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On a discrete risk model with delayed claims and a randomized dividend strategy

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Abstract
In this paper, we consider a discrete risk model with delayed claims and randomized dividend strategy. The expected discounted dividends before ruin are studied. Difference equations for the expected discounted dividends are derived and solved.

Keywords: randomized dividend strategy; expected discounted dividends; difference equations; delayed claims

1 Introduction

In the compound binomial model, the surplus process for an insurance company is described as follows:

$$U_t = u + t - [X_1\xi_1 + X_2\xi_2 + \cdots + X_t\xi_t], \quad t = 1, 2, 3, \ldots,$$

where u is a nonnegative integer denoting the initial surplus, $\{X_t\}_{t=1}^{\infty}$ is a sequence of i.i.d. random variables denoting the individual claim sizes. Let X denote the generic version of X_t's and define their common probability function by

$$f_k = \Pr(X = k), \quad k = 1, 2, \ldots.$$

The Bernoulli sequence $\{\xi_t\}_{t=1}^{\infty}$ is used to denote claim occurrence such that $\xi_t = 1$ if a claim occurs in the time period $(t-1, t]$, and $\xi_t = 0$ if no claim occurs in the time period $(t-1, t]$. It is assumed that

$$\Pr(\xi_1 = 1) = q, \qquad \Pr(\xi_1 = 0) = 1 - q,$$

where $0 < q < 1$.

The compound binomial risk model has been studied by many authors, for example, Gerber [1], Shiu [2], Willmot [3] and Dickson [4]. Recently, some extensions have been made on this model. Yang *et al.* [5] study the ruin probabilities in a discrete Markov risk model. Yang and Zhang [6] consider a discrete renewal risk model with two-sided jumps. Gerber *et al.* [7] modify the compound binomial risk model by dividend payments. Chen *et al.* [8] study the survival probabilities in a discrete semi-Markov risk model.

In reality, insurance claims may be delayed due to various reasons. The compound binomial risk model can be extended by involving two types of insurance claims, namely the main claims and the by-claims. We use $\{X_t\}$ and $\{\xi_t\}$ to denote the main claim sizes and the indicators for their occurrences, respectively. We assume that each main claim induces a by-claim. The by-claim and its associated main claim may occur simultaneously with probability p ($0 < p < 1$), or the occurrence of the by-claim may be delayed to the next time period with probability $1 - p$. Let $\{Y_t\}_{t=1}^{\infty}$ be an i.i.d. sequence to denote by-claim sizes and let Y be a generic variable of the by-claim. Define the probability mass function of Y by

$$g_l = \Pr(Y_1 = l), \quad l = 1, 2, \ldots.$$

Let S_t^X and S_t^Y be, respectively, the total main claims and by-claims up to time t, where the superscripts X and Y are used to indicate main claim and by-claim, respectively. Then the delayed risk model $U^{\infty} = \{U_t^{\infty}\}_{t=0}^{\infty}$ can be described as follows: $U_0^{\infty} = u$ and for $t = 1, 2, \ldots,$

$$U_t^{\infty} = u + t - S_t^X - S_t^Y.$$

For the study on risk models with delayed claims, we refer the interested readers to Yuen and Guo [9], Yuen *et al.* [10] and Xiao and Guo [11].

Recently, risk models with randomized dividend strategy have received a lot of attention in the literature. Albrecher *et al.* [12] study the expected discounted dividends in the compound Poisson model with randomized dividend decision times. Avanzi *et al.* [13] consider a periodic dividend strategy in the dual model. Zhang [14] considers a perturbed compound Poisson risk model with a randomized dividend strategy. Zhang and Cheung [15] investigate the randomized dividend strategy in a Markov additive risk model. For the discrete risk model, Tan and Yang [16] propose a randomized dividend strategy by modifying the compound binomial model. In their model, whenever the surplus process is larger or equal to a barrier b (a positive integer), the company will possibly pay dividends at the end of the next period. He and Yang [17] consider a compound binomial model, where dividends are randomly paid to shareholders and policyholders. In this paper, we employ a randomized dividend strategy to modify the delayed risk model U^{∞}, and denote the modified model by $U^b = \{U_t^b\}_{t=1}^{\infty}$. As in Tan and Yang [16], for $t = 0, 1, \ldots,$ we assume that whenever $U_t^b \geq b$, a dividend of size η_{t+1} is possibly paid at the beginning of the $(t+1)$th period $(t, t+1]$, where $\{\eta_t\}_{t=1}^{\infty}$ is a Bernoulli sequence such that

$$\Pr(\eta_t = 1) = \theta, \qquad \Pr(\eta_t = 0) = 1 - \theta, \quad 0 < \theta < 1.$$

Now the total dividends paid up to time t can be expressed as

$$Z_t = \sum_{j=1}^{t} \eta_j \mathbf{1}_{(U_{j-1}^b \geq b)}, \quad t = 1, 2, \ldots.$$

Starting from the initial surplus $U_0^b = u$, we have for $t = 1, 2, \ldots,$

$$U_t^b = u + t - S_t^X - S_t^Y - Z_t.$$

Associated with the model U^b, we define the ruin time by

$$\tau^b = \inf\{t \geq 1 : U_t^b < 0\},$$

where $\tau^b = \infty$ if $U_t^b \geq 0$ for all t. The total discounted dividends paid off before ruin are given by

$$D = \sum_{t=1}^{\tau^b} v^{t-1} \eta_t \mathbf{1}_{(U_{t-1}^b \geq b)},$$

where $0 < v < 1$ is a discount factor. Given the initial surplus u, we define

$$V(u;b) = E\big[D | U_0^b = u\big]$$

as the expected present value of discounted dividends paid off prior to ruin.

