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Jackson's (−1)-Bessel functions with the Askey-Wilson algebra setting

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Abstract

This work is devoted to the study of some functions arising from a limit transition of the Jackson q-Bessel functions when q tends to −1. These functions coincide with the so-called cas function for particular values of parameters. We prove that there are eigenfunctions of differential-difference operators of Dunkl-type. Also we consider special cases of the Askey-Wilson algebra $AW(3)$, which have these operators (up to constants) as one of their three generators and whose defining relations are given in terms of anti-commutators.

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1 Introduction

In [1], Vinet and Zhedanov introduced new families of orthogonal polynomials by considering appropriate limits when q tends to −1 of the little and big q-Jacobi polynomials. In this work we will study the limit when q tends to −1 of the three q-analogs of the Bessel function, which are introduced by Jackson [2–4]. The first and the second q-Bessel functions are reconsidered and rewritten in modern notations by Ismail [5]. The third q-Bessel function is rediscovered later by Hahn [6] and Exton [7]. This function has an interpretation as matrix elements of irreducible representations of the quantum group of plane motions $E_q(2)$ and satisfies an orthogonality relation that makes it more suitable for harmonic analysis [8–10]. Of course, when q tends to 1, the Jackson q-Bessel functions tend to the standard Bessel function [11].

In this paper we will show that the limit when q tends to −1 of the third q-Bessel functions leads to a new type of nonsymmetric Bessel functions satisfying first order differential-difference equation. Also these functions coincide for a particular value of its parameters with the cas function [12]. Furthermore, by using the limit transition from little q-Jacobi polynomials to the third q-Bessel function and from q-Laguerre to the second q-Bessel function we construct a q-Bessel version of the Askey-Wilson $AW(3)$ algebra.

Notations Throughout we assume $-1 < q < 1$. For q-Pochhammer symbols and q-hypergeometric series we use the notation of [13],

$$(a;q)_0 := 1, \qquad (a;q)_n := \prod_{k=0}^{n-1}(1 - aq^k), \quad n = 1, \dots, \infty. \tag{1.1}$$

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The basic hypergeometric series are defined by

$$_r\Phi_s\left(\begin{matrix} a_2,\ldots,a_r \\ b_1,\ldots,b_s \end{matrix}\,\middle|\, q;z\right) := \sum_{k=0}^{+\infty} \frac{(a_1,\ldots,a_r;q)_k}{(q,b_1,\ldots,b_s;q)_k}\left((-1)^k q^{\binom{k}{2}}\right)^{1+s-r} z^k, \tag{1.2}$$

where

$$(a_1,\ldots,a_r;q)_n := (a_1;q)_n \cdots (a_r;q)_n.$$

2 Askey-Wilson relations for the little q-Jacobi polynomials

2.1 Little q-Jacobi polynomials

The little q-Jacobi polynomial is defined by [14], (18.27(iv))

$$p_n(x) = p_n(x;a,b|q) := {}_2\Phi_1\left(\begin{matrix} q^{-n},abq^{n+1} \\ aq \end{matrix}\,\middle|\, q;qx\right). \tag{2.1}$$

For $0 < a < q^{-1}$ and $b < q^{-1}$, the polynomials $\{p_n(x)\}_n$ satisfy the orthogonality relations

$$\frac{(aq;q)_\infty}{(abq^2;q)_\infty}\sum_{k=0}^\infty p_m(q^k)p_n(q^k)(aq)^k \frac{(bq;q)_k}{(q;q)_k} = \frac{(1-abq)(aq)^n}{1-abq^{2n+1}}\frac{(q,bq;q)_n}{(aq,abq;q)_n}\delta_{m,n}.$$

