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Asymptotic behavior of increasing positive solutions of second order quasilinear ordinary differential equations in the framework of regular variation

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Abstract

The existence and asymptotic behavior at infinity of increasing positive solutions of second order quasilinear ordinary differential equations $(p(t)\varphi(x'(t)))' + q(t)\psi(x(t)) = 0$ are studied in the framework of regular variation, where p and q are continuous functions regularly varying at infinity and φ, ψ are both continuous functions regularly varying at zero and regularly varying at infinity, respectively.

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1 Introduction

It is of particular interest in the theory of qualitative analysis of differential equations to determine the exact asymptotic behavior at infinity of the solutions under the appropriate assumptions for the coefficients of an equation. This problem is extremely complex when the coefficients are general continuous functions. Thus, the recent research shows that the problem should be studied in the framework of regularly varying functions (also known as Karamata functions). This approach was initiated by Avakumović in 1947 (see [1]), and followed by Marić and Tomić (see [2–4]). It turns out that the problem is completely solvable in the case when the coefficients are regularly varying or generalized regularly varying functions. Namely, in this case, it is possible to completely determine the existence of the solutions, as well as their asymptotic behavior at infinity.

In this paper, we study the differential equation of the form

$$\text{(E)}\qquad \big(p(t)\varphi\big(x'(t)\big)\big)' + q(t)\psi\big(x(t)\big) = 0, \qquad t \geq a > 0,$$

under the following assumptions:

(i) $\varphi : (0,\infty) \to (0,\infty)$ is an increasing continuous function which is regularly varying at zero of index $\alpha > 0$;

(ii) $\psi : (0,\infty) \to (0,\infty)$ is a continuous function which is regularly varying at infinity of index $\beta \in (0,\alpha)$;

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(iii) $p : [a,\infty) \to (0,\infty)$ is a continuous function which is regularly varying at infinity of index $\eta \in (0,\alpha)$;
(iv) $q : [a,\infty) \to (0,\infty)$ is a continuous function which is regularly varying at infinity of index $\sigma \in \mathbb{R}$.

By a solution of (E) we mean a function $x(t) : [T,\infty) \to \mathbb{R}$, $T \geq a$ which is continuously differentiable together with $p(t)\varphi(x'(t))$ on $[T,\infty)$ and satisfies (E) at every point of $[T,\infty)$.

It is easily seen (see [5]) that if $x(t)$ is an increasing positive solution of (E), then we have the following classification of increasing positive solutions of (E) into three types according to their asymptotic behavior at infinity:

$$\text{(I)}\quad \lim_{t\to\infty} x(t) = \text{const.} > 0,$$

$$\text{(II)}\quad \lim_{t\to\infty} x(t) = \infty, \qquad \lim_{t\to\infty} p(t)\varphi\bigl(x'(t)\bigr) = 0,$$

$$\text{(III)}\quad \lim_{t\to\infty} \frac{x(t)}{P(t)} = \text{const.} > 0,$$

where the function $P(t)$ is defined as

$$P(t) = \int_a^t \varphi^{-1}\bigl(p(s)^{-1}\bigr)\,ds \tag{1.1}$$

and $\varphi^{-1}(\cdot)$ denotes the inverse function of $\varphi(\cdot)$.

Solutions of type (I), (II), (III) are often called, respectively, *subdominant, intermediate,* and *dominant* solutions.

It is well known (see [5, 6]) that the existence of subdominant and dominant solutions for (E) with continuous coefficients $p(t)$, $q(t)$, $\varphi(s)$, and $\psi(s)$ can be completely characterized by the convergence of the integrals

$$I = \int_a^\infty q(t)\psi\bigl(P(t)\bigr)\,dt, \qquad J = \int_a^\infty \varphi^{-1}\left(p(t)^{-1}\int_t^\infty q(s)\,ds\right)dt.$$

Theorem 1.1 *Let $p(t), q(t) \in C[a,\infty)$ and $\varphi(s), \psi(s) \in C[0,\infty)$.*
(a) *Equation* (E) *has an increasing positive solution of type* (I) *if and only if $J < \infty$.*
(b) *Equation* (E) *has an increasing positive solution of type* (III) *if and only if $I < \infty$.*
(c) *Equation* (E) *has an increasing positive solution of type* (II) *if $J = \infty$ and $I < \infty$.*

For the existence of intermediate solutions for (E), necessary conditions can be obtained with relative ease. But the problem of establishing necessary and sufficient conditions turns out to be extremely difficult and thus has been an open problem for a long time.

In this paper we establish the necessary and sufficient conditions for the existence of intermediate solutions for (E) and precisely determine their behavior at infinity, using the theory of regularly varying functions. The present work was motivated by the recent progress in the asymptotic analysis of differential equations by means of regularly varying functions in the sense of Karamata, which was initiated by the monograph of Marić [7]. Also, the equation under consideration in this paper is a generalization of the equation

$$x''(t) + q(t)\phi\bigl(x(t)\bigr) = 0,$$

considered in [8], as well as of the equation

$$\bigl(p(t)\bigl|x'(t)\bigr|^{\alpha-1}x'(t)\bigr)' + q(t)\bigl|x(t)\bigr|^{\beta-1}x(t) = 0$$

considered in [9, 10]. See also [11–16] for related results regarding second order equations and first order systems, and [17–21] for high-order differential equations and systems.

The main body of the paper is divided into six sections. The definition and basic properties of regularly varying functions are given in Section 2. The main results are stated in Section 3 and proved in Section 5. In Section 4 we collect some preparatory results which will help us to simplify the proof of our main theorems. Finally, some illustrative examples are presented in Section 6.

2 Regularly varying functions

In our analysis we shall extensively use the class of regularly varying functions introduced by Karamata in 1930 by the following.

Definition 2.1 A measurable function $f : [a,\infty) \to (0,\infty)$, $a > 0$ is said to be regularly varying at infinity of index $\rho \in \mathbb{R}$ if

$$\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = \lambda^{\rho} \quad \text{for all } \lambda > 0. \tag{2.1}$$

A measurable function $f : (0,a) \to (0,\infty)$ is said to be regularly varying at zero of index $\rho \in \mathbb{R}$ if

$$\lim_{t\to 0+} \frac{f(\lambda t)}{f(t)} = \lambda^{\rho} \quad \text{for all } \lambda > 0. \tag{2.2}$$

The set of regularly varying functions of index ρ at infinity and at zero, are denoted, respectively, with $\mathrm{RV}(\rho)$ and $\mathrm{RV}_0(\rho)$. If, in particular $\rho = 0$, the function f is called *slowly varying* at infinity or at zero. With SV and SV_0 we denote, respectively, the set of slowly varying functions at infinity and at zero. By an only regularly or a slowly varying function, we mean regularity at infinity.

It follows from Definition 2.1 that any function $f(t) \in \mathrm{RV}(\rho)$ can be written as

$$f(t) = t^{\rho} g(t), \quad g(t) \in \mathrm{SV}, \tag{2.3}$$

and so the class SV of slowly varying functions is of fundamental importance in the theory of regular variation. If, in particular, the function $g(t) \to k > 0$ as $t \to \infty$, it is called a *trivial slowly varying*, denoted by $g(t) \in$ tr-SV, and the function $f(t)$ is called a *trivial regularly varying of index* ρ, denoted by $f(t) \in$ tr-RV(ρ). Otherwise, the function $g(t)$ is called a *nontrivial slowly varying*, denoted by $g(t) \in$ ntr-SV, and the function $f(t)$ is called a *nontrivial regularly varying of index* ρ, denoted by $f(t) \in$ ntr-RV(ρ).

Since regularly variation of $f(\cdot)$ at zero of index α means in fact regularly variation of $f(1/t)$ at infinity of index $-\alpha$, the properties of RV_0 functions can be deduced from theory of RV functions.

For a comprehensive treatise on regular variation the reader is referred to Bingham *et al.* [22]. See also Seneta [23]. However, to help the reader we present here some elementary properties of regularly varying functions and a fundamental result, called *Karamata's integration theorem*, which will be used throughout the paper.

Proposition 2.1 (Karamata's integration theorem) *Let* $L(t) \in \mathrm{SV}$. *Then*:
(i) *If* $\alpha > -1$,

$$\int_a^t s^\alpha L(s)\,ds \sim \frac{t^{\alpha+1}L(t)}{\alpha+1}, \quad t \to \infty.$$

(ii) *If* $\alpha < -1$,

$$\int_t^\infty s^\alpha L(s)\,ds \sim -\frac{t^{\alpha+1}L(t)}{\alpha+1}, \quad t \to \infty.$$

(iii) *If* $\alpha = -1$, *the integral* $\int_a^\infty s^{-1}L(s)\,ds$ *may or may not be convergent. The integral* $m_1(t) = \int_a^t s^{-1}L(s)\,ds$ *is a new slowly varying function and* $L(t)/m_1(t) \to 0$, $t \to \infty$. *In the case* $\int_a^\infty s^{-1}L(s)\,ds < \infty$, *again* $m_2(t) = \int_t^\infty s^{-1}L(s)\,ds \in \mathrm{SV}$ *and* $L(t)/m_2(t) \to 0$, $t \to \infty$.

The symbol $\sim$ denotes the asymptotic equivalence of two positive functions, *i.e.*,

$$f(t) \sim g(t), \quad t \to \infty \quad \Longleftrightarrow \quad \lim_{t\to\infty} \frac{g(t)}{f(t)} = 1.$$

We shall also use the following results:

Proposition 2.2 *Let* $g_1(t) \in \mathrm{RV}(\sigma_1)$, $g_2(t) \in \mathrm{RV}(\sigma_2)$, $g_3(t) \in \mathrm{RV}_0(\sigma_3)$. *Then*:
(i) $(g_1(t))^\alpha \in \mathrm{RV}(\alpha\sigma_1)$ *for any* $\alpha \in \mathbb{R}$;
(ii) $g_1(t) + g_2(t) \in \mathrm{RV}(\sigma)$, $\sigma = \max(\sigma_1, \sigma_2)$;
(iii) $g_1(t)g_2(t) \in \mathrm{RV}(\sigma_1 + \sigma_2)$;
(iv) $g_1(g_2(t)) \in \mathrm{RV}(\sigma_1\sigma_2)$, *if* $g_2(t) \to \infty$, *as* $t \to \infty$; $g_3(g_2(t)) \in \mathrm{RV}(\sigma_3\sigma_2)$, *if* $g_2(t) \to 0$, *as* $t \to \infty$;
(v) *for any* $\varepsilon > 0$ *and* $L(t) \in \mathrm{SV}$ *one has* $t^\varepsilon L(t) \to \infty$, $t^{-\varepsilon}L(t) \to 0$, *as* $t \to \infty$.