2 Difference equations

In this section, we derive difference equations for the expected discounted dividends paid before ruin. First, we introduce an auxiliary process $\bar{U}_t^b$ defined as $\bar{U}_0^b = u$ and for $t = 1, 2, \ldots,$

$$\bar{U}_t^b = u + t - \bar{Y} - S_t^X - S_t^Y - \bar{Z}_t,$$

where $\bar{Y}$ independent of other random variables is distributed like Y, and

$$\bar{Z}_t = \sum_{j=1}^{t} \eta_j \mathbf{1}_{(\bar{U}_{j-1}^b \geq b)}, \quad t = 1, 2, \ldots.$$

Accordingly, we define the ruin time by

$$\bar{\tau}^b = \inf\big\{t \geq 1 : \bar{U}_t^b < 0\big\}$$

with $\bar{\tau}^b = \infty$ if $\bar{U}_t^b \geq 0$ for all t. For the risk model $\bar{U}^b$, the discounted dividends paid before ruin are given by

$$\bar{D} = \sum_{t=1}^{\bar{\tau}^b} v^{t-1} \eta_t \mathbf{1}_{(\bar{U}_{t-1}^b \geq b)}.$$

Define the expected present value of discounted dividends paid before ruin by

$$\bar{V}(u;b) = E\big[\bar{D} | \bar{U}_0^b = u\big].$$

For the surplus process U^b, consider the following situations:

(1) no claim occurs in $(0,1]$ and no dividend is paid in $(0,1]$;

(2) no claim occurs in $(0,1]$ and a dividend of 1 is paid in $(0,1]$ (if $u < b$, this case does not exist);

(3) a main claim and its by-claim occur simultaneously in $(0,1]$, and no dividend is paid in $(0,1]$;

(4) a main claim and its by-claim occur simultaneously in $(0,1]$, and a dividend of 1 is paid in $(0,1]$ (if $u < b$, this case does not exist);

(5) a main claim occurs in $(0,1]$ and its by-claim is delayed to the next period, and no dividend is paid in $(0,1]$;

(6) a main claim occurs in $(0,1]$ and its by-claim is delayed to the next period, and a dividend of 1 is paid in $(0,1]$ (if $u < b$, this case does not exist).

Note that in situations (1)-(4), the surplus process U^b will regenerate itself after the first period; whereas in (5)-(6), U^b will switch to $\bar{U}^b$. For $0 \leq u < b$, no dividends will be paid in the first time period, then we have

$$\begin{aligned} V(u;b) &= v(1-q)V(u+1;b) + vq(1-p)\sum_{k\leq u+1} f_k \bar{V}(u+1-k;b) \\ &\quad + vqp\sum_{k+l\leq u+1} f_k g_l V(u+1-k-l;b), \end{aligned} \tag{2.1}$$

where we use the convention $\sum_{x=i}^{j} \cdot = 0$ for $i > j$. Whereas for $u \geq b$, a dividend will be paid at the beginning of the first time period with probability θ, then we have

$$\begin{aligned} V(u;b) &= \theta + v(1-q)(1-\theta)V(u+1;b) + v(1-q)\theta V(u;b) \\ &\quad + vq(1-p)(1-\theta)\sum_{k\leq u+1} f_k \bar{V}(u+1-k;b) \\ &\quad + vq(1-p)\theta\sum_{k\leq u} f_k \bar{V}(u-k;b) \\ &\quad + vqp(1-\theta)\sum_{k+l\leq u+1} f_k g_l V(u+1-k-l;b) \\ &\quad + vqp\theta\sum_{k+l\leq u} f_k g_l V(u-k-l;b). \end{aligned} \tag{2.2}$$

Similarly, for model $\bar{U}^b$, we have for $0 \leq u < b$,

$$\begin{aligned} \bar{V}(u;b) &= v(1-q)\sum_{l\leq u+1} g_l V(u+1-l;b) + vq(1-p)\sum_{k+l\leq u+1} f_k g_l \bar{V}(u+1-k-l;b) \\ &\quad + vqp\sum_{k+l+m\leq u+1} f_k g_l g_m V(u+1-k-l-m;b), \end{aligned} \tag{2.3}$$

and for $u \geq b$,

$$\begin{aligned} \bar{V}(u;b) &= \theta + v(1-q)(1-\theta)\sum_{l\leq u+1} g_l V(u+1-l;b) + v(1-q)\theta\sum_{l\leq u} g_l V(u-l;b) \\ &\quad + vq(1-p)(1-\theta)\sum_{k+l\leq u+1} f_k g_l \bar{V}(u+1-k-l;b) \\ &\quad + vq(1-p)\theta\sum_{k+l\leq u} f_k g_l \bar{V}(u-k-l;b) \\ &\quad + vqp(1-\theta)\sum_{k+l+m\leq u+1} f_k g_l g_m V(u+1-k-l-m;b) \\ &\quad + vqp\theta\sum_{k+l+m\leq u} f_k g_l g_m V(u-k-l-m;b). \end{aligned} \tag{2.4}$$

3 The case $0 \le u < b$

In this section, we consider the case $0 \le u < b$. In order to simplify (2.1) and (2.3), we define the following auxiliary functions:

$$q_{-,11}(x) = \begin{cases} v(1-q), & x = 0, \\ -1, & x = 1, \\ vqp\sum_{k+l=x} f_k g_l, & x = 2,3,\ldots, \end{cases}$$

$$q_{-,12}(x) = \begin{cases} 0, & x = 0, \\ vq(1-p)f_x, & x = 1,2,\ldots, \end{cases}$$

$$q_{-,21}(x) = \begin{cases} 0, & x = 0, \\ v(1-q)g_x, & x = 1,2, \\ v(1-q)g_x + vqp\sum_{k+l+m=x} f_k g_l g_m, & x = 3,4,\ldots, \end{cases}$$

$$q_{-,22}(x) = \begin{cases} 0, & x = 0, \\ -1, & x = 1, \\ vq(1-p)\sum_{k+l=x} f_k g_l, & x = 2,3,\ldots. \end{cases}$$