There is a q-difference equation for this polynomial of the form

$$Y_{a,b,q}f(x) = \lambda_n f(x), \tag{2.2}$$

where

$$(Y_{a,b,q}f) := a(bq - x^{-1})(f(qx) - f(x)) + (1 - x^{-1})(f(q^{-1}x) - f(x)) \tag{2.3}$$

and

$$\lambda_n = q^{-n}(1-q^n)(1-abq^{n+1}). \tag{2.4}$$

The Askey-Wilson algebra $AW(3)$ involves a nonzero scalar q and three parameters ω_1, ω_2, and ω_3, it was introduced by Zhedanov [15] as an associative algebra generated by X, Y, and Z subject to the following commutation relations:

$$\begin{aligned} YX - qXY &= \mu_3 Z + \omega_3, & ZY - qYZ &= \mu_2 X + \omega_2, \\ XZ - qZXY &= \mu_1 Y + \omega_1. \end{aligned} \tag{2.5}$$

There is a central element Q, which is explicitly given as a polynomial of degree 3 in terms of X, Y, and Z [15]

$$Q = (q^2 - 1)YXZ + \mu_1 Y^2 + \mu_2 qX^2 + \mu_3 Z^2 + (q+1)(\omega_1 Y + \omega_2 qX + \omega_3 Z). \tag{2.6}$$

The limit of orthogonal polynomials in the Askey scheme as $q \to 1$ corresponds to the limit $q \to 1$ of $AW(3)$ to some classical algebras. In particular, the related Askey-Wilson

algebra for the little q-Jacobi polynomial is generated by X, Y, Z with relations [1]

$$YX - qXY = Z + \omega_3, \qquad ZY - qYZ = X + \omega_2, \qquad XZ - qZX = 0, \tag{2.7}$$

where

$$\omega_2 = -\frac{1+b}{(1+q)b}, \qquad \omega_3 = -\frac{1+a}{(1+q)\sqrt{ab}}. \tag{2.8}$$

There is a representation on the space of polynomials of the little q-Jacobi $AW(3)$ defined by relations (2.7) with structure constants given in (2.8) as follows:

$$(Yf)(x) := \mu(Y_{a,b,q} + 1 + qab)f(x), \qquad (Xf)(x) := xf(x),$$
$$(Zf)(x) := \frac{1-x}{q\sqrt{ab}}f(q^{-1}x), \tag{2.9}$$

where

$$\mu = -\frac{1}{(1-q^2)\sqrt{ab}}. $$

The Casimir operator

$$Q = (q^2 - 1)YXZ + q^2X^2 + Z^2 + (q+1)(\omega_2 qX + \omega_3 Z) \tag{2.10}$$

takes the value $Q = -b^{-1}$.

2.2 Little (-1)-Jacobi polynomials

The little (-1)-Jacobi polynomials $P_n^{(\alpha,\beta,-1)}(x)$ have been introduced and investigated in [1] as limits of the little q-Jacobi polynomials (2.1)

$$\lim_{\varepsilon \to 0} p_n(x; -e^{\varepsilon\alpha}, -e^{\varepsilon\beta} \mid -e^{\varepsilon}) = P_n^{(\alpha,\beta,-1)}(x). \tag{2.11}$$

We recall here their basic properties [1]. The polynomial $P_n^{(\alpha,\beta,-1)}(x)$ satisfies the following difference-differential equation

$$(Y_{\alpha,\beta,-1}f)(x) = \lambda_n f(x), \tag{2.12}$$

where

$$(Y_{\alpha,\beta,-1}f)(x) = (x-1)f'(-x) + (\alpha + \beta - \alpha x^{-1})\frac{f(x) - f(-x)}{2}$$

and

$$\lambda_n = \begin{cases} -n, & \text{if } n \text{ is even,} \\ \alpha + \beta + n + 1, & \text{if } n \text{ is odd.} \end{cases}$$

These polynomials have the following expressions in terms of the hypergeometric series:

For n, even

$$P_n^{(\alpha,\beta,-1)}(x) = {}_2F_1\left(\begin{matrix}-\frac{n}{2},\frac{\alpha+\beta+n+2}{2}\\\frac{\alpha+1}{2}\end{matrix};x^2\right) + \frac{nx}{\alpha+1}{}_2F_1\left(\begin{matrix}1-\frac{n}{2},\frac{\alpha+\beta+n+2}{2}\\\frac{\alpha+3}{2}\end{matrix};x^2\right),$$

and for n odd

$$P_n^{(\alpha,\beta,-1)}(x) = {}_2F_1\left(\begin{matrix}-\frac{n-1}{2},\frac{\alpha+\beta+n+1}{2}\\\frac{\alpha+1}{2}\end{matrix};x^2\right) - \frac{(\alpha+\beta+n+1)x}{\alpha+1}\,{}_2F_1\left(\begin{matrix}-\frac{n-1}{2},\frac{\alpha+\beta+n+3}{2}\\\frac{\alpha+3}{2}\end{matrix};x^2\right).$$

We introduce the operators

$$(Yf)(x) := (Y_{\alpha,\beta,-1}f)(x) - \frac{1}{2}(\alpha+\beta+1)f(x), \qquad (Xf)(x) := xf(x),$$

$$(Zf)(x) := (x-1)f(-x).$$

In [1], it was shown that these operators are closed in the framework of the Askey-Wilson algebra and they satisfy the commutation relations

$$\{X,Y\} = Z + \alpha, \qquad \{X,Z\} = 0, \qquad \{Y,Z\} = Y + \beta, \tag{2.13}$$

with $\{A,B\} = AB + BA$ denoting as usual the anti-commutator of A and B.

The Casimir operator is

$$Q = Y^2 + Z^2 \tag{2.14}$$

and takes the value

$$Q = 1.$$

3 The nonsymmetric Hankel transform

For Bessel functions $J_\alpha(x)$ see [14], Chapter 10, and the references given therein. Let us consider the normalized Bessel function $\mathcal{J}_\alpha(x)$, which is given by

$$\mathcal{J}_\alpha(x) := \Gamma(\alpha+1)(2/x)^\alpha J_\alpha(x). \tag{3.1}$$

Then

$$\mathcal{J}_\alpha(x) = \sum_{k=0}^{\infty}\frac{(-\frac{1}{4}x^2)^k}{(\alpha+1)_k k!} = {}_0F_1\left(\begin{matrix}-\\\alpha+1\end{matrix};-\frac{1}{4}x^2\right) \quad (\alpha > -1).$$

$\mathcal{J}_\alpha(x)$ is an entire function and has the simple properties and special cases [14], Section 10.16.9

$$\mathcal{J}_\alpha(x) = \mathcal{J}_\alpha(-x), \qquad \mathcal{J}_\alpha(0) = 1, \qquad \mathcal{J}_{-1/2}(x) = \cos x, \qquad \mathcal{J}_{1/2}(x) = \frac{\sin x}{x}.$$

The function $x \mapsto \mathcal{J}_\alpha(\lambda x)$ satisfies also the eigenvalue equation [14], Section 10.13.5:

$$\left(\frac{d^2}{dx^2} + \frac{2\alpha+1}{x}\frac{d}{dx}\right)\mathcal{J}_\alpha(\lambda x) = -\lambda^2\mathcal{J}_\alpha(\lambda x).$$

The *Hankel transform* pair [14], Section 10.22(v), for f in a suitable function class, is given by

$$\begin{cases} \widehat{f}(\lambda) = \int_0^\infty f(x) \mathcal{J}_\alpha(\lambda x) \frac{x^{2\alpha+1}\,dx}{2^{\alpha+1/2}\Gamma(\alpha+1)}, \\ f(x) = \int_0^\infty \widehat{f}(\lambda) \mathcal{J}_\alpha(\lambda x) \frac{\lambda^{2\alpha+1}\,d\lambda}{2^{\alpha+1/2}\Gamma(\alpha+1)}. \end{cases} \tag{3.2}$$

Now consider the so-called *nonsymmetric Bessel function*, also called *Dunkl-type Bessel function*, in the rank one case (see [16], Section 4):

$$\mathcal{E}_\alpha(x) := \mathcal{J}_\alpha(x) + \frac{ix}{2(\alpha+1)} \mathcal{J}_{\alpha+1}(x). \tag{3.3}$$

