Wang and Guo *Advances in Difference Equations* (2015) 2015:275
DOI 10.1186/s13662-015-0601-9

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

New existence results for periodic boundary value problems with impulsive effects

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Abstract

New results as regards the existence of positive solutions for first order impulsive differential equations are provided. The method of proof relies on the fixed point theorem and degree theory. Some examples are presented to illustrate the main results.

MSC: 34B18; 34B37

Keywords: positive solutions; periodic boundary value problem; fixed point; degree

1 Introduction

In this paper, we study the existence of positive solutions for the following boundary value problem with impulsive effects:

$$\begin{cases} x'(t) + a(t)x(t) = f(t, x(t)), & t \neq t_k, t \in J, \\ \Delta x(t_k) = I_k(x(t_k)), & k = 1, 2, \ldots, p, \\ x(0) = x(T), \end{cases} \tag{1.1}$$

where $J = [0, T]$, $0 = t_0 < t_1 < t_2 < \cdots < t_p < 1 = t_{p+1}$, $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, $x(t_k^+)$, and $x(t_k^-)$ represent the right limit and left limits of $u(t)$ at t_k, respectively.

Impulsive differential equations serve as basic models to study the dynamics of processes that are subject to sudden changes in their states and its theory has developed fast during the past few years. There has been increasing interest in the investigation for boundary value problems of nonlinear impulsive differential equations and much literature has been published about the existence of solutions for impulsive differential equations, see [1–3] and the references therein. There are some common techniques to approach those problems: the fixed point theorems [4–8], the method of upper and lower solutions [9–12], the topological degree theory [13, 14], the variational method [15–17] and so on. Recently, using the fixed point theorem, Zhang *et al.* [8] obtained the existence of a positive solution of (1.1), where they required that the function a is of definite sign. In this paper, we continue to discuss (1.1). By using the fixed point theorem in a cone different from the one in [8] and degree theory, we obtain some new conditions which guarantee the existence of single and multiple positive solutions for (1.1). Our results are different from the results in [8] and are new even if $I_k \equiv 0$.

2 Main results

Let $J^* = J\backslash\{t_1, t_2, \ldots, t_p\}$, $PC(J) = \{u : J \to R | u \in C(J^*), u(t_i^+), u(t_i^-)$ exist and $u(t_i^-) = u(t_i), i = 1, 2, \ldots, p\}$. $PC(J)$ is a Banach space with the norm $\|u\| = \sup\{|u(t)| : t \in J\}$.

Lemma 2.1 [8] *The function $x \in PC(J)$ is a solution of* (1.1) *if and only if x is a solution of the following impulsive integral equation*:

$$x(t) = \int_0^T G(t,s)f\bigl(s, x(s)\bigr)\,ds + \sum_{k=1}^{p} G(t, t_k)I_k\bigl(x(t_k)\bigr),$$

where

$$G(t,s) = \begin{cases} \dfrac{e^{\int_t^s a(r)\,dr}}{e^{\int_0^T a(s)\,ds} - 1}, & \text{if } 0 \le t < s \le T, \\ \dfrac{e^{\int_t^s a(r)\,dr} e^{\int_0^T a(r)\,dr}}{e^{\int_0^T a(s)\,ds} - 1}, & \text{if } 0 \le s \le t \le T. \end{cases}$$

Set

$$\Lambda^+ = \biggl\{a \in C(J,R) : \int_0^T a(r)\,dr > 0\biggr\}, \qquad \Lambda^- = \biggl\{a \in C(J,R) : \int_0^T a(r)\,dr < 0\biggr\},$$
$$M = \max\bigl\{G(t,s) : t, s \in J\bigr\}, \qquad m = \min\bigl\{G(t,s) : t, s \in J\bigr\}.$$

The Green's function G is of definite sign if $a \in \Lambda^+ \cup \Lambda^-$. Moreover, $M > m > 0$ if $a \in \Lambda^+$; $m < M < 0$ if $a \in \Lambda^-$.