It is easily seen that the difference equations (2.1) and (2.3) can be rewritten as follows:

$$\begin{cases} \sum_{x=0}^{u+1} q_{-,11}(x)V(u+1-x;b) + \sum_{x=1}^{u+1} q_{-,12}(x)\bar{V}(u+1-x;b) = 0, \\ \sum_{x=1}^{u+1} q_{-,21}(x)V(u+1-x;b) + \sum_{x=1}^{u+1} q_{-,22}(x)\bar{V}(u+1-x;b) = 0, \end{cases}$$
$$u = 0,1,\ldots,b-1. \tag{3.1}$$

Now we relax the restriction $0 \le u < b$ to $u \ge 0$ in (3.1), and let $(\chi_1(u), \chi_2(u))$ be the corresponding solution, *i.e.*

$$\begin{cases} \sum_{x=0}^{u+1} q_{-,11}(x)\chi_1(u+1-x) + \sum_{x=1}^{u+1} q_{-,12}(x)\chi_2(u+1-x) = 0, \\ \sum_{x=1}^{u+1} q_{-,21}(x)\chi_1(u+1-x) + \sum_{x=1}^{u+1} q_{-,22}(x)\chi_2(u+1-x) = 0, \end{cases}$$
$$u = 0,1,2,\ldots. \tag{3.2}$$

In order to get $(\chi_1(u), \chi_2(u))$, we use the generating function method. In the rest of this paper, we put a hat on top of a function to denote its generating function. For example,

$$\hat{f}(z) = \sum_{k=1}^{\infty} z^k f_k, \qquad \hat{g}(z) = \sum_{l=1}^{\infty} z^l g_l, \quad |z| \le 1.$$

For the convolution

$$f * g(x) = \sum_{k+l=x} f_k g_l,$$

since $f * g(0) = f * g(1) = 0$, its generating function is given by

$$\sum_{x=0}^{\infty} z^x f * g(x) = \sum_{x=2}^{\infty} z^x \sum_{k+l=x} f_k g_l = \sum_{x=2}^{\infty} z^x \sum_{k=1}^{x-1} f_k g_{x-k} = \sum_{k=1}^{\infty} z^k f_k \sum_{x=k+1}^{\infty} z^{x-k} g_{x-k} = \hat{f}(z)\hat{g}(z).$$

It is not hard to see that

$$\hat{q}_{-,11}(z) = v\bigl[1 - q + qp\hat{f}(z)\hat{g}(z)\bigr] - z,$$
$$\hat{q}_{-,12}(z) = vq(1-p)\hat{f}(z),$$
$$\hat{q}_{-,21}(z) = v\hat{g}(z)\bigl[1 - q + qp\hat{f}(z)\hat{g}(z)\bigr],$$
$$\hat{q}_{-,22}(z) = vq(1-p)\hat{f}(z)\hat{g}(z) - z.$$

For example,

$$\begin{aligned}\hat{q}_{-,22}(z) &= \sum_{x=0}^{\infty} z^x q_{-,22}(x) = -z + vq(1-p)\sum_{x=2}^{\infty}\sum_{k+l=x} f_k g_l \\ &= vq(1-p)\hat{f}(z)\hat{g}(z) - z.\end{aligned}$$

For $0 < |z| < 1$, multiplying the first equation in (3.2) by z^u and then summing over u from 0 to ∞, we obtain

$$\begin{aligned}0 &= q_{-,11}(0)\sum_{u=0}^{\infty} z^u \chi_1(u+1) + \sum_{u=0}^{\infty} z^u \sum_{x=1}^{u+1} q_{-,11}(x)\chi_1(u+1-x) \\ &\quad + \sum_{u=0}^{\infty} z^u \sum_{x=1}^{u+1} q_{-,12}(x)\chi_2(u+1-x) \\ &= q_{-,11}(0)\frac{1}{z}\sum_{u=0}^{\infty} z^{u+1}\chi_1(u+1) + \frac{1}{z}\sum_{x=1}^{\infty} z^x q_{-,11}(x)\sum_{u+1=x}^{\infty} z^{u+1-x}\chi_1(u+1-x) \\ &\quad + \frac{1}{z}\sum_{x=1}^{\infty} z^x q_{-,12}(x)\sum_{u+1=x}^{\infty} z^{u+1-x}\chi_2(u+1-x) \\ &= \frac{1}{z}q_{-,11}(0)\bigl[\hat{\chi}_1(z) - \chi_1(0)\bigr] + \frac{1}{z}\bigl[\hat{q}_{-,11}(z) - q_{-,11}(0)\bigr]\hat{\chi}_1(z) + \frac{1}{z}\hat{q}_{-,12}(z)\hat{\chi}_2(z),\end{aligned}$$

which leads to

$$\hat{q}_{-,11}(z)\hat{\chi}_1(z) + \hat{q}_{-,12}(z)\hat{\chi}_2(z) = q_{-,11}(0)\chi_1(0). \tag{3.3}$$

Similarly, from the second equation in (3.2) we can obtain

$$\hat{q}_{-,21}(z)\hat{\chi}_1(z) + \hat{q}_{-,22}(z)\hat{\chi}_2(z) = 0. \tag{3.4}$$

Immediately, solving (3.3) and (3.4) gives

$$\hat{\chi}_1(z) = \frac{q_{-,11}(0)\hat{q}_{-,22}(z)\chi_1(0)}{z^2 - zv[1 - q + q\hat{f}(z)\hat{g}(z)]}, \qquad \hat{\chi}_2(z) = \frac{-q_{-,11}(0)\hat{q}_{-,21}(z)\chi_1(0)}{z^2 - zv[1 - q + q\hat{f}(z)\hat{g}(z)]},$$

where we have used the fact

$$\hat{q}_{-,11}(z)\hat{q}_{-,22}(z) - \hat{q}_{-,12}(z)\hat{q}_{-,21}(z) = z^2 - zv\bigl[1 - q + q\hat{f}(z)\hat{g}(z)\bigr].$$