Proposition 2.3 *If* $f(t) \sim t^\alpha l(t)$ *as* $t \to \infty$ *with* $l(t) \in \mathrm{SV}$, *then* $f(t)$ *is a regularly varying function of index* α *i.e.* $f(t) = t^\alpha l^*(t)$, $l^*(t) \in \mathrm{SV}$, *where in general* $l^*(t) \neq l(t)$, *but* $l^*(t) \sim l(t)$ *as* $t \to \infty$.

Proposition 2.4 *A positive measurable function* $l(t)$ *belongs to* SV *if and only if for every* $\alpha > 0$ *there exist a non-decreasing function* Ψ *and a non-increasing function* ψ *with*

$$t^\alpha l(t) \sim \Psi(t) \quad \textit{and} \quad t^{-\alpha} l(t) \sim \psi(t), \quad t \to \infty.$$

Proposition 2.5 *For the function* $f(t) \in \mathrm{RV}(\alpha)$, $\alpha > 0$, *there exists* $g(t) \in \mathrm{RV}(1/\alpha)$ *such that*

$$f\big(g(t)\big) \sim g\big(f(t)\big) \sim t \quad \textit{as } t \to \infty.$$

Here g is an asymptotic inverse of f (and it is determined uniquely to within asymptotic equivalence).

Note that the same result holds for $t \to 0$ *i.e.* when $f(t) \in \mathrm{RV}_0(\alpha)$, $\alpha > 0$.

Proposition 2.6 *For the function $f(t) \in \mathrm{RV}_0(\alpha)$, $\alpha > 0$, there exists $g(t) \in \mathrm{RV}_0(1/\alpha)$ such that*

$$f\bigl(g(t)\bigr) \sim g\bigl(f(t)\bigr) \sim t \quad \textit{as } t \to 0.$$

Proof Since $f(t) \in \mathrm{RV}_0(\alpha)$, we have $f(1/t) \in \mathrm{RV}(-\alpha)$ and $1/f(1/t) \in \mathrm{RV}(\alpha)$. We can apply the Proposition 2.5 to the function $\tilde{f}(t) = 1/f(1/t)$. Then there exists $\tilde{g} \in \mathrm{RV}(1/\alpha)$ such that

$$\tilde{f}\bigl(\tilde{g}(t)\bigr) \sim \tilde{g}\bigl(\tilde{f}(t)\bigr) \sim t \quad \text{as } t \to \infty.$$

Then it is easy to show that the function $g(t) = 1/\tilde{g}(1/t) \in \mathrm{RV}_0(1/\alpha)$ is an asymptotic inverse of f. □

Next result is proved in [24] and we are going to use it very often in our proofs. It helps us with manipulation of the asymptotic relations.

Let $H = \{x | x : [a,\infty) \to (0,\infty)\}$ and $H_1 = \{x \in H | x(t) \to \infty, t \to \infty\}$. If $\simeq$ denotes the asymptotic similarity of two positive functions, *i.e.*,

$$f(t) \simeq g(t), \quad t \to \infty \quad \Longleftrightarrow \quad \lim_{t\to\infty} \frac{g(t)}{f(t)} = \text{const.} > 0$$

and ρ_1, ρ_2 are arbitrary relations from the set $\{\sim, \simeq\}$, then let $\mathrm{Hom}((H_1,\rho_1);(H,\rho_2))$ be the set of all measurable functions $F : [a,\infty) \to (0,\infty)$ such that

$$x(t)\rho_1 y(t), \quad t \to \infty \quad \Rightarrow \quad F\bigl(x(t)\bigr)\rho_2 F\bigl(y(t)\bigr), \quad t \to \infty.$$

Proposition 2.7 *Let $F : [a,\infty) \to (0,\infty)$ be a measurable function. Then*

$$F \in \mathrm{RV} \quad \Leftrightarrow \quad F \in \mathrm{Hom}\bigl((H_1,\simeq);(H,\simeq)\bigr).$$

To avoid repetitions we state here basic conditions imposed of the functions φ, ψ, p, q. In what follows we always assume

$$\begin{aligned} &\varphi(s) \in \mathrm{RV}_0(\alpha), \quad \alpha > 0; \qquad \psi(s) \in \mathrm{RV}(\beta), \quad \alpha > \beta > 0; \\ &p(t) \in \mathrm{RV}(\eta), \quad \eta \in (0,\alpha); \qquad q(t) \in \mathrm{RV}(\sigma), \quad \sigma \in \mathbb{R}. \end{aligned} \tag{2.4}$$

Using the notation (2.3), we can express $\varphi(s)$, $\psi(s)$, $p(t)$, and $q(t)$ as

$$\varphi(s) = s^{\alpha} L_1(s), \quad L_1(s) \in \mathrm{SV}_0; \qquad \psi(s) = s^{\beta} L_2(s), \quad L_2(s) \in \mathrm{SV}; \tag{2.5}$$

$$p(t) = t^{\eta} l_p(t), \quad l_p(t) \in \mathrm{SV}; \qquad q(t) = t^{\sigma} l_q(t), \quad l_q(t) \in \mathrm{SV}. \tag{2.6}$$

By assumption (i), $\varphi(s)$ is an increasing function, so $\varphi(s)$ has the inverse function, denoted by $\varphi^{-1}(s)$ and from (2.5) we conclude that

$$\varphi^{-1}(s) \in \mathrm{RV}_0(1/\alpha) \quad \Rightarrow \quad \varphi^{-1}(s) = s^{1/\alpha} L(s), \quad L(s) \in \mathrm{SV}_0. \tag{2.7}$$

We also need the additional requirements for the slowly varying parts of φ and ψ:

$$L\bigl(tu(t)\bigr) \sim L(t), \quad t \to 0, \forall u(t) \in \mathrm{SV}_0 \cap C^1(\mathbb{R}); \tag{2.8}$$

$$L_2\bigl(tu(t)\bigr) \sim L_2(t), \quad t \to \infty, \forall u(t) \in \mathrm{SV} \cap C^1(\mathbb{R}). \tag{2.9}$$

It is easy to check that this is satisfied by *e.g.*

$$L_0(t) = \prod_{k=1}^{N} (\log_k t)^{\alpha_k}, \quad \alpha_k \in \mathbb{R}, \quad \text{but not by} \quad L_0(t) = \exp \prod_{k=1}^{N} (\log_k t)^{\beta_k}, \quad \beta_k \in (0,1),$$

where $\log_k t = \log \log_{k-1} t$, $k = 1, 2, \ldots$.

Remark 2.1 The condition (2.8) implies an useful property of the function φ^{-1}. For $u(t) \in \mathrm{SV} \cap C^1(\mathbb{R})$ and $\lambda \in \mathbb{R}^-$, applying Proposition 2.2(iv), we have $u(s^{\frac{1}{\lambda}}) \in \mathrm{SV}_0 \cap C^1(\mathbb{R})$. Using substitution $t^\lambda = s$ ($s \to 0$ as $t \to \infty$) and (2.8) we obtain

$$L\bigl(t^\lambda u(t)\bigr) = L\bigl(su\bigl(s^{\frac{1}{\lambda}}\bigr)\bigr) \sim L(s) = L\bigl(t^\lambda\bigr), \qquad t \to \infty, \forall \lambda \in \mathbb{R}^-, \forall u(t) \in \mathrm{SV} \cap C^1(\mathbb{R}),$$

from which it follows that

$$\varphi^{-1}\bigl(t^\lambda u(t)\bigr) \sim \varphi^{-1}\bigl(t^\lambda\bigr) u(t)^{\frac{1}{\alpha}}, \quad t \to \infty, \forall \lambda \in \mathbb{R}^-, \forall u(t) \in \mathrm{SV} \cap C^1(\mathbb{R}). \tag{2.10}$$

Similarly, the condition (2.9) implies an useful property of the function ψ:

$$\psi\bigl(t^\lambda u(t)\bigr) \sim \psi\bigl(t^\lambda\bigr) u(t)^\beta, \quad t \to \infty, \forall \lambda \in \mathbb{R}^+, \forall u(t) \in \mathrm{SV} \cap C^1(\mathbb{R}). \tag{2.11}$$

3 Main results

This section is devoted to the study of the existence and asymptotic behavior of an intermediate regularly varying solutions of (E) with functions φ, ψ, p, q satisfying (2.4). We seek such solutions $x(t)$ of (E) expressed in the form

$$x(t) = t^\rho l_x(t), \quad l_x(t) \in \mathrm{SV}. \tag{3.1}$$

Since $\eta > 0$, applying Proposition 2.2(v), we have $\lim_{t\to\infty} p(t) = \infty$. Then, applying Proposition 2.2(iv), we get $\varphi^{-1}(p(t)^{-1}) \in \mathrm{RV}_0(-\frac{\eta}{\alpha})$ so that the assumption $\eta < \alpha$ ensures that we may apply Karamata's integration theorem (Proposition 2.1) to the integral in (1.1). Using (2.6), (2.10), (2.7), and Proposition 2.1 we obtain

$$\begin{aligned} P(t) &= \int_a^t \varphi^{-1}\bigl(s^{-\eta} l_p(s)^{-1}\bigr)\, ds \sim \int_a^t \varphi^{-1}\bigl(s^{-\eta}\bigr) l_p(s)^{-\frac{1}{\alpha}}\, ds \\ &= \int_a^t s^{-\frac{\eta}{\alpha}} L\bigl(s^{-\eta}\bigr) l_p(s)^{-\frac{1}{\alpha}}\, ds \sim \frac{\alpha}{\alpha - \eta} t^{1-\frac{\eta}{\alpha}} L\bigl(t^{-\eta}\bigr) l_p(t)^{-\frac{1}{\alpha}}, \quad t \to \infty, \end{aligned} \tag{3.2}$$

implying that $P(t) \in \mathrm{RV}(1 - \frac{\eta}{\alpha})$. Since $\eta < \alpha$ by Proposition 2.2(v) we have $\lim_{t\to\infty} P(t) = \infty$.