Define the operator A and cone K on $PC(J)$ by

$$(Ax)(t) = \int_0^T G(t,s)f\bigl(s, x(s)\bigr)\,ds + \sum_{k=1}^{p} G(t, t_k)I_k\bigl(x(t_k)\bigr),$$
$$K = \bigl\{x \in PC(J) : x(t) \ge \delta\|x\|\bigr\},$$

where $\delta = \frac{m}{M}$ if $a \in \Lambda^+$; $\delta = \frac{M}{m}$ if $a \in \Lambda^-$.

Lemma 2.2 [18] *Let X be a Banach space and K be a cone in X. Suppose Ω_1 and Ω_2 are open subsets of X such that $0 \in \Omega_1 \subset \bar{\Omega}_1 \subset \Omega_2$ and suppose that*

$$\Phi : K \cap (\bar{\Omega}_2 \backslash \Omega_1) \to K$$

is a completely continuous operator such that:

(i) $\inf\|\Phi u\| > 0$, *$u \neq \mu\Phi u$ for $u \in K \cap \partial\Omega_1$ and $\mu \ge 1$, and $u \neq \mu\Phi u$ for $u \in K \cap \partial\Omega_2$ and $0 < \mu \le 1$, or*

(ii) $\inf\|\Phi u\| > 0$, *$u \neq \mu\Phi u$ for $u \in K \cap \partial\Omega_2$ and $\mu \ge 1$, and $u \neq \mu\Phi u$ for $u \in K \cap \partial\Omega_1$ and $0 < \mu \le 1$.*

Then Φ has a fixed point in $K \cap (\bar{\Omega}_2 \backslash \Omega_1)$.

Set

$$\varphi(s) = T \sup_{(t,u)\in J\times[\delta s, s]} \frac{f(t,u)}{u} + \sup\Biggl\{\sum_{k=1}^{p} \ln\biggl(1 + \frac{I_k(v_k)}{v_k}\biggr) : v_k \in [\delta s, s]\Biggr\},$$

$$\psi(s) = T \inf_{(t,u)\in J\times[\delta s,s]} \frac{f(t,u)}{u} + \inf\left\{\sum_{k=1}^{p} \ln\left(1+\frac{I_k(v_k)}{v_k}\right) : v_k \in [\delta s, s]\right\},$$

$$\varphi_0 = \lim_{s\to 0^+} \varphi(s), \qquad \varphi_\infty = \lim_{s\to+\infty} \varphi(s), \qquad \psi_0 = \lim_{s\to 0^+} \psi(s), \qquad \psi_\infty = \lim_{s\to+\infty} \psi(s).$$

The following theorems are the main results of this paper.

Theorem 2.1 *Assume that $a \in \Lambda^+$ and there exist two positive constants $r < R$ such that*

$$f \in C\bigl(J \times [\delta r, R], [0, +\infty)\bigr), \qquad I_k \in C\bigl([\delta r, R], [0, +\infty)\bigr) \quad (1 \le k \le p). \tag{2.1}$$

Then (1.1) *has at least one solution x with $r \le \|x\| \le R$ if one of the following conditions is satisfied*:

(H_1) $\varphi(r) < \int_0^T a(s)\,ds$ *and* $\psi(R) > \int_0^T a(s)\,ds$;
(H_2) $\varphi(R) < \int_0^T a(s)\,ds$ *and* $\psi(r) > \int_0^T a(s)\,ds$.

Proof Here, we only prove the case in which (H_1) is satisfied. Let $\Omega_R = \{x \in K : \|x\| < R\}$, $\Omega_r = \{x \in K : \|x\| < r\}$. At first, we show that $A : \bar{\Omega}_R \setminus \Omega_r \to K$. For any $x \in \bar{\Omega}_R \setminus \Omega_r$, $\delta r \le x(t) \le R$, $t \in J$. From $0 < m \le G(t,s) \le M$ and (2.1), we obtain, for $x \in \bar{\Omega}_R \setminus \Omega_r$,

$$\begin{aligned} 0 \le (Ax)(t) &\le M\left[\int_0^T f\bigl(s, x(s)\bigr)\,ds + \sum_{k=1}^{p} I_k\bigl(x(t_k)\bigr)\right] \\ &\le MT \max\bigl\{f(t,u) : t \in J, \delta r \le u \le R\bigr\} \\ &\quad + pM \max\bigl\{I_k(u) : \delta r \le u \le R, 1 \le k \le p\bigr\} < +\infty, \\ (Ax)(t) &\ge m\left[\int_0^T f\bigl(s, x(s)\bigr)\,ds + \sum_{k=1}^{p} I_k\bigl(x(t_k)\bigr)\right] \ge \delta\|Ax\|. \end{aligned}$$

Hence, $A : \bar{\Omega}_R \setminus \Omega_r \to K$. In addition, one easily checks that A is completely continuous.