Hence, we conclude that the solution to the difference system (3.2) is uniquely determined by the initial value $\chi_1(0)$, from which we know that the solution to (3.1) can be expressed as follows:

$$V(u;b) = \alpha h_{-,1}(u), \qquad \bar{V}(u;b) = \alpha h_{-,2}(u), \quad u = 0,1,\ldots,b-1, \tag{3.5}$$

where α is an unknown constant, $h_{-,1}(u)$ and $h_{-,2}(u)$ are determined by the generating functions

$$\hat{h}_{-,k}(z) = \frac{\hat{w}_{-,k}(z)}{z - \nu[1 - q + q\hat{f}(z)\hat{g}(z)]}, \quad k = 1,2, \tag{3.6}$$

with

$$\hat{w}_{-,1}(z) = \nu(1-q)\hat{q}_{-,22}(z)/z, \qquad \hat{w}_{-,2}(z) = -\nu(1-q)\hat{q}_{-,21}(z)/z.$$

Note that $\hat{w}_{-,1}(z)$, $\hat{w}_{-,2}(z)$ are both analytic inside the unit circle. In fact, since $q_{-,21}(0) = q_{-,22}(0) = 0$, we have

$$\hat{w}_{-,1}(z) = \nu(1-q)\sum_{x=1}^{\infty} z^{x-1}q_{-,22}(x) = \sum_{x=0}^{\infty} z^x \nu(1-q)q_{-,22}(x+1),$$
$$\hat{w}_{-,2}(z) = -\nu(1-q)\sum_{x=1}^{\infty} z^{x-1}q_{-,21}(x) = -\sum_{x=0}^{\infty} z^x \nu(1-q)q_{-,21}(x+1).$$

Hence, upon inverting the above generating functions we obtain

$$w_{-,1}(u) = \nu(1-q)q_{-,22}(u+1), \qquad w_{-,2}(u) = -\nu(1-q)q_{-,21}(u+1), \quad u = 0,1,2,\ldots.$$

To continue, we introduce the discrete Dickson-Hipp operator defined as

$$\mathcal{T}_z f(y) = \sum_{x=y}^{\infty} z^{x-y} f(x) = \sum_{x=0}^{\infty} z^x f(x+y),$$

for some function $f(x)$ defined on $\{0,,1,\ldots\}$. As a matter of fact, $\mathcal{T}_z f(y)$ is the generating function of $f(y+\cdot)$. One of the nice properties of $\mathcal{T}_z$ is the commutative property, *i.e.*

$$\mathcal{T}_s\mathcal{T}_z f(y) = \mathcal{T}_z\mathcal{T}_s f(y) = \frac{s\mathcal{T}_s f(y) - z\mathcal{T}_z f(y)}{s-z}.$$

For more properties on this operator, we refer the interested readers to Li [18].

For $\gamma_-(z) := z - \nu[1 - q + q\hat{f}(z)\hat{g}(z)]$, we have

$$\gamma_-(0) = -\nu(1-q) < 0, \qquad \gamma_-(1) = 1 - \nu > 0,$$

which imply that there is a number $\rho_- \in (0,1)$ such that $\gamma_-(\rho_-) = 0$. Furthermore, note that

$$\hat{f}(z)\hat{g}(z) = \sum_{x=2}^{\infty} z^x f * g(x) = z\sum_{x=2}^{\infty} z^{x-1} f * g(x) = z\mathcal{T}_z(f * g)(1).$$

Then we have

$$\begin{aligned}\gamma_-(z) &= (z-\rho_-)\frac{\gamma(z)-\gamma(\rho_-)}{z-\rho_-} = (z-\rho_-)\left(1 - vq\frac{z\mathcal{T}_z(f*g)(1)-\rho_-\mathcal{T}_{\rho_-}(f*g)(1)}{z-\rho_-}\right)\\ &= (z-\rho_-)\big(1 - vq\mathcal{T}_z\mathcal{T}_{\rho_-}(f*g)(1)\big),\end{aligned}$$

which also yields for $|z| \le 1$,

$$\begin{aligned}\big|vq\mathcal{T}_z\mathcal{T}_{\rho_-}(f*g)(1)\big| &= \left|vq\sum_{x=0}^{\infty} z^x\mathcal{T}_{\rho_-}(f*g)(x+1)\right| \le vq\sum_{x=0}^{\infty}\mathcal{T}_{\rho_-}(f*g)(x+1)\\ &= vq\mathcal{T}_1\mathcal{T}_{\rho_-}(f*g)(1) = 1-\frac{\gamma_-(1)}{1-\rho_-} = 1-\frac{1-v}{1-\rho_-} < 1.\end{aligned}$$

Hence, we conclude that $\phi(x) := vq\mathcal{T}_{\rho_-}(f*g)(x+1)$ is a defective probability function.

Now for $k = 1,2$,

$$\begin{aligned}\hat{h}_{-,k}(z) &= \frac{\hat{w}_{-,k}(z)}{(z-\rho_-)(1-vq\mathcal{T}_z\mathcal{T}_{\rho_-}(f*g)(1))}\\ &= \sum_{j=0}^{\infty}\big[vq\mathcal{T}_z\mathcal{T}_{\rho_-}(f*g)(1)\big]^j\frac{\hat{w}_{-,k}(z)}{z-\rho_-}.\end{aligned}$$

After inverting the generating functions in the above formula, we obtain

$$h_{-,k}(u) = \sum_{j=0}^{\infty}\phi^{*j} * \bar{w}_{-,k}(u), \quad k = 1,2, \tag{3.7}$$

where

$$\bar{w}_{-,k}(u) = -\sum_{x=0}^{u}\rho_-^{-(u-x)-1}w_{-,k}(x), \quad u = 0,1,2,\ldots.$$

The j-fold convolution $\phi^{*j}(x)$ in (3.7) is recursively defined as

$$\phi^{*j}(x) = \sum_{y=0}^{x}\phi^{*(j-1)}(x-y)\phi(y)$$

with the starting point $\phi^{*0}(x) = \mathbf{1}_{(x=0)}$.