We emphasize that we exclude the case $\eta = \alpha$ because of the computational difficulty and the fact that the integral

$$\int_a^t \varphi^{-1}\big(p(s)^{-1}\big)\,ds = \int_a^t s^{-1}L\big(s^{-\eta}\big)l_p(s)^{-\frac{1}{\alpha}}\,ds$$

might be either convergent or divergent.

Since there are positive constants c_1 and c_2 such that $c_1 \le x(t) \le c_2P(t)$, for all large t, the regularity index ρ of $x(t)$ must satisfy $0 \le \rho \le 1 - \frac{\eta}{\alpha}$. Therefore, the class of intermediate regularly varying solutions of (E) is divided into three types of subclasses:

$$\text{ntr-SV}, \qquad \text{RV}(\rho), \quad \rho \in \left(0, 1 - \frac{\eta}{\alpha}\right), \qquad \text{ntr-RV}\left(1 - \frac{\eta}{\alpha}\right).$$

To state our main results, we will need the function

$$\Psi(y) = \int_0^y \frac{dv}{\psi(v)^{\frac{1}{\alpha}}}, \quad y > 0, \tag{3.3}$$

which is clearly increasing on $(0,\infty)$. From (2.5), (3.3), and Proposition 2.1 we get

$$\Psi(y) = \int_0^y v^{-\frac{\beta}{\alpha}}L_2(v)^{-\frac{1}{\alpha}}\,dv \sim \frac{\alpha}{\alpha-\beta}y^{1-\frac{\beta}{\alpha}}L_2(y)^{-\frac{1}{\alpha}} = \frac{\alpha}{\alpha-\beta}\frac{y}{\psi(y)^{\frac{1}{\alpha}}}, \quad y \to \infty, \tag{3.4}$$

implying $\Psi(y) \in \text{RV}(\frac{\alpha-\beta}{\alpha})$ and $\Psi^{-1}(y) \in \text{RV}(\frac{\alpha}{\alpha-\beta})$ with $\frac{\alpha-\beta}{\alpha} > 0$.

Theorem 3.1 *Suppose that* (2.4), (2.8), *and* (2.9) *hold. Equation* (E) *possesses intermediate solutions* $x(t) \in$ ntr-SV *if and only if*

$$\sigma = \eta - \alpha - 1 \quad \text{and} \quad \int_a^\infty \varphi^{-1}\left(p(t)^{-1}\int_t^\infty q(s)\,ds\right)dt = \infty, \tag{3.5}$$

in which case any such solution $x(t)$ *has the asymptotic behavior* $x(t) \sim X_1(t)$, $t \to \infty$, *where*

$$X_1(t) = \Psi^{-1}\left(\int_a^t \varphi^{-1}\left(p(s)^{-1}\int_s^\infty q(r)\,dr\right)ds\right), \quad t \ge t_0. \tag{3.6}$$

Theorem 3.2 *Suppose that* (2.4), (2.8), *and* (2.9) *hold. Equation* (E) *possesses intermediate solutions* $x(t) \in \text{RV}(\rho)$ *with* $\rho \in (0, 1 - \frac{\eta}{\alpha})$ *if and only if*

$$\eta - \alpha - 1 < \sigma < \frac{\beta}{\alpha}\eta - \beta - 1, \tag{3.7}$$

in which case ρ *is given by*

$$\rho = \frac{\sigma + \alpha + 1 - \eta}{\alpha - \beta} \tag{3.8}$$

and any such solution $x(t)$ *has the asymptotic behavior* $x(t) \sim X_2(t)$, $t \to \infty$, *where*

$$X_2(t) = \Psi^{-1}\left(\frac{\alpha}{\alpha-\beta}\frac{t^{2-\rho+\frac{1}{\alpha}}}{\rho[\alpha(1-\rho)-\eta]^{\frac{1}{\alpha}}}\varphi^{-1}\big(t^{\alpha(\rho-1)}\big)p(t)^{-\frac{1}{\alpha}}q(t)^{\frac{1}{\alpha}}\right), \quad t \ge t_0. \tag{3.9}$$

Theorem 3.3 *Suppose that* (2.4), (2.8), *and* (2.9) *hold*. *Equation* (E) *possesses intermediate solutions* $x(t) \in \text{ntr-RV}(1 - \frac{\eta}{\alpha})$ *if and only if*

$$\sigma = \frac{\beta}{\alpha}\eta - \beta - 1 \quad \text{and} \quad \int_a^\infty q(t)\psi\bigl(P(t)\bigr)\,dt < \infty, \tag{3.10}$$

in which case any such solution $x(t)$ *has the asymptotic behavior* $x(t) \sim X_3(t)$, $t \to \infty$, *where*

$$X_3(t) = P(t)\biggl(\frac{\alpha - \beta}{\alpha}\int_t^\infty q(s)\psi\bigl(P(s)\bigr)\,ds\biggr)^{\frac{1}{\alpha-\beta}}, \quad t \geq t_0. \tag{3.11}$$

4 Preparatory results

Let $x(t)$ be an intermediate solution of (E) defined on $[t_0, \infty)$. Since $\lim_{t\to\infty} p(t)\varphi(x'(t)) = \lim_{t\to\infty} x'(t) = 0$, $\lim_{t\to\infty} x(t) = \infty$, integrating of (E) first on (t_0, ∞) and then on $[t_0, t]$ gives

$$x(t) = x(t_0) + \int_{t_0}^t \varphi^{-1}\biggl(p(s)^{-1}\int_s^\infty q(r)\psi\bigl(x(r)\bigr)\,dr\biggr)\,ds, \quad t \geq t_0. \tag{4.1}$$

It follows therefore that $x(t)$ satisfies the integral asymptotic relation

$$x(t) \sim \int_b^t \varphi^{-1}\biggl(p(s)^{-1}\int_s^\infty q(r)\psi\bigl(x(r)\bigr)\,dr\biggr)\,ds, \quad t \to \infty, \tag{4.2}$$

for any $b \geq a$, which is regarded as an 'approximation' of (4.1) at infinity. A common way of determining the desired intermediate solution of (E) would be by solving the integral equation (4.1) with the help of a fixed point technique. For this purpose the Schauder-Tychonoff fixed point theorem should be applied to the integral operator

$$\mathcal{F}x(t) = x_0 + \int_{t_0}^t \varphi^{-1}\biggl(p(s)^{-1}\int_s^\infty q(r)\psi\bigl(x(r)\bigr)\,dr\biggr)\,ds, \quad t \geq t_0, x_0 \in \mathbb{R},$$

acting on some closed convex subsets $\mathcal{X}$ of $C[t_0, \infty)$, which should be chosen in such a way that $\mathcal{F}$ is a continuous self-map on $\mathcal{X}$ and send it into a relatively compact subset of $C[t_0, \infty)$. That such choices of $\mathcal{X}$ are feasible is guaranteed by the existence of three types of regularly varying functions that determine exactly the asymptotic behavior of all possible solutions of (4.2).

The purpose of this section is to collect preparatory results which will help us to simplify the proof of both 'if' and 'only if' parts of our main theorems. We begin by proving three results verifying that regularly varying functions $X_i(t)$, $i = 1, 2, 3$ defined, respectively by (3.6), (3.9), and (3.11) satisfy the integral asymptotic relation (4.2).

Lemma 4.1 *Suppose that* (3.5) *holds*. *The function* $X_1(t)$ *given by* (3.6) *satisfies the asymptotic relation* (4.2).

Proof Let (3.5) hold. Since $\eta < \alpha$, from (3.5) we have $\sigma < -1$, so we can apply Proposition 2.1 to the integral

$$\int_t^\infty q(s)\,ds = \int_t^\infty s^\sigma l_q(s)\,ds \sim \bigl(-(\sigma + 1)\bigr)^{-1} t^{\sigma+1} l_q(t), \quad t \to \infty.$$

Using the above relation, (2.6), (2.10), and (2.7) we get

$$\begin{aligned}
&\varphi^{-1}\left(p(t)^{-1}\int_t^\infty q(s)\,ds\right)\\
&\quad=\varphi^{-1}\left(t^{-\eta}l_p(t)^{-1}\int_t^\infty s^\sigma l_q(s)\,ds\right)\\
&\quad\sim\bigl(-(\sigma+1)\bigr)^{-\frac{1}{\alpha}}\varphi^{-1}\bigl(t^{\sigma+1-\eta}\bigr)l_p(t)^{-\frac{1}{\alpha}}l_q(t)^{\frac{1}{\alpha}}\\
&\quad=\bigl(-(\sigma+1)\bigr)^{-\frac{1}{\alpha}}t^{\frac{\sigma+1-\eta}{\alpha}}L\bigl(t^{\sigma+1-\eta}\bigr)l_p(t)^{-\frac{1}{\alpha}}l_q(t)^{\frac{1}{\alpha}},\quad t\to\infty.
\end{aligned}\tag{4.3}$$

Since $\sigma=\eta-\alpha-1$ we can rewrite (4.3) in the form

$$\varphi^{-1}\left(p(t)^{-1}\int_t^\infty q(s)\,ds\right)\sim(\alpha-\eta)^{-\frac{1}{\alpha}}t^{-1}L\bigl(t^{-\alpha}\bigr)l_p(t)^{-\frac{1}{\alpha}}l_q(t)^{\frac{1}{\alpha}},\quad t\to\infty.\tag{4.4}$$

Application of Proposition 2.1(iii) to (4.4) gives

$$\int_{t_0}^t\varphi^{-1}\left(p(s)^{-1}\int_s^\infty q(r)\,dr\right)ds\in\mathrm{SV}.\tag{4.5}$$

From (3.6) and (4.5), by Proposition 2.2(iv), we find that $X_1(t)\in$ ntr-SV and $\psi(X_1(t))\in$ ntr-SV. We integrate $q(t)\psi(X_1(t))$ on $[t,\infty)$. Applying Proposition 2.1 (which is possible since $\sigma<-1$) and using (2.6) we obtain

$$\begin{aligned}
\int_t^\infty q(s)\psi\bigl(X_1(s)\bigr)\,ds&=\int_t^\infty s^\sigma l_q(s)\psi\bigl(X_1(s)\bigr)\,ds\sim\frac{t^{\sigma+1}}{-(\sigma+1)}l_q(t)\psi\bigl(X_1(t)\bigr)\\
&=\frac{t^{\eta-\alpha}}{\alpha-\eta}l_q(t)\psi\bigl(X_1(t)\bigr),
\end{aligned}$$

as $t\to\infty$, from which it readily follows that

$$p(t)^{-1}\int_t^\infty q(s)\psi\bigl(X_1(s)\bigr)\,ds\sim\frac{t^{-\alpha}}{\alpha-\eta}l_p(t)^{-1}l_q(t)\psi\bigl(X_1(t)\bigr),\quad t\to\infty.$$