Next, we show that:

(a) $x \ne \mu Ax$ for $x \in K \cap \partial\Omega_r$ and $0 < \mu \le 1$,
(b) $\inf \|Ax\| > 0$, $x \ne \mu Au$ for $x \in K \cap \partial\Omega_R$ and $\mu \ge 1$.

If (a) is not true, there exist $x \in K \cap \partial\Omega_r$ and $0 < \mu \le 1$ with $x = \mu Ax$. Hence,

$$\begin{cases} x'(t) + a(t)x(t) = \mu f(t, x(t)), & t \ne t_k, t \in J, \\ x(t_k^+) = x(t_k) + \mu I_k(x(t_k)), & k = 1, 2, \ldots, p, \\ x(0) = x(T). \end{cases} \tag{2.2}$$

Since $x(t) \ge \delta r > 0$, we rewrite (2.2) as

$$\begin{cases} (\ln x(t))' + a(t) = \frac{\mu f(t,x(t))}{x(t)}, & t \ne t_k, t \in J, \\ \Delta \ln x(t_k) = \ln(1 + \frac{\mu I_k(x(t_k))}{x(t_k)}), & k = 1, 2, \ldots, p, \\ x(0) = x(T). \end{cases} \tag{2.3}$$

Integrating the first equality in (2.3) from 0 to T, we obtain

$$\begin{aligned}\int_0^T a(t)\,dt &= \mu\int_0^T \frac{f(t,x(t))}{x(t)}\,dt + \sum_{k=1}^{p}\ln\left(1+\frac{\mu I_k(x(t_k))}{x(t_k)}\right)\\ &\le \int_0^T \frac{f(t,x(t))}{x(t)}\,dt + \sum_{k=1}^{p}\ln\left(1+\frac{I_k(x(t_k))}{x(t_k)}\right)\\ &\le \varphi(r),\end{aligned}$$

which is a contradiction.

Suppose that $\inf_{x\in K\cap\partial\Omega_R}\|Ax\| = 0$. There exists the sequence $x_n \in K\cap\partial\Omega_R$ such that $\|Ax_n\| \to 0$ as $n\to\infty$. Noting $f(t,x_n(t)) \ge 0$, $I_k(x_n(t_k)) \ge 0$ and

$$0 \le m\left[\int_0^T f\bigl(s,x_n(s)\bigr)\,ds + \sum_{k=1}^{p} I_k\bigl(x_n(t_k)\bigr)\right] \le \|Ax_n\| \to 0,$$

we obtain $f(s,x_n(s)) \to 0$, $I_k(x_n(t_k)) \to 0$ as $n\to\infty$. Hence,

$$\int_0^T \frac{f(t,x_n(t))}{x_n(t)}\,dt + \sum_{k=1}^{p}\ln\left(1+\frac{I_k(x_n(t_k))}{x_n(t_k)}\right) \to 0 \quad \text{as } n\to\infty,$$

which implies that $\psi(R) = 0$, a contradiction.

Suppose that there exist $u \in K\cap\partial\Omega_R$ and $\mu \ge 1$ with $u = \mu Au$. Then

$$\begin{cases}(\ln u(t))' + a(t) = \frac{\mu f(t,u(t))}{u(t)}, & t \ne t_k, t\in J,\\ \Delta \ln u(t_k) = \ln(1+\frac{\mu I_k(u(t_k))}{x(t_k)}), & k = 1,2,\dots,p,\\ u(0) = u(T).\end{cases} \tag{2.4}$$

Integrating the first equality in (2.4) from 0 to T, we obtain

$$\begin{aligned}\int_0^T a(t)\,dt &= \mu\int_0^T \frac{f(t,u(t))}{u(t)}\,dt + \sum_{k=1}^{p}\ln\left(1+\frac{\mu I_k(u(t_k))}{u(t_k)}\right)\\ &\ge \int_0^T \frac{f(t,u(t))}{u(t)}\,dt + \sum_{k=1}^{p}\ln\left(1+\frac{I_k(u(t_k))}{u(t_k)}\right)\\ &\ge \psi(R),\end{aligned}$$

which is a contradiction.