4 The case $u \ge b$

In this section, we consider the case $u \ge b$. First, we introduce the following auxiliary functions to simplify (2.2) and (2.4):

$$q_{+,11}(x) = \begin{cases} v(1-q)(1-\theta), & x = 0,\\ -1+v(1-q)\theta, & x = 1,\\ vqp(1-\theta)f_1g_1, & x = 2,\\ vqp[(1-\theta)\sum_{k+l=x}f_kg_l + \theta\sum_{k+l=x-1}f_kg_l], & x = 3,4,\ldots,\end{cases}$$

$$q_{+,12}(x)=\begin{cases}0, & x=0,\\ vq(1-p)(1-\theta)f_1, & x=1,\\ vq(1-p)[(1-\theta)f_x+\theta f_{x-1}], & x=2,3,\ldots,\end{cases}$$

$$q_{+,21}(x)=\begin{cases}0, & x=0,\\ v(1-q)(1-\theta)g_1, & x=1,\\ v(1-q)[(1-\theta)g_2+\theta g_1], & x=2,\\ v(1-q)[(1-\theta)g_3+\theta g_2]+vqp(1-\theta)f_1g_1g_1, & x=3,\\ v(1-q)[(1-\theta)g_x+\theta g_{x-1}] & \\ \quad +vqp[(1-\theta)\sum_{k+l+m=x}f_kg_lg_m+\theta\sum_{k+l+m=x-1}f_kg_lg_m], & x=4,5,\ldots,\end{cases}$$

$$q_{+,22}(x)=\begin{cases}0, & x=0,\\ -1, & x=1,\\ vq(1-p)(1-\theta)f_1g_1, & x=2,\\ vq(1-p)[(1-\theta)\sum_{k+l=x}f_kg_l+\theta\sum_{k+l=x-1}f_kg_l], & x=3,4,\ldots.\end{cases}$$

Immediately, (2.2) and (2.4) are simplified to be

$$\begin{cases}\sum_{x=0}^{u+1}q_{+,11}(x)V(u+1-x;b)+\sum_{x=1}^{u+1}q_{+,12}(x)\bar{V}(u+1-x;b)+\theta=0,\\ \sum_{x=1}^{u+1}q_{+,21}(x)V(u+1-x;b)+\sum_{x=1}^{u+1}q_{+,22}(x)\bar{V}(u+1-x;b)+\theta=0,\end{cases}$$
$$u=b,b+1,\ldots. \tag{4.1}$$

We use generating function method to solve (4.1). By some straightforward calculations, we obtain

$$\begin{aligned}\hat{q}_{+,11}(z)&=\sum_{x=0}^{\infty}z^xq_{+,11}(x)\\ &=v(1-q)(1-\theta)-z+v(1-q)\theta z\\ &\quad+vqp(1-\theta)\sum_{x=2}^{\infty}z^x\sum_{k+l=x}f_kg_l+vqp\theta\sum_{x=3}^{\infty}z^x\sum_{k+l=x-1}f_kg_l\\ &=v(1-q)(1-\theta)-z+v(1-q)\theta z+vqp(1-\theta)\hat{f}(z)\hat{g}(z)+vqp\theta z\hat{f}(z)\hat{g}(z)\\ &=v\bigl[1-q+qp\hat{f}(z)\hat{g}(z)\bigr](1-\theta+\theta z)-z.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\hat{q}_{+,12}(z)&=vq(1-p)\hat{f}(z)(1-\theta+\theta z),\\ \hat{q}_{+,21}(z)&=v\hat{g}(z)\bigl[1-q+qp\hat{f}(z)\hat{g}(z)\bigr](1-\theta+\theta z),\\ \hat{q}_{+,22}(z)&=vq(1-p)\hat{f}(z)\hat{g}(z)(1-\theta+\theta z)-z.\end{aligned}$$

For $0<|z|<1$, we have

$$\begin{aligned}&\sum_{u=b}^{\infty}z^{u-b}\sum_{x=0}^{u+1}q_{+,11}(x)V(u+1-x;b)\\ &\quad=\sum_{u=b}^{\infty}z^{u-b}q_{+,11}(0)V(u+1;b)+\sum_{u=b}^{\infty}z^{u-b}\sum_{x=1}^{u-b+1}q_{+,11}(x)V(u+1-x;b)\end{aligned}$$

$$
\begin{aligned}
&+\sum_{u=b}^{\infty} z^{u-b} \sum_{x=u-b+2}^{u+1} q_{+,11}(x) V(u+1-x; b) \\
&= \frac{1}{z} q_{+,11}(0) \sum_{u=b}^{\infty} z^{u+1-b} V(u+1; b) + \sum_{x=1}^{\infty} z^x q_{+,11}(x) \sum_{u=x+b-1}^{\infty} z^{u-x-b} V(u+1-x; b) \\
&\quad + \sum_{x=0}^{b-1} \sum_{u=b}^{\infty} z^{u-b} q_{+,11}(u+1-x) V(x; b) \\
&= \frac{1}{z} q_{+,11}(0)\left[\mathcal{T}_z V(b; b) - V(b; b)\right] + \frac{1}{z}\left[\hat{q}_{+,11}(z) - q_{+,11}(0)\right]\mathcal{T}_z V(b; b) \\
&\quad + \sum_{x=0}^{b-1} \mathcal{T}_z q_{+,11}(b+1-x) V(x; b) \\
&= -\frac{1}{z} q_{+,11}(0) V(b; b) + \frac{1}{z} \hat{q}_{+,11}(z) \mathcal{T}_z V(b; b) + \sum_{x=0}^{b-1} \mathcal{T}_z q_{+,11}(b+1-x) V(x; b),
\end{aligned}
$$

and similarly

$$
\sum_{u=b}^{\infty} z^{u-b} \sum_{x=1}^{u+1} q_{+,12}(x) \bar{V}(u+1-x; b) = \frac{1}{z} \hat{q}_{+,12}(z) \mathcal{T}_z \bar{V}(b; b) + \sum_{x=0}^{b-1} \mathcal{T}_z q_{+,12}(b+1-x) \bar{V}(x; b).
$$