From the above relation, using Proposition 2.7, (2.10), and (2.7) we conclude

$$\begin{aligned}
&\varphi^{-1}\left(p(t)^{-1}\int_t^\infty q(s)\psi\bigl(X_1(s)\bigr)\,ds\right)\\
&\quad\sim\varphi^{-1}\bigl((\alpha-\eta)^{-1}t^{-\alpha}l_p(t)^{-1}l_q(t)\psi\bigl(X_1(t)\bigr)\bigr)\\
&\quad\sim(\alpha-\eta)^{-\frac{1}{\alpha}}\varphi^{-1}\bigl(t^{-\alpha}\bigr)l_p(t)^{-\frac{1}{\alpha}}l_q(t)^{\frac{1}{\alpha}}\psi\bigl(X_1(t)\bigr)^{\frac{1}{\alpha}}\\
&\quad=(\alpha-\eta)^{-\frac{1}{\alpha}}t^{-1}L\bigl(t^{-\alpha}\bigr)l_p(t)^{-\frac{1}{\alpha}}l_q(t)^{\frac{1}{\alpha}}\psi\bigl(X_1(t)\bigr)^{\frac{1}{\alpha}},\quad t\to\infty.
\end{aligned}\tag{4.6}$$

In view of (4.4), integrating (4.6) from t_0 to t, we get

$$\begin{aligned}
&\int_{t_0}^t\varphi^{-1}\left(p(s)^{-1}\int_s^\infty q(r)\psi\bigl(X_1(r)\bigr)\,dr\right)ds\\
&\quad\sim\int_{t_0}^t\varphi^{-1}\left(p(s)^{-1}\int_s^\infty q(r)\,dr\right)\psi\bigl(X_1(s)\bigr)^{\frac{1}{\alpha}}\,ds,\quad t\to\infty.
\end{aligned}\tag{4.7}$$

On the other hand, we rewrite (3.6) as

$$\Psi\big(X_1(t)\big) = \int_{t_0}^{t} \varphi^{-1}\left(p(s)^{-1}\int_s^\infty q(r)\,dr\right)ds, \quad t \ge t_0. \tag{4.8}$$

Since

$$\Psi\big(X_1(t)\big) = \int_0^{X_1(t)} \frac{dv}{\psi(v)^{\frac{1}{\alpha}}},$$

differentiation of (4.8) gives

$$X_1'(t) = \varphi^{-1}\left(p(t)^{-1}\int_t^\infty q(s)\,ds\right)\psi\big(X_1(t)\big)^{\frac{1}{\alpha}}, \quad t \ge t_0. \tag{4.9}$$

Integrating (4.9) on $[t_0, t]$ and combining with (4.7) we obtain

$$X_1(t) \sim \int_{t_0}^{t} X_1'(s)\,ds \sim \int_{t_0}^{t} \varphi^{-1}\left(p(s)^{-1}\int_s^\infty q(r)\psi\big(X_1(r)\big)\,dr\right)ds, \quad t \to \infty.$$

This completes the proof of Lemma 4.1. □

Lemma 4.2 *Suppose that* (3.7) *holds and let* ρ *be defined by* (3.8). *The function* $X_2(t)$ *given by* (3.9) *satisfies the asymptotic relation* (4.2).

Proof Let (3.7) hold. Using (2.6) and (2.7) we rewrite (3.9) in the form

$$\Psi\big(X_2(t)\big) = \frac{\alpha}{\alpha-\beta}\frac{t^{\frac{\sigma+\alpha+1-\eta}{\alpha}}}{\rho[\alpha(1-\rho)-\eta]^{\frac{1}{\alpha}}}L\big(t^{\alpha(\rho-1)}\big)l_p(t)^{-\frac{1}{\alpha}}l_q(t)^{\frac{1}{\alpha}}, \quad t \ge t_0, \tag{4.10}$$

from which using (3.4) follows

$$\frac{X_2(t)}{\psi(X_2(t))^{\frac{1}{\alpha}}} \sim \frac{t^{\frac{\sigma+\alpha+1-\eta}{\alpha}}}{\rho[\alpha(1-\rho)-\eta]^{\frac{1}{\alpha}}}L\big(t^{\alpha(\rho-1)}\big)l_p(t)^{-\frac{1}{\alpha}}l_q(t)^{\frac{1}{\alpha}}, \quad t \to \infty. \tag{4.11}$$

Since $\frac{\sigma+\alpha+1-\eta}{\alpha} > 0$, by Proposition 2.2(v), we conclude that the function on the right-hand side of (4.10) tends to ∞ as $t \to \infty$. From (4.10) using the previous conclusion and $\Psi^{-1} \in \mathrm{RV}(\frac{\alpha}{\alpha-\beta})$ with application of Proposition 2.2(iv), we obtain $X_2(t) \in \mathrm{RV}(\rho)$, with ρ given by (3.8). Thus, $X_2(t)$ is expressed as $X_2(t) = t^\rho l_2(t)$, $l_2(t) \in \mathrm{SV}$. Then, using (4.11), we get

$$\begin{aligned}
&\int_t^\infty q(s)\psi\big(X_2(s)\big)\,ds \\
&\quad= \int_t^\infty q(s)\frac{\psi(X_2(s))}{X_2(s)^\alpha}X_2(s)^\alpha\,ds \\
&\quad\sim \rho^\alpha\big[\alpha(1-\rho)-\eta\big]\int_t^\infty q(s)s^{-\sigma-\alpha-1+\eta}L\big(s^{\alpha(\rho-1)}\big)^{-\alpha}l_p(s)l_q(s)^{-1}X_2(s)^\alpha\,ds \\
&\quad= \rho^\alpha\big[\alpha(1-\rho)-\eta\big]\int_t^\infty s^{\alpha(\rho-1)+\eta-1}L\big(s^{\alpha(\rho-1)}\big)^{-\alpha}l_p(s)l_2(s)^\alpha\,ds, \quad t \to \infty.
\end{aligned} \tag{4.12}$$

Since $\sigma + \beta + 1 < \frac{\beta}{\alpha}\eta$, we have $\alpha(\rho - 1) + \eta < 0$, implying that we can apply Proposition 2.1 on the last integral in (4.12) and then multiplying the result with $p(t)^{-1}$ we obtain

$$p(t)^{-1}\int_t^\infty q(s)\psi\bigl(X_2(s)\bigr)\,ds \sim \rho^\alpha t^{\alpha(\rho-1)}L\bigl(t^{\alpha(\rho-1)}\bigr)^{-\alpha}l_2(t)^\alpha, \quad t \to \infty,$$

from which, applying Proposition 2.7, it readily follows as $t \to \infty$ that

$$\varphi^{-1}\biggl(p(t)^{-1}\int_t^\infty q(s)\psi\bigl(X_2(s)\bigr)\,ds\biggr) \sim \rho\varphi^{-1}\bigl(t^{\alpha(\rho-1)}\bigr)L\bigl(t^{\alpha(\rho-1)}\bigr)^{-1}l_2(t) \sim \rho t^{\rho-1}l_2(t),$$

where we use (2.7) and (2.10) in the two last steps. Integration on the above relation from t_0 to t with application of Proposition 2.1 (which is possible since $\rho > 0$) then yields

$$\begin{aligned}&\int_{t_0}^t \varphi^{-1}\biggl(p(s)^{-1}\int_s^\infty q(r)\psi\bigl(X_2(r)\bigr)\,dr\biggr)\,ds\\ &\quad\sim \rho\int_{t_0}^t s^{\rho-1}l_2(s)\,ds \sim t^\rho l_2(t) = X_2(t), \quad t \to \infty.\end{aligned}$$

This completes the proof of Lemma 4.2. □

Lemma 4.3 *Suppose that* (3.10) *holds. The function* $X_3(t)$ *given by* (3.11) *satisfies the asymptotic relation* (4.2).

Proof Let (3.10) hold. Since $\sigma = \frac{\beta}{\alpha}\eta - \beta - 1$, using (2.5), (2.6), and (3.2), by Proposition 2.2 we get $q(t)\psi(P(t)) \in \mathrm{RV}(-1)$ so that $\int_t^\infty q(s)\psi(P(s))\,ds \in \mathrm{SV}$ by Proposition 2.1(iii). In view of (3.2) and (3.11), we conclude that $X_3(t) \in \text{ntr-RV}(1 - \frac{\eta}{\alpha})$. Using (2.11) and (3.2) we have

$$\int_t^\infty q(s)\psi\bigl(P(s)\bigr)\,ds \sim \int_t^\infty s^{\beta(\frac{\eta}{\alpha}-1)}q(s)\psi\bigl(s^{1-\frac{\eta}{\alpha}}\bigr)P(s)^\beta\,ds, \quad t \to \infty. \tag{4.13}$$

This, combined with (3.11), gives the following expression for $X_3(t)$:

$$X_3(t) \sim P(t)\biggl(\frac{\alpha-\beta}{\alpha}\int_t^\infty s^{\beta(\frac{\eta}{\alpha}-1)}q(s)\psi\bigl(s^{1-\frac{\eta}{\alpha}}\bigr)P(s)^\beta\,ds\biggr)^{\frac{1}{\alpha-\beta}}, \quad t \to \infty. \tag{4.14}$$

Next, we integrate $q(t)\psi(X_3(t))$ on $[t,\infty)$. Since $X_3(t) = t^{1-\frac{\eta}{\alpha}}l_3(t)$, $l_3(t) \in \mathrm{SV}$, due to (2.11), we obtain

$$\begin{aligned}&\int_t^\infty q(s)\psi\bigl(X_3(s)\bigr)\,ds\\ &\quad= \int_t^\infty q(s)\psi\bigl(s^{1-\frac{\eta}{\alpha}}l_3(s)\bigr)\,ds\\ &\quad\sim \int_t^\infty q(s)\psi\bigl(s^{1-\frac{\eta}{\alpha}}\bigr)l_3(s)^\beta\,ds\\ &\quad= \int_t^\infty s^{\beta(\frac{\eta}{\alpha}-1)}q(s)\psi\bigl(s^{1-\frac{\eta}{\alpha}}\bigr)X_3(s)^\beta\,ds, \quad t \to \infty.\end{aligned} \tag{4.15}$$