In particular,

$$\mathcal{E}_{-1/2}(x) = e^{ix}.$$

The nonsymmetric Hankel transform pair takes the form

$$\begin{cases} \widehat{f}(\lambda) = \int_{\mathbb{R}} f(x) \mathcal{E}_\alpha(\lambda x) \frac{|x|^{2\alpha+1}\,dx}{2^{\alpha+1}\Gamma(\alpha+1)}, \\ f(x) = \int_{\mathbb{R}} \widehat{f}(\lambda) \mathcal{E}_\alpha(\lambda x) \frac{|\lambda|^{2\alpha+1}\,d\lambda}{2^{\alpha+1}\Gamma(\alpha+1)}. \end{cases} \tag{3.4}$$

The transform pair (3.4) follows immediately from (3.2). For some given α let us define the differential-reflection operator

$$(T_\alpha f)(x) := f'(x) + \left(\alpha + \frac{1}{2}\right)\frac{f(x) - f(-x)}{x}, \tag{3.5}$$

called the *Dunkl operator* for root system A_1 (see [17], Definition 4.4.2). We have the eigenvalue equation

$$T_\alpha\big(\mathcal{E}_\alpha(\lambda x)\big) = i\lambda \mathcal{E}_\alpha(\lambda x). \tag{3.6}$$

If in (3.6) we substitute (3.3), compare even and odd parts, and then substitute (3.1), then we see that (3.6) is equivalent to a pair of lowering and raising differentiation formulas for Bessel functions (see [14], (10.6.2)):

$$J_\alpha'(x) - \frac{\alpha}{x}J_\alpha(x) = -J_{\alpha+1}(x), \qquad J_{\alpha+1}'(x) + \frac{\alpha+1}{x}J_{\alpha+1}(x) = J_\alpha(x).$$

The double degeneration of the double affine Hecke algebra $\mathcal{H}''$ is generated by D, Z, and s with relations [18]

$$sZs^{-1} = -Z, \qquad sDs^{-1} = -D, \qquad [D, Z] = 1 + 2ks. \tag{3.7}$$

The operators D, s, and Z act on the polynomial $f(x)$ as

$$(Df)(x) = (T_\alpha f)(x), \qquad (sf)(x) = f(-x), \qquad (Zf)(x) = xf(x). \tag{3.8}$$

The operators defined in (3.8) are also known as para-Bose operators and the algebra (3.7) is equivalent to the para-Bose algebra [19, 20].

Another important generalization of the exponential function is the so-called cas function, which is given by [12, 21]

$$\text{cas}(x) = \cos(x) + \sin(x). \tag{3.9}$$

It is evident that the function $y(x) = \text{cas}(\lambda x)$ satisfies

$$(\Lambda y)(x) = \lambda y(x), \qquad y(0) = 1, \tag{3.10}$$

where $\Lambda = s\partial$ and ∂ is the derivative operator.

4 Limit $q \to -1$ of the third q-Bessel function

The third q-Bessel function was introduced in [6, 7], see also [7, 9, 10], and is defined by

$$J_\nu^{(3)}(x;q) := \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty} \left(\frac{1}{2}x\right)^\nu {}_1\Phi_1\left({0 \atop q^{\nu+1}}\Big| q; \frac{1}{4}qx^2\right) \quad (x > 0).$$

We will consider a slightly different function $\mathcal{J}_3(x, a; q)$, called a normalized third q-Bessel function, which is defined by

$$\mathcal{J}_3(x, a; q) = {}_1\Phi_1\left({0 \atop aq}\Big| q; qx\right). \tag{4.1}$$

It is easy to see that

$$\lim_{q \to 1} \mathcal{J}_3\left((1-q)^2 x, q^\alpha; q\right) = \mathcal{J}_\alpha(2\sqrt{x}).$$