By Lemma 2.2, there exists $x \in \bar{\Omega}_R\backslash\Omega_r$ with $Ax = x$, which is the positive solution of (1.1). The proof is complete. □

Corollary 2.1 *Assume that $a \in \Lambda^+$ and $f \in C(J\times[0,+\infty],[0,+\infty))$, $I_k \in C([0,+\infty),[0,+\infty))$ $(1\le k\le p)$. Then* (1.1) *has at least one positive solution if one of the following conditions is satisfied*:

(1) $\varphi_0 < \int_0^T a(s)\,ds$ *and* $\psi_\infty > \int_0^T a(s)\,ds$;

(2) $\varphi_\infty < \int_0^T a(s)\,ds$ *and* $\psi_0 > \int_0^T a(s)\,ds$.

Theorem 2.2 *Assume that $a \in \Lambda^+$ and there exist $N+1$ positive constants $p_1 < p_2 < \cdots < p_N < p_{N+1}$ such that*

$$f \in C\big(J \times [\delta p_1, p_{N+1}], [0, +\infty)\big), \qquad I_k \in C\big([\delta p_1, p_{N+1}], [0, +\infty)\big) \quad (1 \le k \le p).$$

Further suppose that one of the following conditions is satisfied:

(1) $\varphi(p_{2k-1}) < \int_0^T a(s)\,ds$, $k = 1, 2, \ldots, [(N+2)/2]$, $\psi(p_{2k}) > \int_0^T a(s)\,ds$, $k = 1, 2, \ldots, [(N+1)/2]$, *or*

(2) $\psi(p_{2k-1}) > \int_0^T a(s)\,ds$, $k = 1, 2, \ldots, [(N+2)/2]$, $\varphi(p_{2k}) < \int_0^T a(s)\,ds$, $k = 1, 2, \ldots, [(N+1)/2]$,

where $[d]$ denotes the integer part of d. Then (1.1) *has at least N positive solutions $x_k \in X$, $k = 1, 2, \ldots, N$ with $p_k < \|x_k\| < p_{k+1}$.*

Proof Assume that (1) holds. The case in which (2) holds is similar. Since φ, ψ are continuous functions, for any $1 \le j \le N$, there exist r_j, R_j such that $p_j < r_j < R_j < p_{j+1}$ and

$$\begin{aligned} &\varphi(r_j) < \int_0^T a(s)\,ds, \qquad \psi(R_j) > \int_0^T a(s)\,ds, \quad j \text{ is odd}, \\ &\varphi(R_j) < \int_0^T a(s)\,ds, \qquad \psi(r_j) > \int_0^T a(s)\,ds, \quad j \text{ is even}. \end{aligned}$$

By Theorem 2.1, (1.1) has at least one positive solution x_j with $r_j \le \|x_j\| \le R_j$. This ends the proof. □

Theorem 2.3 *Assume that $a \in \Lambda^-$ and there exist two positive constants $r < R$ such that*

$$f \in C\big(J \times [\delta r, R], (-\infty, 0]\big), \qquad I_k \in C\big([\delta r, R], (-\infty, 0]\big) \quad (1 \le k \le p). \tag{2.5}$$

Further suppose that (H_1) *or* (H_2) *is satisfied, where $\ln \alpha = -\infty$ if $\alpha \le 0$. Then* (1.1) *has at least one solution x with $r \le \|x\| \le R$.*

The proof of Theorem 2.3 is similar to that of Theorem 2.1 and we omit it.