Changing (4.14) in the last integral in (4.15), by a simple calculation we have

$$\begin{aligned}
&\int_t^\infty q(s)\psi\big(X_3(s)\big)\,ds \\
&\quad\sim \left(\frac{\alpha-\beta}{\alpha}\right)^{\frac{\beta}{\alpha-\beta}} \\
&\qquad\times \int_t^\infty s^{\beta(\frac{\eta}{\alpha}-1)}q(s)\psi\big(s^{1-\frac{\eta}{\alpha}}\big)P(s)^\beta\left(\int_s^\infty r^{\beta(\frac{\eta}{\alpha}-1)}q(r)\psi\big(r^{1-\frac{\eta}{\alpha}}\big)P(r)^\beta\,dr\right)^{\frac{\beta}{\alpha-\beta}}ds \\
&\quad= \left(\frac{\alpha-\beta}{\alpha}\int_t^\infty s^{\beta(\frac{\eta}{\alpha}-1)}q(s)\psi\big(s^{1-\frac{\eta}{\alpha}}\big)P(s)^\beta\,ds\right)^{\frac{\alpha}{\alpha-\beta}} \\
&\quad\sim \left(\frac{\alpha-\beta}{\alpha}\int_t^\infty q(s)\psi\big(P(s)\big)\,ds\right)^{\frac{\alpha}{\alpha-\beta}}, \quad t\to\infty,
\end{aligned}\tag{4.16}$$

where we use (4.13) in the last step. Since $\int_t^\infty q(s)\psi(X_3(s))\,ds\in\mathrm{SV}$, (2.6), (2.7), and (2.10) give

$$\begin{aligned}
&\varphi^{-1}\left(p(t)^{-1}\int_t^\infty q(s)\psi\big(X_3(s)\big)\,ds\right) \\
&\quad= \varphi^{-1}\left(t^{-\eta}l_p(t)^{-1}\int_t^\infty q(s)\psi\big(X_3(s)\big)\,ds\right) \\
&\quad\sim \varphi^{-1}\big(t^{-\eta}\big)l_p(t)^{-\frac{1}{\alpha}}\left(\int_t^\infty q(s)\psi\big(X_3(s)\big)\,ds\right)^{\frac{1}{\alpha}} \\
&\quad= t^{-\frac{\eta}{\alpha}}L\big(t^{-\eta}\big)l_p(t)^{-\frac{1}{\alpha}}\left(\int_t^\infty q(s)\psi\big(X_3(s)\big)\,ds\right)^{\frac{1}{\alpha}},
\end{aligned}\tag{4.17}$$

as $t\to\infty$. Integrating (4.17) from t_0 to t, we conclude via Proposition 2.1 that

$$\begin{aligned}
&\int_{t_0}^t \varphi^{-1}\left(p(s)^{-1}\int_s^\infty q(r)\psi\big(X_3(r)\big)\,dr\right)ds \\
&\quad\sim \frac{\alpha}{\alpha-\eta}t^{1-\frac{\eta}{\alpha}}L\big(t^{-\eta}\big)l_p(t)^{-\frac{1}{\alpha}}\left(\int_t^\infty q(s)\psi\big(X_3(s)\big)\,ds\right)^{\frac{1}{\alpha}},
\end{aligned}$$

as $t\to\infty$. This, combined with (3.2) and (4.16), shows that $X_3(t)$ satisfies the asymptotic relation (4.2). This completes the proof of Lemma 4.3. □

After the construction of intermediate solutions with the help of the Schauder-Tychonoff fixed point theorem, to finish the proof of the 'if' part of our main results we prove the regularity of those solutions using the generalized L'Hospital rule (see [25]).

Lemma 4.4 *Let $f,g\in C^1[T,\infty)$. Let*

$$\lim_{t\to\infty} g(t)=\infty \quad \textit{and} \quad g'(t)>0 \quad \textit{for all large } t. \tag{4.18}$$

Then

$$\liminf_{t\to\infty}\frac{f'(t)}{g'(t)}\le\liminf_{t\to\infty}\frac{f(t)}{g(t)}\le\limsup_{t\to\infty}\frac{f(t)}{g(t)}\le\limsup_{t\to\infty}\frac{f'(t)}{g'(t)}.$$

If we replace (4.18) *with condition*

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} g(t) = 0 \qquad g'(t) < 0 \quad \textit{for all large } t,$$

then the same conclusion holds.

5 Proof of main results

Proof of the 'only if' part of Theorems 3.1, 3.2, 3.3 Suppose that (E) has an intermediate solution $x(t) \in \mathrm{RV}(\rho)$ with $\rho \in [0, 1 - \frac{\eta}{\alpha}]$ defined on $[t_0, \infty)$. Since $\lim_{t\to\infty} p(t)\varphi(x'(t)) = 0$, integration of (E) on (t, ∞) using (2.5), (2.6), and (3.1) gives

$$p(t)\varphi\bigl(x'(t)\bigr) = \int_t^\infty q(s)\psi\bigl(x(s)\bigr)\,ds = \int_t^\infty s^{\sigma+\beta\rho} l_q(s) l_x(s)^\beta L_2\bigl(x(s)\bigr)\,ds, \quad t \ge t_0, \tag{5.1}$$

implying the convergence of the last integral in (5.1) *i.e.* implying that $\sigma + \beta\rho \le -1$. We distinguish the two cases:

$$\text{(a)} \quad \sigma + \beta\rho = -1, \qquad \text{(b)} \quad \sigma + \beta\rho < -1.$$

Assume that (a) holds. Multiplying (5.1) with $p(t)^{-1}$ we get

$$\varphi\bigl(x'(t)\bigr) = p(t)^{-1}\xi(t), \quad \text{where } \xi(t) = \int_t^\infty s^{-1} l_q(s) l_x(s)^\beta L_2\bigl(x(s)\bigr)\,ds. \tag{5.2}$$

Clearly, $\xi(t) \in \mathrm{SV}$ and $\lim_{t\to\infty} \xi(t) = 0$. From (5.2), using (2.6) and (2.10) we have

$$x'(t) = \varphi^{-1}\bigl(p(t)^{-1}\xi(t)\bigr) = \varphi^{-1}\bigl(t^{-\eta} l_p(t)^{-1}\xi(t)\bigr) \sim \varphi^{-1}\bigl(t^{-\eta}\bigr) l_p(t)^{-\frac{1}{\alpha}} \xi(t)^{\frac{1}{\alpha}}, \quad t \to \infty. \tag{5.3}$$

Integrating (5.3) from t_0 to t and using (2.7) we get

$$x(t) \sim \int_{t_0}^t \varphi^{-1}\bigl(s^{-\eta}\bigr) l_p(s)^{-\frac{1}{\alpha}} \xi(s)^{\frac{1}{\alpha}}\,ds = \int_{t_0}^t s^{-\frac{\eta}{\alpha}} L\bigl(s^{-\eta}\bigr) l_p(s)^{-\frac{1}{\alpha}} \xi(s)^{\frac{1}{\alpha}}\,ds, \quad t \to \infty. \tag{5.4}$$

From (5.4) we find via Karamata's integration theorem that

$$x(t) \sim \frac{\alpha}{\alpha - \eta} t^{1-\frac{\eta}{\alpha}} L\bigl(t^{-\eta}\bigr) l_p(t)^{-\frac{1}{\alpha}} \xi(t)^{\frac{1}{\alpha}} \in \mathrm{RV}\left(1 - \frac{\eta}{\alpha}\right), \quad t \to \infty. \tag{5.5}$$

Using (3.2) we rewrite (5.5) in the form

$$x(t) \sim P(t)\xi(t)^{\frac{1}{\alpha}}, \quad t \to \infty. \tag{5.6}$$

Assume that (b) holds. Applying Proposition 2.1 to the last integral in (5.1) we have

$$p(t)\varphi\bigl(x'(t)\bigr) \sim \frac{t^{\sigma+\beta\rho+1}}{-(\sigma + \beta\rho + 1)} l_q(t) l_x(t)^\beta L_2\bigl(x(t)\bigr), \quad t \to \infty. \tag{5.7}$$

Multiplying (5.7) with $p(t)^{-1}$ and using (2.6) we get

$$\varphi\bigl(x'(t)\bigr) \sim \frac{t^{\sigma+\beta\rho+1-\eta}}{-(\sigma + \beta\rho + 1)} l_p(t)^{-1} l_q(t) l_x(t)^\beta L_2\bigl(x(t)\bigr), \quad t \to \infty. \tag{5.8}$$

Using Proposition 2.7, (2.10), and (2.7) we have

$$\begin{aligned} x'(t) &\sim \varphi^{-1}\big(t^{\sigma+\beta\rho+1-\eta}\big(-(\sigma+\beta\rho+1)\big)^{-1} l_p(t)^{-1} l_q(t) l_x(t)^{\beta} L_2\big(x(t)\big)\big) \\ &\sim \varphi^{-1}\big(t^{\sigma+\beta\rho+1-\eta}\big)\big(-(\sigma+\beta\rho+1)\big)^{-\frac{1}{\alpha}} l_p(t)^{-\frac{1}{\alpha}} l_q(t)^{\frac{1}{\alpha}} l_x(t)^{\frac{\beta}{\alpha}} L_2\big(x(t)\big)^{\frac{1}{\alpha}} \\ &= \big(-(\sigma+\beta\rho+1)\big)^{-\frac{1}{\alpha}} t^{\frac{\sigma+\beta\rho+1-\eta}{\alpha}} L\big(t^{\sigma+\beta\rho+1-\eta}\big) l_p(t)^{-\frac{1}{\alpha}} l_q(t)^{\frac{1}{\alpha}} l_x(t)^{\frac{\beta}{\alpha}} L_2\big(x(t)\big)^{\frac{1}{\alpha}}, \end{aligned} \tag{5.9}$$

as $t \to \infty$. Integration of (5.9) on $[t_0, t]$ leads to

$$\begin{aligned} x(t) &\sim \big(-(\sigma+\beta\rho+1)\big)^{-\frac{1}{\alpha}} \\ &\quad \times \int_{t_0}^{t} s^{\frac{\sigma+\beta\rho+1-\eta}{\alpha}} L\big(s^{\sigma+\beta\rho+1-\eta}\big) l_p(s)^{-\frac{1}{\alpha}} l_q(s)^{\frac{1}{\alpha}} l_x(s)^{\frac{\beta}{\alpha}} L_2\big(x(s)\big)^{\frac{1}{\alpha}}\, ds, \end{aligned} \tag{5.10}$$

as $t \to \infty$. Since the above integral tends to infinity as $t \to \infty$ (note that $x(t) \to \infty$, $t \to \infty$), we consider the following two cases separately.