The function $\mathcal{J}_3(\lambda x, a; q)$ is a solution of the q-difference

$$(Y_{a,q}f)(x) = -\lambda f(x), \tag{4.2}$$

where

$$(Y_{a,q}f)(x) := \frac{a}{x}\left(f(qx) - f(x)\right) + \frac{1}{x}\left(f\left(q^{-1}x\right) - f(x)\right). \tag{4.3}$$

Next, we describe the construction of new function by the limiting process from normalized third q-Bessel as $q \to -1$ and from the (-1)-Jacobi polynomials as $n \to \infty$. Moreover, we have the following diagram for limit relations between these special functions and orthogonal polynomials.
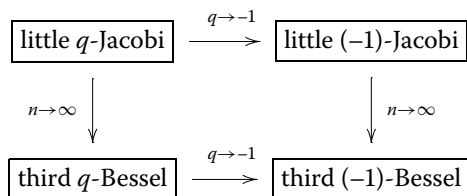
The upper limit in the previous diagram is studied by [1]. Next we are concerned only by the three other limits.

There is a well-known limit from Jacobi polynomials to Bessel functions, see [14], (18.11.5),

$$\lim_{n\to\infty} \frac{\Gamma(\alpha+1)}{n^\alpha} P_n^{(\alpha,\beta)}\left(1 - \frac{x^2}{2n^2}\right) = \mathcal{J}_\alpha(x). \tag{4.4}$$

The *q*-analog of this limit transition starts with the little *q*-Jacobi polynomials (2.1). From Proposition A.1 in [9] we have

$$\lim_{n\to\infty} p_n\big(q^n x; a, b | q\big) = \mathcal{J}_3(x, a; q). \tag{4.5}$$

The operators X, Y, and Z defined in (2.9) have also a limit for $n \to \infty$ after the rescaling $x \to q^n x$. More precisely, let us denote

$$(Xf)(x) := \lim_{n\to\infty} q^{-n}(Xf)\big(q^n x\big),$$

$$(Yf)(x) := \lim_{n\to\infty} \sqrt{ab}\, q^n (Yf)\big(q^n x\big),$$

$$(Zf)(x) := \lim_{n\to\infty} \sqrt{ab}(Zf)\big(q^n x\big).$$

Then in the limit the operators X, Y, and Z are given by

$$(Xf)(x) = xf(x), \qquad (Yf)(x) = \frac{1}{q^2-1}(Y_{a,q}f)(x), \qquad (Zf)(x) = q^{-1}f\big(q^{-1}x\big), \tag{4.6}$$

and (2.13) become

$$YX - qXY = Z + \omega_3, \qquad ZY - qYZ = 0, \qquad XZ - qZX = 0, \tag{4.7}$$

where

$$\omega_3 = -\frac{1+a}{1+q}. \tag{4.8}$$

The resulting algebra generated by X, Y, and Z with (4.7) and (4.8) is called the *q*-Bessel $AW(3)$ algebra. In this case the Casimir operator becomes

$$Q = \big(q^2 - 1\big)YXZ + Z^2 - (1+a)Z \tag{4.9}$$

and takes the value $Q = -a$.

Equations (4.7) hold in the limit $q \to -1$. Indeed, let us take the parametrization

$$a = -e^{\varepsilon(2\alpha+1)} \quad \text{and} \quad q = -e^\varepsilon.$$

Since

$$f\big(-e^{\pm\varepsilon}z\big) = f(-z) \mp zf(-z)\varepsilon + o(\varepsilon),$$

$$q = -1 - \varepsilon + o(\varepsilon),$$

$$a = -1 - (2\alpha + 1)\varepsilon + o(\varepsilon),$$

the q-difference equation (4.2) tends formally as $\varepsilon \to 0$ to the differential-difference equation

$$(Y_\alpha f)(x) = \lambda f(x), \tag{4.10}$$

where

$$(Y_\alpha f)(x) = f'(-x) + \left(\alpha + \frac{1}{2}\right)\frac{f(x) - f(-x)}{x}. \tag{4.11}$$

The operator Y_α is a difference-differential operator of the first order containing reflection terms. Notice the important property of the operator Y_α: it sends the linear space of polynomials of dimension $n + 1$ to the space of dimension n. In particular, this means that there are no polynomial eigenfunction of the operator Y_α.