Now multiplying both sides of the first equation in (4.1) and summing over u from b to ∞, we obtain

$$
\hat{q}_{+,11}(z) \mathcal{T}_z V(b; b) + \hat{q}_{+,12}(z) \mathcal{T}_z \bar{V}(b; b) = q_{+,11}(0) V(b; b) - \alpha \hat{\varphi}_1(z) - \theta z (1-z)^{-1}, \tag{4.2}
$$

where

$$
\hat{\varphi}_1(z) = \sum_{x=0}^{b-1} \mathcal{T}_z q_{+,11}(b+1-x) h_{-,1}(x) + \sum_{x=0}^{b-1} \mathcal{T}_z q_{+,12}(b+1-x) h_{-,2}(x).
$$

Applying exactly the same arguments to the second equation in (4.1) gives

$$
\hat{q}_{+,21}(z) \mathcal{T}_z V(b; b) + \hat{q}_{+,22}(z) \mathcal{T}_z \bar{V}(b; b) = -\alpha \hat{\varphi}_2(z) - \theta z (1-z)^{-1} \tag{4.3}
$$

with

$$
\hat{\varphi}_2(z) = \sum_{x=0}^{b-1} \mathcal{T}_z q_{+,21}(b+1-x) h_{-,1}(x) + \sum_{x=0}^{b-1} \mathcal{T}_z q_{+,22}(b+1-x) h_{-,2}(x).
$$

After inverting the generating functions $\hat{\varphi}_1(z)$, $\hat{\varphi}_2(z)$, we obtain for $u = 0, 1, \dots,$

$$
\begin{aligned}
\varphi_1(u) &= \sum_{x=0}^{b-1} q_{+,11}(u+b+1-x) h_{-,1}(x) + \sum_{x=0}^{b-1} q_{+,12}(u+b+1-x) h_{-,2}(x), \\
\varphi_2(u) &= \sum_{x=0}^{b-1} q_{+,21}(u+b+1-x) h_{-,1}(x) + \sum_{x=0}^{b-1} q_{+,22}(u+b+1-x) h_{-,2}(x).
\end{aligned}
$$

Note that

$$\hat{q}_{+,11}(z)\hat{q}_{+,22}(z) - \hat{q}_{+,12}(z)\hat{q}_{+,21}(z) = z^2 - zv\hat{a}(z),$$

where $\hat{a}(z) = [1 - q + q\hat{f}(z)\hat{g}(z)](1 - \theta + \theta z)$ is a probability generating function with the corresponding probability function given by

$$a(x) = \begin{cases} (1-q)(1-\theta), & x = 0, \\ (1-q)\theta, & x = 1, \\ q(1-\theta)f_1 g_1, & x = 2, \\ q(1-\theta)\sum_{k+l=x} f_k g_l + q\theta \sum_{k+l=x-1} f_k g_l, & x = 3, 4, \ldots. \end{cases}$$

Then solving (4.2) and (4.3) results in

$$\mathcal{T}_z V(b;b) = \frac{q_{+,11}(0)V(b;b)\hat{q}_{+,22}(z)/z + \alpha\hat{q}_{+,12}(z)\hat{\varphi}_2(z) - \alpha\hat{q}_{+,22}(z)\hat{\varphi}_1(z) + \theta(1-z)^{-1}[\hat{q}_{+,12}(z) - \hat{q}_{+,22}(z)]}{z - v\hat{a}(z)}, \tag{4.4}$$

$$\mathcal{T}_z \bar{V}(b;b) = \frac{-q_{+,11}(0)V(b;b)\hat{q}_{+,21}(z)/z + \alpha\hat{q}_{+,21}(z)\hat{\varphi}_1(z) - \alpha\hat{q}_{+,11}(z)\hat{\varphi}_2(z) + \theta(1-z)^{-1}[\hat{q}_{+,21}(z) - \hat{q}_{+,11}(z)]}{z - v\hat{a}(z)}. \tag{4.5}$$

For $\gamma_+(z) := z - v\hat{a}(z)$, we have

$$\gamma_+(0) = -v(1-q)(1-\theta) < 0, \qquad \gamma_+(1) = 1 - v > 0,$$

then there exists a number $\rho_+ \in (0,1)$ such that $\gamma_+(\rho_+) = 0$, which also implies that ρ_+ is the zero point of the common denominator of (4.4) and (4.5). Note that $V(u;b)$ cannot grow with an exponential rate, then we conclude that ρ_+ is also zero point of the numerators of (4.4) and (4.5), and this leads to

$$\begin{aligned} &q_{+,11}(0)V(b;b) \\ &\quad = \alpha\bigl[\rho_+\hat{\varphi}_1(\rho_+) - \rho_+\hat{\varphi}_2(\rho_+)\hat{q}_{+,12}(\rho_+)/\hat{q}_{+,22}(\rho_+)\bigr] + \theta\rho_+(1-\rho_+)^{-1}\bigl[1 - \hat{q}_{+,12}(\rho_+)/\hat{q}_{+,22}(\rho_+)\bigr] \\ &\quad = \alpha\bigl[\rho_+\hat{\varphi}_1(\rho_+) - \rho_+\hat{\varphi}_2(\rho_+)\hat{q}_{+,11}(\rho_+)/\hat{q}_{+,21}(\rho_+)\bigr] + \theta\rho_+(1-\rho_+)^{-1}\bigl[1 - \hat{q}_{+,11}(\rho_+)/\hat{q}_{+,21}(\rho_+)\bigr], \end{aligned}$$