$$\text{(b.1)}\quad \frac{\sigma+\beta\rho+1-\eta}{\alpha} > -1, \qquad \text{(b.2)}\quad \frac{\sigma+\beta\rho+1-\eta}{\alpha} = -1.$$

Assume that (b.1) holds. Applying Proposition 2.1 to the integral in (5.10), we get

$$\begin{aligned} x(t) &\sim \frac{\alpha}{\sigma+\beta\rho+1-\eta+\alpha}\big(-(\sigma+\beta\rho+1)\big)^{-\frac{1}{\alpha}} t^{\frac{\sigma+\beta\rho+1-\eta+\alpha}{\alpha}} \\ &\quad \times L\big(t^{\sigma+\beta\rho+1-\eta}\big) l_p(t)^{-\frac{1}{\alpha}} l_q(t)^{\frac{1}{\alpha}} l_x(t)^{\frac{\beta}{\alpha}} L_2\big(x(t)\big)^{\frac{1}{\alpha}} \in \mathrm{RV}\bigg(\frac{\sigma+\beta\rho+1-\eta+\alpha}{\alpha}\bigg), \\ &\quad t \to \infty. \end{aligned} \tag{5.11}$$

Assume that (b.2) holds. Then (5.10) shows that $x(t) \in \mathrm{SV}$, that is, $\rho = 0$, and hence $\sigma = \eta - \alpha - 1$. Since $\sigma + \beta\rho + 1 = \eta - \alpha$, (5.10) reduced to

$$x(t) \sim (\alpha-\eta)^{-\frac{1}{\alpha}} \int_{t_0}^{t} s^{-1} L\big(s^{-\alpha}\big) l_p(s)^{-\frac{1}{\alpha}} l_q(s)^{\frac{1}{\alpha}} l_x(s)^{\frac{\beta}{\alpha}} L_2\big(x(s)\big)^{\frac{1}{\alpha}}\, ds \in \mathrm{SV}, \quad t \to \infty. \tag{5.12}$$

Let us now suppose that $x(t)$ is an intermediate solution of (E) belonging to ntr-SV. From the above observation this is possible only when the case (b.2) holds, in which case $\rho = 0$, $\sigma = \eta - \alpha - 1$, and $x(t) = l_x(t)$ must satisfy the asymptotic behavior (5.12). Denote the right-hand side of (5.12) by $\mu(t)$. Then $\mu(t) \to \infty$, $t \to \infty$ and it satisfies

$$\begin{aligned} \mu'(t) &= (\alpha-\eta)^{-\frac{1}{\alpha}} t^{-1} L\big(t^{-\alpha}\big) l_p(t)^{-\frac{1}{\alpha}} l_q(t)^{\frac{1}{\alpha}} l_x(t)^{\frac{\beta}{\alpha}} L_2\big(x(t)\big)^{\frac{1}{\alpha}} \\ &= (\alpha-\eta)^{-\frac{1}{\alpha}} t^{-1} L\big(t^{-\alpha}\big) l_p(t)^{-\frac{1}{\alpha}} l_q(t)^{\frac{1}{\alpha}} \psi\big(x(t)\big)^{\frac{1}{\alpha}}, \quad t \ge t_0, \end{aligned}$$

where we use (2.5) in the last step. Since (5.12) is equivalent to $x(t) \sim \mu(t)$, $t \to \infty$, from the above using (4.4) we obtain

$$\frac{\mu'(t)}{\psi(\mu(t))^{\frac{1}{\alpha}}} \sim \varphi^{-1}\bigg(p(t)^{-1} \int_t^{\infty} q(s)\, ds\bigg), \quad t \to \infty.$$

An integration of the last relation over $[t_0, t]$ gives

$$\int_{\mu(t_0)}^{\mu(t)} \frac{dv}{\psi(v)^{\frac{1}{\alpha}}} \sim \Psi\bigl(\mu(t)\bigr) \sim \int_{t_0}^{t} \varphi^{-1}\biggl(p(s)^{-1}\int_s^\infty q(r)\,dr\biggr)\,ds, \quad t \to \infty,$$

or

$$x(t) \sim \mu(t) \sim \Psi^{-1}\biggl(\int_{t_0}^{t} \varphi^{-1}\biggl(p(s)^{-1}\int_s^\infty q(r)\,dr\biggr)\,ds\biggr), \quad t \to \infty.$$

Thus, it has been shown that $x(t) \sim X_1(t)$, $t \to \infty$, where $X_1(t)$ is given by (3.6). Notice that the verification of (3.5) is included in the above discussions. This proves the 'only if' part of Theorem 3.1.

Next, suppose that $x(t)$ is an intermediate solution of (E) belonging to $\mathrm{RV}(\rho)$, $\rho \in (0, 1 - \frac{\eta}{\alpha})$. This is possible only when (b.1) holds, in which case $x(t)$ must satisfy the asymptotic relation (5.11). Therefore,

$$\rho = \frac{\sigma + \beta\rho + 1 - \eta + \alpha}{\alpha} \quad \Rightarrow \quad \rho = \frac{\sigma + \alpha + 1 - \eta}{\alpha - \beta},$$

which justifies (3.8). An elementary calculation shows that

$$0 < \rho < 1 - \frac{\eta}{\alpha} \quad \Rightarrow \quad \eta - \alpha - 1 < \sigma < \frac{\beta}{\alpha}\eta - \beta - 1,$$

which determines the range (3.7) of σ. Since $\sigma + \beta\rho + 1 - \eta + \alpha = \alpha\rho$ and $-(\sigma + \beta\rho + 1) = \alpha(1-\rho) - \eta$, (5.11) reduced to

$$\begin{aligned} x(t) &\sim \frac{t^{\rho}}{\rho(\alpha(1-\rho)-\eta)^{\frac{1}{\alpha}}} L\bigl(t^{\alpha(\rho-1)}\bigr) l_p(t)^{-\frac{1}{\alpha}} l_q(t)^{\frac{1}{\alpha}} l_x(t)^{\frac{\beta}{\alpha}} L_2\bigl(x(t)\bigr)^{\frac{1}{\alpha}} \\ &= \frac{t^{2-\rho+\frac{1}{\alpha}}}{\rho(\alpha(1-\rho)-\eta)^{\frac{1}{\alpha}}} \varphi^{-1}\bigl(t^{\alpha(\rho-1)}\bigr) p(t)^{-\frac{1}{\alpha}} q(t)^{\frac{1}{\alpha}} \psi\bigl(x(t)\bigr)^{\frac{1}{\alpha}}, \quad t \to \infty, \end{aligned} \tag{5.13}$$

where we use (2.5), (2.6), (2.7), and (3.1) in the last step. From (5.13) using (3.4) we get

$$\begin{aligned} \Psi\bigl(x(t)\bigr) &\sim \frac{\alpha}{\alpha-\beta} \frac{x(t)}{\psi(x(t))^{\frac{1}{\alpha}}} \\ &\sim \frac{\alpha}{\alpha-\beta} \frac{t^{2-\rho+\frac{1}{\alpha}}}{\rho(\alpha(1-\rho)-\eta)^{\frac{1}{\alpha}}} \varphi^{-1}\bigl(t^{\alpha(\rho-1)}\bigr) p(t)^{-\frac{1}{\alpha}} q(t)^{\frac{1}{\alpha}}, \quad t \to \infty. \end{aligned}$$

Thus, we conclude that $x(t)$ enjoys the asymptotic formula $x(t) \sim X_2(t)$, $t \to \infty$, where $X_2(t)$ is given by (3.9). This proves the 'only if' part of Theorem 3.2.

Finally, suppose that $x(t)$ is an intermediate solution of (E) belonging to $\text{ntr-RV}(1 - \frac{\eta}{\alpha})$. Then the case (a) is the only possibility for $x(t)$, which means that $\rho = 1 - \frac{\eta}{\alpha}$, $\sigma = \frac{\beta}{\alpha}\eta - \beta - 1$, and (5.6) is satisfied by $x(t)$. Differentiation of $\xi(t)$, defined in (5.2), using (2.5), (2.6), and (3.1), leads to

$$\xi'(t) \sim -t^{-1} l_q(t) l_x(t)^{\beta} L_2\bigl(x(t)\bigr) \sim -q(t)\psi\bigl(x(t)\bigr), \quad t \to \infty.$$

Noting that $x(t) \sim P(t)\xi(t)^{\frac{1}{\alpha}}$, $t \to \infty$ and using (2.11), one can transform the above relation into

$$\xi'(t) \sim -q(t)\psi\big(P(t)\xi(t)^{\frac{1}{\alpha}}\big) \sim -q(t)\psi\big(P(t)\big)\xi(t)^{\frac{\beta}{\alpha}}, \quad t \to \infty.$$

So, we get the differential asymptotic relation for $\xi(t)$:

$$\xi(t)^{-\frac{\beta}{\alpha}}\xi'(t) \sim -q(t)\psi\big(P(t)\big), \quad t \to \infty. \tag{5.14}$$

Due to fact that $\alpha - \beta > 0$ and $\xi(t) \to 0$ as $t \to \infty$, the left-hand side of (5.14) can be integrated over (t,∞), assuring the integrability of $q(t)\psi(P(t))$ on (t,∞), which implies the convergence of the integral in (3.10). Integration of (5.14) on (t,∞) yields

$$\xi(t) \sim \left(\frac{\alpha-\beta}{\alpha}\int_t^\infty q(s)\psi\big(P(s)\big)\,ds\right)^{\frac{\alpha}{\alpha-\beta}}, \quad t \to \infty. \tag{5.15}$$

Combining (5.15) with (5.6) gives us $x(t) \sim X_3(t)$, $t \to \infty$, where $X_3(t)$ is given by (3.11). This completes the 'only if' part of the proof of Theorem 3.3. □