Theorem 2.4 *Assume that $a \in \Lambda^+$ and the following conditions are satisfied*:

(D_1) *There exist constants $0 < \alpha < \beta$ such that $f(t, u)$, $I_k(u)$ are nondecreasing in $u \in [\alpha, \beta]$ and*

$$\int_0^T f(t, \alpha)\,dt + \sum_{k=1}^p I_k(\alpha) > \frac{\alpha}{m}, \qquad \int_0^T f(t, \beta)\,dt + \sum_{k=1}^p I_k(\beta) < \frac{\beta}{M}.$$

(D_2) *There exists $\gamma > \beta$ such that*

$$\begin{aligned} &f \in C\big(J \times [0, \gamma], [0, +\infty)\big), \qquad I_k \in C\big([0, \gamma], [0, +\infty)\big), \\ &\int_0^T \sup_{0 \le u \le \gamma} f(t, u)\,dt + \sum_{k=1}^p \sup_{0 \le u \le \gamma} I_k(u) \le \frac{\gamma}{M}. \end{aligned}$$

(D_3)

$$\lim_{x\to 0^+}\frac{f(t,x)}{x}=0,\qquad \lim_{x\to 0^+}\frac{I_k(u)}{u}=0.$$

Then (1.1) *has at least two positive solutions in* $\overline{U}_\gamma=\{x\in PC(J):\|x\|\le\gamma\}$.

Proof Set

$$\widetilde{f}=\begin{cases}f(t,|u|), & |u|\le\gamma,\\ f(t,\gamma) & |u|>\gamma,\end{cases}\qquad \widetilde{I}_k=\begin{cases}I_k(|u|), & |u|\le\gamma,\\ I_k(\gamma), & |u|>\gamma,\end{cases}$$

$$(T_\lambda u)(t)=\lambda\int_0^T G(t,s)\widetilde{f}\big(s,u(s)\big)\,ds+\lambda\sum_{k=1}^p G(t,t_k)\widetilde{I}_k\big(u(t_k)\big),\quad \lambda\in[0,1],u\in PC(J),$$

$$\Phi_\lambda=I-T_\lambda,\qquad V=\big\{x\in PC(J),\alpha<x<\beta\big\},$$

where I denotes the identity map. It is easy to check that $T_\lambda:[0,1]\times PC(J)\to PC(J)$ is compact. We show:

(1) $T_1(\overline{V})\subset V$.
(2) $\Phi_\lambda(u)=0$ implies that $\|u\|\le\gamma$.
(3) There exists $c\in(0,\alpha)$ such that $\Phi_\lambda(u)=0$ admits only the trivial solution in $\overline{U}_c$.

The proof of (1). From (D_1), we obtain, for $\forall u\in V$,

$$f(t,\alpha)\le\widetilde{f}(t,u)\le f(t,\beta),$$
$$I_k(\alpha)\le\widetilde{I}_k(u)\le I_k(\beta).$$

Thus, for all $u\in V$,

$$T_1(u)=\int_0^T G(t,s)\widetilde{f}\big(s,u(s)\big)\,ds+\sum_{k=1}^p G(t,t_k)\widetilde{I}_k(u)\le M\left[\int_0^T f(s,\beta)\,ds+\sum_{k=1}^p I_k(\beta)\right]<\beta,$$

$$T_1(u)\ge m\left[\int_0^T f(s,\alpha)\,ds+\sum_{k=1}^p I_k(\alpha)\right]>\alpha.$$

That is, $T_1(\overline{V})\subset V$.

The proof of (2). If $\varphi_\lambda(u)=0$, then

$$\begin{aligned}|u|&=\big|T_\lambda(u)\big|=\int_0^T G(t,s)\widetilde{f}\big(s,u(s)\big)\,ds+\sum_{k=1}^p G(t,t_k)\widetilde{I}_k\big(u(t_k)\big)\\&\le M\int_0^T\sup_{0\le u\le\gamma}f(s,u)\,ds+M\sum_{k=1}^p\sup_{0\le u\le\gamma}I_k(u)\le\gamma.\end{aligned}$$

