delta}\left(\frac{|A|(\log\eta)^{\gamma+\alpha}}{\Gamma(\gamma+\alpha+1)} + \frac{|B|}{\Gamma(\alpha+1)}\right) \approx 1.197596,\\
&\bigl|f(t,x) - f(t,y)\bigr| \leq \frac{L}{2}\left|\sin x + \frac{|x|^3}{1+|x|^3} - \sin y - \frac{|y|^3}{1+|y|^3}\right| \leq L|x-y|.
\end{aligned}
$$

With $L < \frac{1}{\omega} \approx 0.835006$, all the assumptions of Theorem 3.1 are satisfied. Hence, problem (3.8) has a unique solution on $[1,e]$.

Example 3.8 Consider problem (3.8) with

$$f(t,x) = \frac{e^{-t}}{3}\left(\frac{(1+x)^2}{1+(1+x)^2} + x\right). \tag{3.9}$$

Clearly $|f(t,x)| \leq \frac{1}{3e}(1+\|x\|)$. By assumption (H_4),

$$\frac{M}{\psi(M)\|p\|\omega + \frac{|c|}{|\Delta|}} > 1,$$

we find that $M > 2.275971$. Thus, by Theorem 3.5, there exists at least one solution for problem (3.8) with $f(t,x)$ given by (3.9).

Example 3.9 Consider problem (3.8) with

$$f(t,x) = \sin(ax) + \sqrt{\log(t) + 3}, \quad a > 0. \tag{3.10}$$

It is obvious that $|f(t,x)| = |\sin(ax) + \sqrt{\log(t) + 3}| \leq a|x| + 2$. With $a < 1/\omega \approx 0.835006$, the assumptions of Theorem 3.6 are satisfied, and in consequence problem (3.8) with $f(t,x)$ given by (3.10) has a solution on $[1,e]$.

4 A companion problem

In this section, we consider a companion boundary value problem by replacing the nonlocal integral boundary condition in (1.1) by $AI^\gamma x(e) + Bx(\eta) = c$. Precisely we consider the following problem:

$$\begin{cases} D^\alpha x(t) = f(t,x(t)), & 1 < t < e, 1 < \alpha \leq 2, \\ x(1) = 0, & AI^\gamma x(e) + Bx(\eta) = c, \quad 1 < \eta < e. \end{cases} \tag{4.1}$$

In this case, we have an operator of the form $\mathcal{T} : C([1,e],\mathbb{R}) \to C([1,e],\mathbb{R})$ by

$$\begin{aligned} \mathcal{T}x(t) = {} & \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{f(s,x(s))}{s}\, ds \\ & + \frac{(\log t)^{\alpha-1}}{\Theta} \left\{ c - \frac{A}{\Gamma(\gamma+\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\gamma+\alpha-1} \frac{f(s,x(s))}{s}\, ds \right. \\ & \left. - \frac{B}{\Gamma(\alpha)} \int_1^\eta \left(\log \frac{\eta}{s}\right)^{\alpha-1} \frac{f(s,x(s))}{s}\, ds \right\}, \quad t \in [1,e], \end{aligned} \tag{4.2}$$

where

$$\Theta = B(\log \eta)^{\alpha-1} + \frac{A}{\Gamma(\gamma+1)}.$$

The existence results (Theorems 3.1, 3.3, 3.5, 3.6) for problem (4.1) can be obtained using the strategy followed in Section 3 by replacing the value of ω with the following one:

$$\omega_1 = \frac{1}{\Gamma(\alpha+1)} + \frac{1}{|\Theta|} \left\{ \frac{|A|}{\Gamma(\gamma+\alpha+1)} + \frac{|B|(\log \eta)^\alpha}{\Gamma(\alpha+1)} \right\}. \tag{4.3}$$

5 Conclusions

We have investigated the existence and uniqueness of solutions for a semi-linear Hadamard type fractional differential equation supplemented with nonlocal non-conserved boundary conditions involving Hadamard integral. The uniqueness result is proved by applying Banach's fixed point theorem, while the three existence results are established by means of Krasnoselskii's fixed point theorem, Leray-Schauder's nonlinear alternative, and Leray-Schauder degree, respectively. We have also discussed a companion problem (4.1) by replacing the condition $AI^\gamma x(\eta) + Bx(e) = c$ with $AI^\gamma x(e) + Bx(\eta) = c$ in problem (1.1). The results presented in this paper are more general and correspond to several known and new results by fixing the parameters involved in problem (1.1) appropriately. For instance, we have:

- By taking $A = 0$, $c = 0$, $B \neq 0$, our results correspond to the ones for Hadamard type fractional differential equations with Dirichlet boundary conditions.
- Letting $A = 1$, $B = -1$, $c = 0$, and $\eta \to e$ in the results of this paper, we obtain the ones presented in [32].
- With $A \neq 0$, $B = 0$, $c = 0$, our problem becomes an 'average type' nonlocal boundary value problem in the sense of Hadamard integral (in the classical sense for $\gamma = 1$). This reduced integral condition can also be termed as a 'conserved' condition in the sense of Hadamard. In this case, the operator $Q : C([1,e],\mathbb{R}) \to C([1,e],\mathbb{R})$ takes the form

$$\begin{aligned} Qx(t) = {}& \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{f(s,x(s))}{s}\, ds \\ & - \frac{A(\log t)^{\alpha-1}}{\Delta\Gamma(\gamma+\alpha)} \int_1^\eta \left(\log \frac{\eta}{s}\right)^{\gamma+\alpha-1} \frac{f(s,x(s))}{s}\, ds, \quad t \in [1,e]. \end{aligned}$$

In relation to problem (4.1), we can make similar observations.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Each of the authors, AA, SKN, BA, and AH, contributed to each part of this work equally and read and approved the final version of the manuscript.

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