Proof of the 'if' part of Theorems 3.1, 3.2, 3.3 Suppose that (3.5), (3.7) or (3.10) holds. From Lemmas 4.1, 4.2, and 4.3 it is well known that each $X_i(t)$, $i = 1,2,3$, defined by (3.6), (3.9), and (3.11), satisfies the asymptotic relation (4.2) for any $b \geq a$. We perform the simultaneous proof for $X_i(t)$, $i = 1,2,3$ so the subscript $i = 1,2,3$ will be deleted in the rest of proof. By (4.2) there exists $T_0 > a$ such that

$$\int_{T_0}^t \varphi^{-1}\left(p(s)^{-1}\int_s^\infty q(r)\psi\big(X(r)\big)\,dr\right)ds \leq 2X(t), \quad t \geq T_0. \tag{5.16}$$

Let such a T_0 be fixed. We may assume that $X(t)$ is increasing on $[T_0,\infty)$. Since (4.2) is satisfied with $b = T_0$, there exists $T > T_0$ such that

$$\int_{T_0}^t \varphi^{-1}\left(p(s)^{-1}\int_s^\infty q(r)\psi\big(X(r)\big)\,dr\right)ds \geq \frac{1}{2}X(t), \quad t \geq T. \tag{5.17}$$

Applying Proposition 2.4 to the function $\psi(s) \in \mathrm{RV}(\beta)$, $\beta > 0$ we see that there exists a constant $A > 1$ such that

$$\psi(s_1) \leq A\psi(s_2) \quad \text{for each } 0 \leq s_1 \leq s_2. \tag{5.18}$$

Now we choose positive constants m and M such that

$$m^{1-\frac{\beta}{\alpha}} \leq \frac{1}{4(2A)^{1/\alpha}}, \qquad M^{1-\frac{\beta}{\alpha}} \geq 8(2A)^{1/\alpha}, \qquad 2mX(T) \leq MX(T_0). \tag{5.19}$$

In addition, since $X(t) \to \infty$ as $t \to \infty$, from (2.1), for $\lambda > 0$ we have

$$\frac{\lambda^\beta}{2}\psi\big(X(t)\big) \leq \psi\big(\lambda X(t)\big) \leq 2\lambda^\beta\psi\big(X(t)\big), \quad \text{for all sufficiently large } t. \tag{5.20}$$

Also, since $Q(t) = 1/p(t) \int_t^\infty q(s)\psi(X(s))\,ds \to 0$ as $t \to \infty$, from (2.2), for $\lambda > 0$ we have

$$\frac{\lambda^{1/\alpha}}{2}\varphi^{-1}\bigl(Q(t)\bigr) \le \varphi^{-1}\bigl(\lambda Q(t)\bigr) \le 2\lambda^{1/\alpha}\varphi^{-1}\bigl(Q(t)\bigr), \quad \text{for all sufficiently large } t. \tag{5.21}$$

Define the integral operator $\mathcal{F}$ by

$$\mathcal{F}x(t) = x_0 + \int_{T_0}^t \varphi^{-1}\left(p(s)^{-1}\int_s^\infty q(r)\psi\bigl(x(r)\bigr)\,dr\right)ds, \quad t \ge T_0, \tag{5.22}$$

where x_0 is constant such that

$$mX(T) \le x_0 \le \frac{M}{2}X(T_0), \tag{5.23}$$

and let it act on the set

$$\mathcal{X} := \bigl\{x(t) \in C[T_0,\infty) : mX(t) \le x(t) \le MX(t), t \ge T_0\bigr\}. \tag{5.24}$$

It is clear that $\mathcal{X}$ is a closed convex subset of the locally convex space $C[T_0,\infty)$ equipped with the topology of uniform convergence on compact subintervals of $[T_0,\infty)$.

Let $x(t) \in \mathcal{X}$. Using first (5.18) and (5.24) and then (5.20) and (5.23) we get

$$\begin{aligned}\mathcal{F}x(t) &\le x_0 + \int_{T_0}^t \varphi^{-1}\left(Ap(s)^{-1}\int_s^\infty q(r)\psi\bigl(MX(r)\bigr)\,dr\right)ds \\ &\le \frac{M}{2}X(T_0) + \int_{T_0}^t \varphi^{-1}\left(2AM^\beta p(s)^{-1}\int_s^\infty q(r)\psi\bigl(X(r)\bigr)\,dr\right)ds, \quad t \ge T_0,\end{aligned}$$

from which, using (5.21), (5.16), and (5.19), it follows that

$$\begin{aligned}\mathcal{F}x(t) &\le \frac{M}{2}X(T_0) + 2\bigl(2AM^\beta\bigr)^{1/\alpha}\int_{T_0}^t \varphi^{-1}\left(p(s)^{-1}\int_s^\infty q(r)\psi\bigl(X(r)\bigr)\,dr\right)ds \\ &\le \frac{M}{2}X(t) + 4\bigl(2AM^\beta\bigr)^{1/\alpha}X(t) \le \frac{M}{2}X(t) + \frac{M}{2}X(t) = MX(t), \quad t \ge T_0.\end{aligned}$$

On the other hand, using (5.23) we have

$$\mathcal{F}x(t) \ge x_0 \ge mX(T) \ge mX(t) \quad \text{for } T_0 \le t \le T,$$

and using (5.24), (5.18), and (5.20) we obtain

$$\begin{aligned}\mathcal{F}x(t) &\ge \int_{T_0}^t \varphi^{-1}\left(\frac{p(s)^{-1}}{A}\int_s^\infty q(r)\psi\bigl(mX(r)\bigr)\,dr\right)ds \\ &\ge \int_{T_0}^t \varphi^{-1}\left(\frac{m^\beta p(s)^{-1}}{2A}\int_s^\infty q(r)\psi\bigl(X(r)\bigr)\,dr\right)ds, \quad t \ge T.\end{aligned}$$

From the above using (5.21), (5.17), and (5.19) we conclude

$$\begin{aligned}\mathcal{F}x(t) &\geq \frac{1}{2}\left(\frac{m^\beta}{2A}\right)^{\frac{1}{\alpha}} \int_{T_0}^t \varphi^{-1}\left(p(s)^{-1}\int_s^\infty q(r)\psi\big(X(r)\big)\,dr\right)ds \\ &\geq \frac{1}{4}\left(\frac{m^\beta}{2A}\right)^{\frac{1}{\alpha}} X(t) \geq mX(t), \quad t \geq T.\end{aligned}$$

This shows that $\mathcal{F}x(t) \in \mathcal{X}$, that is, $\mathcal{F}$ maps $\mathcal{X}$ into itself.

Furthermore it can be verified (similarly to the proof of Theorem 3 in [5]) that $\mathcal{F}$ is a continuous mapping and that $\mathcal{F}(\mathcal{X})$ is relatively compact in $C[T_0,\infty)$.

Thus, all the hypotheses of the Schauder-Tychonoff fixed point theorem are fulfilled and so there exists a fixed point $x(t) \in \mathcal{X}$ of $\mathcal{F}$, which satisfies integral equation

$$x(t) = x_0 + \int_{T_0}^t \varphi^{-1}\left(p(s)^{-1}\int_s^\infty q(r)\psi\big(x(r)\big)\,dr\right)ds, \quad t \geq T_0.$$

Differentiating the above twice shows that $x(t)$ is a solution of (E) on $[T_0,\infty)$. It is clear from (5.24) that $x(t)$ is an intermediate solution of (E).

Therefore, the existence of three types of intermediate solutions of (E) has been established. The proof of our main results will be completed with the verification that the intermediate solutions of (E) constructed above are actually regularly varying functions.

We define the function

$$J(t) = \int_{T_0}^t \varphi^{-1}\left(p(s)^{-1}\int_s^\infty q(r)\psi\big(X(r)\big)\,dr\right)ds, \quad t \geq T_0,$$

and put

$$l = \liminf_{t\to\infty}\frac{x(t)}{J(t)}, \qquad L = \limsup_{t\to\infty}\frac{x(t)}{J(t)}.$$

Since $x(t) \in \mathcal{X}$, it is clear that $0 < l \leq L < \infty$. By Lemmas 4.1, 4.2, and 4.3 we have

$$J(t) \sim X(t), \quad t \to \infty. \tag{5.25}$$

Using Lemma 4.4 and (2.5) we see that

$$\begin{aligned}\liminf_{t\to\infty}\frac{\int_t^\infty q(s)\psi(x(s))\,ds}{\int_t^\infty q(s)\psi(X(s))\,ds} &\geq \liminf_{t\to\infty}\frac{\psi(x(t))}{\psi(X(t))} = \liminf_{t\to\infty}\frac{x(t)^\beta L_2(x(t))}{X(t)^\beta L_2(X(t))} \\ &\geq \liminf_{t\to\infty}\left(\frac{x(t)}{X(t)}\right)^\beta \liminf_{t\to\infty}\frac{L_2(\frac{x(t)}{X(t)}X(t))}{L_2(X(t))}.\end{aligned} \tag{5.26}$$

Since $m \leq \frac{x(t)}{X(t)} \leq M$, $t \geq T_0$, using the uniform convergence theorem ([22], Theorem 1.2.1) we conclude

$$\left|\frac{L_2(\frac{x(t)}{X(t)}X(t))}{L_2(X(t))} - 1\right| \leq \sup_{\lambda\in[m,M]}\left|\frac{L_2(\lambda X(t))}{L_2(X(t))} - 1\right| \longrightarrow 0, \quad t \to \infty. \tag{5.27}$$

From (5.26), using (5.27) and (5.25), we get

$$\liminf_{t\to\infty} \frac{\int_t^\infty q(s)\psi(x(s))\,ds}{\int_t^\infty q(s)\psi(X(s))\,ds} \geq \left(\liminf_{t\to\infty} \frac{x(t)}{X(t)}\right)^\beta = \left(\liminf_{t\to\infty} \frac{x(t)}{J(t)}\right)^\beta = l^\beta. \tag{5.28}$$

Similarly, we conclude that

$$\limsup_{t\to\infty} \frac{\int_t^\infty q(s)\psi(x(s))\,ds}{\int_t^\infty q(s)\psi(X(s))\,ds} \leq L^\beta. \tag{5.29}$$