Now, let us introduce the operators

$$(Xf)(x) := xf(x), \qquad (Yf)(x) := (Y_\alpha f)(x), \qquad (Zf)(x) := f(-x). \tag{4.12}$$

Then it is elementary to verify that the operators X, Y, Z satisfy the relations

$$\{X, Y\} = -Z - 2\alpha - 1, \qquad \{X, Z\} = 0, \qquad \{Y, Z\} = 0, \tag{4.13}$$

which corresponds to the $AW(3)$ algebra with parameters

$$q = -1, \qquad \omega_3 = -2\alpha - 1, \qquad \mu_3 = -1, \qquad \omega_2 = \omega_3 = \mu_1 = \mu_2 = 0.$$

It is easily verified that the Casimir operator is

$$Q = Z^2. \tag{4.14}$$

In the case of the realization (4.12) of the operators X, Y Z, the Casimir operator becomes the identity operator.

Theorem 4.1 *For each $\lambda \in \mathbb{C}$. The differential-difference equation* (4.10) *under the initial condition $y(0) = 1$, admits unique C^∞-solution denoted $\mathcal{J}_{\alpha,-1}(\lambda x)$, which is expressed in terms of the normalized q-Bessel function* (3.1) *by*

$$\mathcal{J}_{\alpha,-1}(\lambda x) = \mathcal{J}_\alpha(\lambda x) + \frac{\lambda x}{2(\alpha + 1)}\mathcal{J}_{\alpha+1}(\lambda x). \tag{4.15}$$

Proof From the decomposition in the form $f = f_1 + f_2$ where f_1 is even and f_2 is odd. Equation (3.1) is equivalent to the following system:

$$\begin{cases} -f_1'(x) = \lambda f_2(x), \\ f_2'(x) + \frac{2\alpha+1}{x}f_2(x) = \lambda f_1(x), \\ f_1(0) = 1, \qquad f_1'(0) = 0. \end{cases}$$

Thus

$$\begin{cases} -f_1'(x) = \lambda f_2(x), \\ f_1''(x) + \frac{2\alpha+1}{x}f_1'(x) = -\lambda^2 f_1(x), \\ f_1(0) = 1, \qquad f_1'(0) = 0. \end{cases}$$

Hence,

$$f_1(x) = \mathcal{J}_\alpha(\lambda x) \quad \text{and} \quad f_2(x) = -\frac{1}{\lambda}\frac{d}{dx}\mathcal{J}_\alpha(\lambda x). \qquad \square$$

In particular, for $\alpha = -1/2$, the (-1)-Bessel function $\mathcal{J}_{\alpha,-1}(\lambda x)$ coincides with the cas function (3.9)

$$\mathcal{J}_{-1/2,-1}(\lambda x) = \cos(\lambda x).$$

The function $\mathcal{J}_{\alpha,-1}(\lambda x)$ is related to the Dunkl function (3.3) by

$$\mathcal{J}_{\alpha,-1}(\lambda x) = \frac{1}{2}\big((1+i)\mathcal{E}_\alpha(\lambda x) + (1-i)\mathcal{E}_\alpha(-\lambda x)\big). \tag{4.16}$$

The (-1)-Bessel transform pair takes the form

$$\begin{cases} \widehat{f}(\lambda) = \frac{1}{2^{\alpha+1}\Gamma(\alpha+1)}\int_{-\infty}^{\infty} f(x)\mathcal{J}_{\alpha,-1}(-\lambda x)|x|^{2\alpha+1}\,dx, \\ f(x) = \frac{1}{2^{\alpha+1}\Gamma(\alpha+1)}\int_{-\infty}^{\infty} \widehat{f}(\lambda)\mathcal{J}_{\alpha,-1}(\lambda x)|\lambda|^{2\alpha+1}\,d\lambda. \end{cases} \tag{4.17}$$

The transform pair (4.17) follows immediately from (3.2) by putting $f(x) = f_1(x) + xf_2(x)$ in (4.17) with f_1 and f_2 even.