where the second equality holds since $\hat{q}_{+,11}(\rho_+)\hat{q}_{+,22}(\rho_+) - \hat{q}_{+,12}(\rho_+)\hat{q}_{+,21}(\rho_+) = 0$ thanks to $\gamma_+(\rho_+) = 0$. For convenience, set

$$\begin{aligned} k_0 &= \theta\rho_+(1-\rho_+)^{-1}\bigl[1 - \hat{q}_{+,12}(\rho_+)/\hat{q}_{+,22}(\rho_+)\bigr], \\ k_1 &= \rho_+\hat{\varphi}_1(\rho_+) - \rho_+\hat{\varphi}_2(\rho_+)\hat{q}_{+,12}(\rho_+)/\hat{q}_{+,22}(\rho_+). \end{aligned}$$

Then we have

$$q_{+,11}(0)V(b;b) = k_0 + k_1\alpha. \tag{4.6}$$

It follows from (4.6) that the numerators in (4.4) and (4.5) reduce to

$$\begin{aligned}
&q_{+,11}(0)V(b;b)\hat{q}_{+,22}(z)/z + \alpha\hat{q}_{+,12}(z)\hat{\varphi}_2(z) - \alpha\hat{q}_{+,22}(z)\hat{\varphi}_1(z) \\
&\qquad + \theta(1-z)^{-1}\bigl[\hat{q}_{+,12}(z) - \hat{q}_{+,22}(z)\bigr] \\
&\quad = \hat{\zeta}_{10}(z) + \alpha\hat{\zeta}_{11}(z)
\end{aligned} \tag{4.7}$$

and

$$\begin{aligned}
&-q_{+,11}(0)V(b;b)\hat{q}_{+,21}(z)/z + \alpha\hat{q}_{+,21}(z)\hat{\varphi}_1(z) - \alpha\hat{q}_{+,11}(z)\hat{\varphi}_2(z) \\
&\qquad + \theta(1-z)^{-1}\bigl[\hat{q}_{+,21}(z) - \hat{q}_{+,11}(z)\bigr] \\
&\quad = \hat{\zeta}_{20}(z) + \alpha\hat{\zeta}_{21}(z),
\end{aligned} \tag{4.8}$$

where

$$\begin{aligned}
\hat{\zeta}_{10}(z) &= k_0\hat{q}_{+,22}(z)/z + \theta(1-z)^{-1}\bigl[\hat{q}_{+,12}(z) - \hat{q}_{+,22}(z)\bigr], \\
\hat{\zeta}_{11}(z) &= k_1\hat{q}_{+,22}(z)/z + \hat{q}_{+,12}(z)\hat{\varphi}_2(z) - \hat{q}_{+,22}(z)\hat{\varphi}_1(z), \\
\hat{\zeta}_{20}(z) &= -k_0\hat{q}_{+,21}(z)/z + \theta(1-z)^{-1}\bigl[\hat{q}_{+,21}(z) - \hat{q}_{+,11}(z)\bigr], \\
\hat{\zeta}_{21}(z) &= -k_1\hat{q}_{+,21}(z)/z + \hat{q}_{+,21}(z)\hat{\varphi}_1(z) - \hat{q}_{+,11}(z)\hat{\varphi}_2(z).
\end{aligned}$$

After inverting the generating functions in the above formulas, we obtain for $x = 0, 1, 2 \ldots,$

$$\begin{aligned}
\zeta_{10}(x) &= k_0 q_{+,22}(x+1) + \theta\sum_{y=0}^{x}\bigl[q_{+,12}(y) - q_{+,22}(y)\bigr], \\
\zeta_{11}(x) &= k_1 q_{+,22}(x+1) + q_{+,12} * \varphi_2(x) - q_{+,22}(z)\varphi_1(x), \\
\zeta_{20}(x) &= -k_0 q_{+,21}(x+1) + \theta\sum_{y=0}^{x}\bigl[q_{+,21}(y) - q_{+,11}(y)\bigr], \\
\zeta_{21}(x) &= -k_1\hat{q}_{+,21}(x+1) + q_{+,21} * \varphi_1(x) - q_{+,11} * \varphi_2(x).
\end{aligned}$$

Furthermore, using $\hat{\zeta}_{j0}(\rho_+) + \alpha\hat{\zeta}_{j1}(\rho_+) = 0$, $j = 1, 2$, we have

$$\begin{aligned}
\hat{\zeta}_{j0}(z) + \alpha\hat{\zeta}_{j1}(z) &= \hat{\zeta}_{j0}(z) + \alpha\hat{\zeta}_{j1}(z) - \bigl[\hat{\zeta}_{j0}(\rho_+) + \alpha\hat{\zeta}_{j1}(\rho_+)\bigr] \\
&= \sum_{x=0}^{\infty} z^x\bigl[\zeta_{j0}(x) + \alpha\zeta_{j1}(x)\bigr] - \sum_{x=0}^{\infty}\rho_+^x\bigl[\zeta_{j0}(x) + \alpha\zeta_{j1}(x)\bigr] \\
&= z\sum_{x=1}^{\infty} z^{x-1}\bigl[\zeta_{j0}(x) + \alpha\zeta_{j1}(x)\bigr] - \rho_+\sum_{x=1}^{\infty}\rho_+^{x-1}\bigl[\zeta_{j0}(x) + \alpha\zeta_{j1}(x)\bigr] \\
&= z\mathcal{T}_z\zeta_{j0}(1) - \rho_+\mathcal{T}_{\rho_+}\zeta j1(1) + \alpha\bigl[z\mathcal{T}_z\zeta_{j1}(1) - \rho_+\mathcal{T}_{\rho_+}\zeta_{j1}(1)\bigr] \\
&= (z-\rho_+)\bigl[\mathcal{T}_z\mathcal{T}_{\rho_+}\zeta_{j0}(1) + \alpha\mathcal{T}_z\mathcal{T}_{\rho_+}\zeta_{j1}(1)\bigr].
\end{aligned} \tag{4.9}$$