We denote $\hat{x}(t) = p(t)^{-1}\int_t^\infty q(s)\psi(x(s))\,ds$ and $\hat{X}(t) = p(t)^{-1}\int_t^\infty q(s)\psi(X(s))\,ds$. Using Lemma 4.4 and (2.7) we obtain

$$l \geq \liminf_{t\to\infty} \frac{x'(t)}{J'(t)} = \liminf_{t\to\infty} \frac{\varphi^{-1}(\hat{x}(t))}{\varphi^{-1}(\hat{X}(t))} \geq \liminf_{t\to\infty}\left(\frac{\hat{x}(t)}{\hat{X}(t)}\right)^{\frac{1}{\alpha}} \liminf_{t\to\infty} \frac{L(\frac{\hat{x}(t)}{\hat{X}(t)}\hat{X}(t))}{L(\hat{X}(t))}.$$

From (5.28) and (5.29) we see that $\frac{\hat{x}(t)}{\hat{X}(t)}$ is bounded. So, we can apply the uniform convergence again, identically to (5.27), to get

$$l \geq \liminf_{t\to\infty}\left(\frac{\hat{x}(t)}{\hat{X}(t)}\right)^{\frac{1}{\alpha}} = \left(\liminf_{t\to\infty} \frac{\int_t^\infty q(s)\psi(x(s))\,ds}{\int_t^\infty q(s)\psi(X(s))\,ds}\right)^{\frac{1}{\alpha}}. \tag{5.30}$$

In view of (5.28) and (5.30) we have $l \geq l^{\frac{\beta}{\alpha}}$, implying that $l \geq 1$ because $\alpha > \beta$. If we argue similarly by taking the superior limits instead of the inferior limits, we are led to the inequality $L \leq L^{\frac{\beta}{\alpha}}$, which implies that $L \leq 1$. Thus we conclude that $l = L = 1$, *i.e.* $\lim_{t\to\infty} x(t)/J(t) = 1$. This combined with (5.25) shows that $x(t) \sim X(t)$, $t \to \infty$, which shows that $x(t)$ is a regularly varying function whose regularity index ρ is 0, $\frac{\sigma+\alpha+1-\eta}{\alpha-\beta}$, or $1 - \frac{\eta}{\alpha}$ according to whether $\sigma = \eta - \alpha - 1$, $\eta - \alpha - 1 < \sigma < \frac{\beta}{\alpha}\eta - \beta - 1$, or $\sigma = \frac{\beta}{\alpha}\eta - \beta - 1$. □

6 Examples

Example 6.1 Consider the equation

$$\text{(E)} \quad \bigl(p(t)\varphi\bigl(x'(t)\bigr)\bigr)' + q(t)\psi\bigl(x(t)\bigr) = 0, \quad t \geq e = a,$$

where $p(t) = t^{\frac{\alpha}{2}}(\log t)^\alpha \in \mathrm{RV}(\frac{\alpha}{2})$, $\varphi(s) = s^\alpha \in \mathrm{RV}_0(\alpha)$, and $\psi(s) = s^\beta \log s \in \mathrm{RV}(\beta)$, $\alpha > \beta > 0$. We have $\eta = \frac{\alpha}{2} \in (0,\alpha)$, $P(t) \sim 2\sqrt{t}(\log t)^{-1}$, and the functions $\varphi^{-1}(s)$ and $\psi(s)$ satisfy the additional requirements (2.8) and (2.9), respectively.

(i) Suppose that

$$q(t) \sim \frac{\alpha}{2^{\alpha+1}} t^{-1-\frac{\alpha}{2}} \frac{r(t)(\log t)^{\frac{\alpha-\beta}{2}}}{\log\sqrt{\log t}}, \quad t \to \infty, \tag{6.1}$$

where $r(t)$ is continuous function on $[a,\infty)$ such that $\lim_{t\to\infty} r(t) = 1$. Then $q(t) \in \mathrm{RV}(-1 - \frac{\alpha}{2})$, so that $\sigma = \eta - \alpha - 1$ and we see that

$$\begin{aligned}\int_a^t \varphi^{-1}\left(p(s)^{-1}\int_s^\infty q(r)\,dr\right) ds &\sim \frac{1}{2}\int_a^t (\log s)^{-\frac{\alpha+\beta}{2\alpha}} (\log\sqrt{\log s})^{-\frac{1}{\alpha}} \frac{ds}{s} \\ &\sim \frac{\alpha}{\alpha-\beta}(\log t)^{\frac{\alpha-\beta}{2\alpha}} (\log\sqrt{\log t})^{-\frac{1}{\alpha}} \longrightarrow \infty, \quad t \to \infty,\end{aligned}$$

implying that (3.5) holds. Therefore, by Theorem 3.1 there exist nontrivial slowly varying solutions of (E), and any such solution $x(t)$ has asymptotic behavior

$$\Psi\big(x(t)\big) \sim \frac{\alpha}{\alpha-\beta}(\log t)^{\frac{\alpha-\beta}{2\alpha}}(\log\sqrt{\log t})^{-\frac{1}{\alpha}}, \quad t\to\infty.$$

In view of (3.4) we have

$$x(t)^{\frac{\alpha-\beta}{\alpha}}\big(\log x(t)\big)^{-\frac{1}{\alpha}} \sim (\sqrt{\log t})^{\frac{\alpha-\beta}{\alpha}}(\log\sqrt{\log t})^{-\frac{1}{\alpha}}, \quad t\to\infty$$

implying that $x(t)\sim\sqrt{\log t}$, $t\to\infty$. If in (6.1) instead of $\sim$ one has =, and in particular $r(t) = 1-\frac{1}{\log t}$, then (E) possesses an exact increasing nontrivial SV-solution $x(t) = \sqrt{\log t}$ on $[e,\infty)$.

(ii) Suppose that

$$q(t) \sim \frac{\alpha}{6\cdot 3^{\alpha}} t^{-\frac{\alpha}{6}-\frac{\beta}{3}-1}\frac{r(t)(\log t)^{\beta}}{\log\frac{\sqrt[3]{t}}{\log t}}, \quad t\to\infty, \tag{6.2}$$

where $r(t)$ is continuous function on $[a,\infty)$ such that $\lim_{t\to\infty} r(t) = 1$. It is clear that $q(t)$ is regularly varying function of index

$$\sigma = -\frac{\alpha}{6}-\frac{\beta}{3}-1 \in \left(\eta-\alpha-1, \frac{\beta}{\alpha}\eta-\beta-1\right) = (-1-\alpha/2, -1-\beta/2)$$

and that $\rho = \frac{\sigma+\alpha+1-\eta}{\alpha-\beta} = \frac{1}{3}$. By Theorem 3.2 there exist regularly varying solutions of index ρ of (E) and any such solution $x(t)$ has asymptotic behavior

$$\Psi\big(x(t)\big) \sim \frac{\alpha}{\alpha-\beta} t^{\frac{\alpha-\beta}{3\alpha}}(\log t)^{\frac{\beta}{\alpha}-1}\left(\log\frac{\sqrt[3]{t}}{\log t}\right)^{-\frac{1}{\alpha}}, \quad t\to\infty.$$

In view of (3.4) we have

$$x(t)^{\frac{\alpha-\beta}{\alpha}}\big(\log x(t)\big)^{-\frac{1}{\alpha}} \sim \left(\frac{\sqrt[3]{t}}{\log t}\right)^{\frac{\alpha-\beta}{\alpha}}\left(\log\frac{\sqrt[3]{t}}{\log t}\right)^{-\frac{1}{\alpha}}, \quad t\to\infty,$$

implying that

$$x(t) \sim \frac{\sqrt[3]{t}}{\log t}, \quad t\to\infty.$$

Observe that in (6.2) instead of $\sim$ one has = and

$$r(t) = \left(1-\frac{6}{\log t}\right)\left(1+\frac{3}{\log t}\right)\left(1-\frac{3}{\log t}\right)^{\alpha-1},$$

then $x(t) = \sqrt[3]{t}(\log t)^{-1}$ on $[e^6,\infty)$ is an exact increasing solution.

(iii) Suppose that

$$q(t) \sim \frac{\alpha}{2^{\alpha}} t^{-1-\frac{\beta}{2}}\frac{r(t)(\log t)^{2\beta-\alpha-1}}{\log\frac{\sqrt{t}}{\log^2 t}}, \quad t\to\infty, \tag{6.3}$$

where $r(t)$ is continuous function on $[a,\infty)$ such that $\lim_{t\to\infty} r(t) = 1$. Here, $q(t) \in \mathrm{RV}(-1-\frac{\beta}{2})$. Therefore, $\sigma = \frac{\beta}{\alpha}\eta - \beta - 1$ and

$$q(t)\psi\big(P(t)\big) \sim \frac{\alpha}{2^{\alpha-\beta}} t^{-1}(\log t)^{\beta-\alpha-1} \frac{\log \frac{2\sqrt{t}}{\log t}}{\log \frac{\sqrt{t}}{\log^2 t}} \sim \frac{\alpha}{2^{\alpha-\beta}} t^{-1}(\log t)^{\beta-\alpha-1}, \quad t\to\infty,$$

from which it follows that

$$\begin{aligned}\int_t^\infty q(s)\psi\big(P(s)\big)\,ds &\sim \frac{\alpha}{2^{\alpha-\beta}}\int_t^\infty (\log s)^{\beta-\alpha-1}\frac{ds}{s}\\ &\sim \frac{1}{2^{\alpha-\beta}}\frac{\alpha}{\alpha-\beta}(\log t)^{\beta-\alpha} \longrightarrow 0, \quad t\to\infty,\end{aligned}$$

implying that (3.10) holds. Therefore, by Theorem 3.3 there exist nontrivial regularly varying solutions of index $1-\frac{\eta}{\alpha}=\frac{1}{2}$ of (E) and any such solution $x(t)$ has asymptotic behavior

$$x(t) \sim 2\sqrt{t}(\log t)^{-1}\left(\frac{\alpha-\beta}{\alpha}\frac{1}{2^{\alpha-\beta}}\frac{\alpha}{\alpha-\beta}(\log t)^{\beta-\alpha}\right)^{\frac{1}{\alpha-\beta}} \sim \frac{\sqrt{t}}{\log^2 t}, \quad t\to\infty.$$

If in (6.3) instead of $\sim$ one has = and in particular

$$r(t) = \left(1-\frac{4}{\log t}\right)^{\alpha-1}\left(1-\frac{8}{\log t}\right),$$

then (E) possesses an exact increasing solution $x(t) = \sqrt{t}(\log t)^{-2}$ on $[e^8,\infty)$.

Competing interests
The author declares that she has no competing interests.

Author's contributions
All of the new results in this paper were achieved by JM.

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