The (-1)-Bessel function $\mathcal{J}_{\alpha,-1}(\lambda x)$ can be obtained also as limit case of the normalized third q-Bessel function (4.1)

$$\lim_{\varepsilon\to 0} \mathcal{J}_3\big(x, -e^{\varepsilon(2\alpha+1)}; -e^\varepsilon\big) = \mathcal{J}_{\alpha,-1}(x). \tag{4.18}$$

Indeed, from (4.1) we can expand $\mathcal{J}(x, -e^{\varepsilon(2\alpha+1)}; -e^\varepsilon)$ in a power series of x as follows:

$$\mathcal{J}\big(x, -e^{\varepsilon(2\alpha+1)}; -e^\varepsilon\big) = \sum_{n=0}^{\infty} c_{n,\alpha}(\varepsilon)x^n,$$

where

$$c_{n,\alpha}(\varepsilon) = (-1)^n \frac{(-1)^{\frac{n(n-1)}{2}}(1-e^{2\varepsilon})^n}{4^n(e^{\varepsilon(2\alpha+2)}, -e^\varepsilon; -e^\varepsilon)_n}.$$

Then the result follows from the following elementary limits involving q-shifted factorials:

$$\lim_{\varepsilon\to 0} \varepsilon^{-[n/2]}\big(-e^{\varepsilon\alpha}; -e^\varepsilon\big)_n = (-1)^{[n/2]}2^n\big((\alpha+1)/2\big)_{[n/2]},$$

$$\lim_{\varepsilon\to 0} \varepsilon^{-[(n+1)/2]}\big(e^{\varepsilon\alpha}; -e^\varepsilon\big)_n = (-1)^{[(n+1)/2]}2^n(\alpha/2)_{[(n+1)/2]}.$$

5 The second q-Bessel function case

5.1 q-Laguerre polynomials

The q-Laguerre polynomials $\{L_n(x,a;q)\}_n$ are defined by

$$L_n(x,a;q) := \frac{(aq;q)_n}{(q;q)_n}\, {}_1\phi_1 \left(\begin{matrix} q^{-n} \\ aq \end{matrix} \middle| q, -aq^{n+1}x \right), \tag{5.1}$$

we have used slightly different notations (see [14], (18.27.15)). They satisfy the recurrence relations

$$-aq^{2n+1}xL_n(x,a;q) = \left(1 - q^{n+1}\right)L_{n+1}(x,a;q) - \left[\left(1 - q^{n+1}\right)\right.$$
$$\left. + q\left(1 - aq^n\right)\right]L_n(x,a;q) + q\left(1 - aq^n\right)L_{n-1}(x,a;q). \tag{5.2}$$

There is a q-difference equation of the form

$$(L_{a,q}y)(x) = -a\left(1 - q^n\right)y(x), \tag{5.3}$$

where

$$(L_{a,q}y)(x) := a\left(1 + x^{-1}\right)y(qx) - \left[x^{-1} + a\left(1 + x^{-1}\right)\right]y(x) + x^{-1}y\left(q^{-1}x\right). \tag{5.4}$$

When $q \to 1$ ($a = q^\alpha$) the q-Laguerre polynomial $L_n(x,a;q)$ becomes the ordinary Laguerre polynomial

$$L_n^\alpha(x) := \frac{(\alpha + 1)_n}{n!}\, {}_1F_1 \left(\begin{matrix} -n \\ \alpha + 1 \end{matrix} \middle| x \right). \tag{5.5}$$

There is a limit transition from little q-Jacobi to q-Laguerre (see [22], Section 4.12.2),

$$L_n(x,a;q) = \lim_{b\to\infty} \frac{(qa;q)_\infty}{(q;q)_\infty} p_n\left(-\frac{x}{qb}, a, b \middle| q\right). \tag{5.6}$$