The proof of (3). From (D_3), there exists $0<c<\alpha$ such that

$$f(t,u)\le\frac{u}{3MT},\qquad I_k(u)\le\frac{u}{3Mp},\quad 0\le u\le c.$$

If $u \in \overline{U}_c$ with $u = T_\lambda u$, we have

$$\begin{aligned} |u| &= \bigl|T_\lambda(u)\bigr| \le \sup_{t\in[0,T]}\left[\int_0^T G(t,s)\widetilde{f}\bigl(s,u(s)\bigr)\,ds + \sum_{k=1}^p G(t,t_k)\widetilde{I}_k\bigl(u(t_k)\bigr)\right] \\ &\le \sup_{t\in[0,T]}\left[\int_0^T G(t,s)\frac{u(s)}{3MT}\,ds + M\sum_{k=1}^p \widetilde{I}_k\bigl(u(t_k)\bigr)\right] \le \frac{2}{3}\|u\|, \end{aligned}$$

which implies that $u \equiv 0$.

Let $U_c^\gamma = U_\gamma/\overline{U}_c$ and the degree of Φ_λ at 0 relative to open set D be $d(\Phi_\lambda, D, 0)$. Since $\overline{V}$ is a closed, convex set in $PC(J)$ and $T_1(\partial V) \subset V$, by Schauder's fixed point theorem, there exists $u_1 \in V$ such that $T_1(u_1) = u_1$ and

$$d(\Phi_1, V, 0) = 1.$$

For $\lambda \in [0,1]$, $u \in \partial U_c^\gamma$, $\Phi_\lambda(u) \ne 0$, by the homotopy property of the degree, we obtain

$$\begin{aligned} &d\bigl(\Phi_\lambda, U_c^\gamma, 0\bigr) \equiv \text{ constant}, \quad \lambda \in [0,1], \\ &d\bigl(\Phi_1, U_c^\gamma, 0\bigr) = d\bigl(\Phi_0, U_c^\gamma, 0\bigr) = d\bigl(I, U_c^\gamma, 0\bigr) = 0. \end{aligned}$$

By the additivity property of the degree, one obtains

$$d\bigl(\Phi_1, U_c^\gamma, 0\bigr) = d(\Phi_1, V, 0) + d\bigl(\Phi_1, U_c^\gamma/\overline{V}, 0\bigr).$$

Hence,

$$d\bigl(\Phi_1, U_c^\gamma/\overline{V}, 0\bigr) = d\bigl(\Phi_1, U_c^\gamma, 0\bigr) - d(\Phi_1, V, 0) = 0 - 1 = -1,$$

which implies that T_1 has at least a fixed point $u_2 \in U_c^\gamma/\overline{V}$. Clearly, $c \le \|u_2\| \le \gamma$. In addition,

$$\begin{aligned} u_2(t) &= \int_0^T G(t,s)\widetilde{f}\bigl(s,u_2(s)\bigr)\,ds + \sum_{k=1}^p G(t,t_k)\widetilde{I}_k\bigl(u_2(t_k)\bigr) \ge 0, \\ u_2(t) &= \int_0^T G(t,s)f\bigl(s,u_2(s)\bigr)\,ds + \sum_{k=1}^p G(t,t_k)I_k\bigl(u_2(t_k)\bigr) \\ &\ge m\left(\int_0^T f\bigl(s,u_2(s)\bigr)\,ds + \sum_{k=1}^p I_k\bigl(u_2(t_k)\bigr)\right) \ge 0. \end{aligned}$$

If there is $t^* \in J$ such that $u_2(t^*) = 0$, by the above inequality, we obtain $f(s,u_2(s)) \equiv 0$, $I_k(u_2(t_k)) \equiv 0$, which implies that $u_2(t) \equiv 0$, a contradiction. Hence, (1.1) has at least two positive solutions in $\overline{U}_\gamma$. The proof is complete. □

Example 2.1 Consider the differential equation with singularity

$$\begin{cases} x'(t) + \kappa x(t) = \frac{a}{x^\lambda(t)} + h(t), & t \ne t_k, t \in J, \\ \Delta x(t_k) = c_k x(t_k), & k = 1,2,\ldots,p, \\ x(0) = x(T), \end{cases} \tag{2.6}$$

where κ, a, λ are positive constants, $c_k > 0$ and $h \in C(J,R)$.