Similarly, for the common denominator in (4.4) and (4.5), using $\gamma_+(\rho_+) = 0$ we obtain

$$z - v\hat{a}(z) = z - \rho_+ - v\bigl[\hat{a}(z) - \hat{a}(\rho_+)\bigr] = (z-\rho_+)\bigl[1 - v\mathcal{T}_z\mathcal{T}_{\rho+}a(1)\bigr], \tag{4.10}$$

which also gives for $|z| \le 1$,

$$\begin{aligned}\left|\nu \mathcal{T}_z \mathcal{T}_{\rho+} a(1)\right| &= \left|\nu \sum_{x=0}^{\infty} z^x \mathcal{T}_{\rho_+} a(x+1)\right| \le \left|\nu \sum_{x=0}^{\infty} \mathcal{T}_{\rho_+} a(x+1)\right| \\ &= \left|\nu \mathcal{T}_1 \mathcal{T}_{\rho+} a(1)\right| = 1 - \frac{1-\nu}{1-\rho_+} < 1.\end{aligned}$$

Hence, $\beta(x) := \nu \mathcal{T}_{\rho_+} a(x+1)$ is a defective probability function.

Now plugging (4.9) and (4.10) into (4.4) gives

$$\begin{aligned}\mathcal{T}_z V(b;b) &= \frac{\mathcal{T}_z \mathcal{T}_{\rho_+} \zeta_{j0}(1) + \alpha \mathcal{T}_z \mathcal{T}_{\rho_+} \zeta_{j1}(1)}{1 - \nu \mathcal{T}_z \mathcal{T}_{\rho+} a(1)} \\ &= \sum_{j=0}^{\infty} \left[\nu \mathcal{T}_z \mathcal{T}_{\rho+} a(1)\right]^j \left[\mathcal{T}_z \mathcal{T}_{\rho_+} \zeta_{10}(1) + \alpha \mathcal{T}_z \mathcal{T}_{\rho_+} \zeta_{11}(1)\right],\end{aligned} \tag{4.11}$$

upon inversion, which yields

$$V(b+u;b) = \psi_{10}(u) + \alpha \psi_{11}(u), \quad u = 0,1,\ldots, \tag{4.12}$$

where

$$\psi_{10}(u) = \sum_{j=0}^{\infty} \sum_{x=0}^{u} \beta^{*j}(u-x) \mathcal{T}_{\rho_+} \zeta_{10}(x), \qquad \psi_{11}(u) = \sum_{j=0}^{\infty} \sum_{x=0}^{u} \beta^{*j}(u-x) \mathcal{T}_{\rho_+} \zeta_{11}(x).$$

Similarly, from (4.5) we can obtain

$$\bar{V}(b+u;b) = \psi_{20}(u) + \alpha \psi_{21}(u), \quad u = 0,1,\ldots, \tag{4.13}$$

where

$$\psi_{20}(u) = \sum_{j=0}^{\infty} \sum_{x=0}^{u} \beta^{*j}(u-x) \mathcal{T}_{\rho_+} \zeta_{20}(x), \qquad \psi_{21}(u) = \sum_{j=0}^{\infty} \sum_{x=0}^{u} \beta^{*j}(u-x) \mathcal{T}_{\rho_+} \zeta_{21}(x),$$

where the j-fold convolution β^{*j} is defined as in ϕ^{*j}.

It remains to determine the unknown constant α. To this end, we set $u = b-1$ in the first equation in (3.1) to obtain

$$\sum_{x=0}^{b} q_{-,11}(b-x) V(x;b) + \sum_{x=0}^{b-1} q_{-,12}(b-x) \bar{V}(x;b) = 0.$$

Then plugging (3.5) and (4.12) into the above equation gives

$$\alpha = -\frac{q_{-,11}(0)\psi_{10}(0)}{q_{-,11}(0)\psi_{11}(0) + \sum_{x=0}^{b-1} q_{-,11}(b-x) h_{-,1}(x) + \sum_{x=0}^{b-1} q_{-,12}(b-x) h_{-,2}(x)}. \tag{4.14}$$

Finally, we summarize the main results in the following theorem.

Theorem 1 *The expected present values of dividends $V(u;b)$, $\bar{V}(u;b)$ can be expressed as follows*:

$$V(u;b) = \begin{cases} \alpha h_{-,1}(u), & u = 0,1,\ldots,b-1, \\ \psi_{10}(u-b) + \alpha\psi_{11}(u-b), & u = b,b+1,\ldots, \end{cases}$$

$$\bar{V}(u;b) = \begin{cases} \alpha h_{-,2}(u), & u = 0,1,\ldots,b-1, \\ \psi_{20}(u-b) + \alpha\psi_{21}(u-b), & u = b,b+1,\ldots, \end{cases}$$

where α is given by (4.14).

5 Conclusion

Dividend problems are hot topics in insurance risk theory. In this paper, we consider a compound binomial model with delayed claims. Suppose that the insurance company will possibly pay dividends when the surplus level is larger than a given barrier b. The expected present values of dividends paid before ruin are studied. We derive systems of difference equations for $V(u;b)$ and $\bar{V}(u;b)$, and get the solutions by generating function method. The main results given in Theorem 1 show that the analytic expressions for $V(u;b)$ and $\bar{V}(u;b)$ can be obtained.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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