Starting with the operators X, Y, and Z given by (2.9) we can also obtain the following operators:

$$(Xf)(x) := -q \lim_{b\to\infty} b(Xf)(-x/qb),$$

$$(Yf)(x) := \sqrt{a} \lim_{b\to\infty} \frac{1}{\sqrt{b}}(Yf)(-x/qb),$$

$$(Zf)(x) := -q \lim_{b\to\infty} \sqrt{ab}(Zf)(-x/qb).$$

Then

$$(Xf)(x) := xf(x),$$
$$(Yf)(x) := \frac{q}{q^2 - 1}(L_{a,q} + a)f(x), \tag{5.7}$$
$$(Zf)(x) := -f\left(q^{-1}x\right),$$

where the operator $L_{a,q}$ is defined in (5.4). A simple computation shows that the operators X, Y, Z satisfy the relations

$$YX - qXY = Z + \omega_3, \qquad ZY - qYZ = \omega_2, \qquad XZ - qZX = 0, \tag{5.8}$$

where

$$\omega_2 = \frac{aq}{1+q}, \qquad \omega_3 = \frac{q(1+a)}{1+q}. \tag{5.9}$$

The Casimir operator

$$Q = (q^2 - 1)YXZ + Z^2 + (q+1)(\omega_2 qX + \omega_3 Z) \tag{5.10}$$

takes the value $Q = -aq^2$.

5.2 Second Jackson's *q*-Bessel function
The second Jackson's q-Bessel function is defined as follows:

$$J_\nu^{(2)}(x;q) := \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty}\left(\frac{1}{2}x\right)^\nu {}_0\Phi_1\left({0 \atop q^{\nu+1}}\Big|q;-\frac{1}{4}q^{\nu+1}x^2\right) \quad (x > 0).$$

This notation is from [5] and is deferent from Jackson's notation [2–4]. The classical Bessel function J_ν is recovered by letting $q \to 1$ in $J_\nu^{(2)}(x;q)$. Similarly to (4.1), we defined the second normalized q-Bessel function $\mathcal{J}_2(x;a;q)$ by

$$\mathcal{J}_2(x;a;q) = {}_0\Phi_1\left({0 \atop qa}\Big|q;-qax\right). \tag{5.11}$$

There is a well-known limit from q-Laguerre [22] to the second normalized q-Bessel function as $n \to \infty$

$$\mathcal{J}_2(x;a;q) = \lim_{n\to\infty} L_n(x,a;q). \tag{5.12}$$

From (5.12) and (5.3) is not difficult to establish the q-difference equation for $\mathcal{J}_2(\lambda x,a;q)$

$$(Y_{a,q,2}y)(x) = -a\lambda y(x), \tag{5.13}$$

where

$$(Y_{a,q,2}y)(x) = \frac{aq}{x}y(x) - q\frac{a+1}{x}y(q^{-1}x) + \frac{q}{x}y(q^{-2}x). \tag{5.14}$$

Furthermore, the q-Bessel operator $Y_{a,q,2}$ is related to the q-Laguerre operator $L_{a,q}$ defined in (5.4) by

$$(L_{a,q} + a)f(q^{-1}x) = (Y_{a,q,2} + a)f(x). \tag{5.15}$$

This allows us to construct a Askey-Wilson algebra type that has the q-Bessel operator $Y_{a,q,2}$ as one of its three generators. A straightforward computation shows that the operators X, Y, Z given by

$$(Xf)(x) := xf(x),$$

$$(Yf)(x) := \frac{q}{q^2 - 1}(Y_{a,q,2} + a)f(x),$$ (5.16)

$$(Zf)(x) := -f(q^{-1}x),$$

satisfy the relations

$$YX - q^2 XY = \mu_1 Z + \mu_2 X + \mu_3, \qquad ZY - qYZ = \mu_4 Z, \qquad XZ - qZX = 0,$$ (5.17)

where

$$\mu_1 = -q\frac{1+a}{(1+q)}, \qquad \mu_2 = -aq, \qquad \mu_3 = -aq^2, \qquad \mu_4 = -\frac{aq}{(1+q)}.$$ (5.18)

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