Set $h_* = \min\{h(t) : t \in J\}$ and $h^* = \max\{h(t) : t \in J\}$. We claim that (2.6) has at least one positive solution provided that the following conditions hold:

$$\gamma_1 = \sum_{k=1}^{p} \ln(1 + c_k) < \kappa T, \quad \text{if } h_* \geq 0, \tag{2.7}$$

$$\gamma_2 =: \left(h^* - e^{\lambda\kappa T} h_*\right) e^{\kappa T} \left(\frac{-h_*}{a}\right)^{\frac{1}{\lambda}} < \kappa - \frac{1}{T} \sum_{k=1}^{p} \ln(1 + c_k), \quad \text{if } h_* < 0. \tag{2.8}$$

In fact, $f(t,u) = au^{-\lambda} + h(t)$, $\psi_0 = +\infty$. Moreover, $\varphi_\infty = \gamma_1$ if $h_* \geq 0$. If $h_* < 0$, choose $b = \sqrt[\lambda]{a/(-h_*)}$, $0 \leq f(t,u) \leq au^{-\lambda} + h^* \leq \gamma_2 u$ for $(t,u) \in J \times [\delta b, b]$ and $\varphi(b) = \gamma_2 T + \gamma_1 < \kappa T$. By Theorem 2.1, (2.6) has one positive solution.

Remark 2.1 The conditions (2.7) and (2.8) are different from those of Corollary 4.7 in [8].

Example 2.2 Consider the differential equation

$$\begin{cases} x'(t) + \frac{1}{2}x(t) = x^3(t)\exp(-0.1x(t)), & t \in [0,t_1) \cup (t_1,1], \\ \Delta x(t_1) = \frac{10x^2(t_1)}{1+x^2(t_1)}, \\ x(0) = x(1). \end{cases} \tag{2.9}$$

Let $f(t) = t^3 e^{-0.1t}$, $I_1(t) = 10t^2/(1+t^2)$, $\alpha = 0.1$, $\beta = 30$, $\gamma = 10^4$. It is easy to check that the conditions (D_1)-(D_3) hold. Hence, (2.9) has at least two positive solutions.

3 Application

In this section, we consider the differential equation

$$\begin{cases} x'(t) = -\frac{a(t)}{x^\lambda(t)} + f(x(t)), & t \neq t_k, t \in J, \\ \Delta x(t_k) = c_k x(t_k), & k = 1,2,\ldots,p, \\ x(0) = x(T), \end{cases} \tag{3.1}$$

where $\lambda \geq 0$.

Set $y = \exp(x^{\lambda+1}(t)/(\lambda+1))$ and $F(u) = u^\lambda f(u)$, then

$$\begin{cases} y'(t) + a(t)y(t) = y(t)F(((\lambda+1)\ln y(t))^{\frac{1}{\lambda+1}}), & t \neq t_k, t \in J, \\ \Delta y(t_k) = y(t_k)^{(1+c_k)^{\lambda+1}} - y(t_k), & k = 1,2,\ldots,p, \\ y(0) = y(T). \end{cases} \tag{3.2}$$

If (3.2) has a solution $y(t) > 1$, $t \in J$, then $x(t) = ((\lambda+1)\ln y(t))^{\frac{1}{\lambda+1}}$ is the positive solution of (3.1). By Theorem 2.1, we have the following result.

Theorem 3.1 *Assume that $a \in \Lambda^+$, $f \in C((0,+\infty),[0,+\infty))$, $c_k > 0$. If there exist constants $\delta^{-1} < r < R$ such that*

$$T \sup_{\delta r \leq u \leq r} F\left(\left((\lambda+1)\ln u\right)^{\frac{1}{\lambda+1}}\right) + r\sum_{k=1}^{p}\left[(1+c_k)^{\lambda+1} - 1\right] < \int_0^T a(t)\,dt,$$

$$T \inf_{\delta R \le u \le R} F\bigl(\bigl((\lambda+1)\ln u\bigr)^{\frac{1}{\lambda+1}}\bigr) + \bigl(\ln \delta^{-1} R\bigr) \sum_{k=1}^{p} \bigl[(1+c_k)^{\lambda+1} - 1\bigr] > \int_0^T a(t)\,dt$$

or

$$T \sup_{\delta R \le u \le R} F\bigl(\bigl((\lambda+1)\ln u\bigr)^{\frac{1}{\lambda+1}}\bigr) + \ln R \sum_{k=1}^{p} \bigl[(1+c_k)^{\lambda+1} - 1\bigr] < \int_0^T a(t)\,dt,$$

$$T \inf_{\delta r \le u \le r} F\bigl(\bigl((\lambda+1)\ln u\bigr)^{\frac{1}{\lambda+1}}\bigr) + \bigl(\ln \delta^{-1} r\bigr) \sum_{k=1}^{p} \bigl[(1+c_k)^{\lambda+1} - 1\bigr] > \int_0^T a(t)\,dt,$$

then (3.1) *has at least one positive solution.*

Example 3.1 Consider the differential equation

$$\begin{cases} x'(t) = -\frac{a(t)}{x^{\lambda}(t)} + \mu f(x(t)), & t \neq t_k, t \in J, \\ \Delta x(t_k) = \mu x(t_k), & k = 1, 2, \ldots, p, \\ x(0) = x(T), \end{cases} \tag{3.3}$$

where $\lambda \ge 0$, $a \in \Lambda^+$, $f \in C((0,+\infty),[0,+\infty))$, μ is a positive real parameter.

Corollary 3.1 *There exists* $\mu^* > 0$ *such that* (3.3) *has at least a positive solution for any* $\mu \in (0, \mu^*)$.

Proof Choosing $r = \delta^{-1} + 1$. Since $u^{\lambda} f(u)$ is continuous in $(0,+\infty)$ and $\ln u > 0$ for $u \in [\delta r, r]$, we obtain

$$0 \le \sup_{\delta r \le u \le r} \bigl\{\bigl((\lambda+1)\ln u\bigr)^{\frac{\lambda}{1+\lambda}} f\bigl(\bigl((\lambda+1)\ln u\bigr)^{\frac{1}{\lambda+1}}\bigr)\bigr\} \le C$$

for some $C > 0$. On the other hand, since $[(1+\mu)^{\lambda+1} - 1]/\mu \to 1 + \lambda$ as $\mu \to 0^+$, there is $\sigma > 0$ such that

$$(1+\mu)^{\lambda+1} - 1 \le 2(1+\lambda)\mu, \quad 0 < \mu \le \sigma.$$

Set $\mu^* = \min\{\sigma, \int_0^T a(s)\,ds / [CT + 2p(\lambda+1)\ln r]\}$, for $\mu < \mu^*$, we have

$$\begin{aligned} &T \sup_{\delta r \le u \le r} F\bigl(\bigl((\lambda+1)\ln u\bigr)^{\frac{1}{\lambda+1}}\bigr) + \ln r \sum_{k=1}^{p} \bigl[(1+\mu)^{\lambda+1} - 1\bigr] \\ &\quad < \mu^* \bigl[CT + 2p(\lambda+1)\ln r\bigr] < \int_0^T a(t)\,dt. \end{aligned}$$

Taking

$$R = \max\biggl\{\exp \frac{\int_0^T a(t)\,dt}{p(1+\mu)^{1+\lambda} - p}, \delta\biggr\},$$

one easily checks that

$$T \inf_{\delta R \le u \le R} F\Big(\big((\lambda+1)\ln u\big)^{\frac{1}{\lambda+1}}\Big) + \big(\ln \delta^{-1} R\big) \sum_{k=1}^{p} \big[(1+\mu)^{\lambda+1} - 1\big] > \int_0^T a(t)\,dt.$$

By Theorem 3.1, (3.3) has one positive solution. □

Remark 3.1 Corollary 3.1 admits the case that the function a changes sign. As far as we know, no paper discussed (3.3) when a changes sign.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Authors typed, read, and approved the final draft.

Acknowledgements
The authors wish to express their thanks to the referee for his/her very valuable suggestions and careful corrections. The work is supported by the NNSF of China (11171085), Hunan Provincial Natural Science Foundation of China (2015JJ2068).

Received: 1 March 2015 Accepted: 10 August 2015 Published online: 04 September 2015

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