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Invariance of Hyers-Ulam stability of linear differential equations and its applications

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Abstract
We prove that the generalized Hyers-Ulam stability of linear differential equations of nth order (defined on I) is invariant under any monotone one-to-one correspondence $\tau : I \to J$ which is n times continuously differentiable. Moreover, using this result, we investigate the generalized Hyers-Ulam stability of the linear differential equation of second order and the Cauchy-Euler equation.

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1 Introduction

Throughout this paper, let n be a positive integer and let I and J be non-degenerate intervals of $\mathbf{R}$. We will consider the (linear) differential equation of nth order

$$\mathcal{F}\bigl(y^{(n)}, y^{(n-1)}, \ldots, y', y, x\bigr) = 0 \tag{1}$$

defined on I, where $y : I \to \mathbf{R}$ is an n times continuously differentiable function.

For an arbitrary $\varepsilon > 0$, assume that an n times continuously differentiable function $y : I \to \mathbf{R}$ satisfies the differential inequality

$$\bigl|\mathcal{F}\bigl(y^{(n)}, y^{(n-1)}, \ldots, y', y, x\bigr)\bigr| \le \varepsilon \tag{2}$$

for all $x \in I$. If for each function $y : I \to \mathbf{R}$ satisfying the inequality (2), there exists a solution $y_0 : I \to \mathbf{R}$ of the differential equation (1) such that

$$\bigl|y(x) - y_0(x)\bigr| \le K(\varepsilon) \tag{3}$$

for any $x \in I$, where $K(\varepsilon)$ depends on ε only and satisfies $\lim_{\varepsilon\to 0} K(\varepsilon) = 0$, then we say that the differential equation (1) satisfies (or has) the Hyers-Ulam stability (or the local Hyers-Ulam stability if the domain I is not the whole space $\mathbf{R}$). When the above statement also holds even though we replace ε and $K(\varepsilon)$ with some appropriate $\varphi(x)$ and $\Phi(x)$, respectively, then we say that the differential equation (1) satisfies the generalized Hyers-Ulam stability (or the Hyers-Ulam-Rassias stability). For a more detailed definition of the Hyers-Ulam stability, refer to [1–4].

Obłoza seems to be the first author who investigated the Hyers-Ulam stability of linear differential equations (see [5, 6]): Given real-valued constants a and b, let $g, r : (a,b) \to \mathbf{R}$ be continuous functions with $\int_a^b |g(x)|\,dx < \infty$. Assume that $\varepsilon > 0$ is an arbitrary real number. Obłoza proved that if a differentiable function $y : (a,b) \to \mathbf{R}$ satisfies the inequality $|y'(x) + g(x)y(x) - r(x)| \le \varepsilon$ for all $x \in (a,b)$ and if a function $y_0 : (a,b) \to \mathbf{R}$ satisfies $y_0'(x) + g(x)y_0(x) = r(x)$ for all $x \in (a,b)$ and $y(\tau) = y_0(\tau)$ for some $\tau \in (a,b)$, then there exists a constant $\delta > 0$ such that $|y(x) - y_0(x)| \le \delta$ for all $x \in (a,b)$.

Thereafter, Alsina and Ger [7] proved that if a differentiable function $y : (a,b) \to \mathbf{R}$ satisfies the differential inequality $|y'(x) - y(x)| \le \varepsilon$, then there exists a function $y_0 : (a,b) \to \mathbf{R}$ such that $y_0'(x) = y_0(x)$ and $|y(x) - y_0(x)| \le 3\varepsilon$ for all $x \in (a,b)$. This result of Alsina and Ger was generalized by Takahasi *et al.* [8]. Indeed, they proved the Hyers-Ulam stability of the Banach space valued differential equation $y'(x) = \lambda y(x)$ (see also [9–19]).

Assume that there exists a monotone one-to-one correspondence $\tau : I \to J$, which is n times continuously differentiable. Let $\sigma : J \to I$ be the inverse of τ. If we make a change of variable $t = \tau(x)$ and define an m times continuously differentiable function $z : J \to \mathbf{R}$ by $z(t) = y(\sigma(t))$, where m is an appropriate positive integer (possibly $m = n$), then we can substitute $x = \sigma(t)$, $y(x) = z(t)$, and

$$y^{(k)}(x) = \sum_{i=1}^{k} a_{k,i} z^{(i)}(t) \prod_{j=1}^{k} \tau^{(j)}(x)^{b_{k,j}}$$

in (1) for each $k \in \{1, 2, \ldots, n\}$, where $a_{k,i} \in \mathbf{N}_0$ and $b_{k,j} \in \{0, 1, \ldots, k\}$, to reduce the linear differential equation (1) to another equation of the form

$$\mathcal{G}\bigl(z^{(m)}, z^{(m-1)}, \ldots, z', z, t\bigr) = 0 \tag{4}$$

defined on J. For this case, an n times continuously differentiable function $y : I \to \mathbf{R}$ is a solution of the differential equation (1) if and only if the function $z : J \to \mathbf{R}$ is a solution of the differential equation (4).

The main goal of this paper is to prove that the (generalized) Hyers-Ulam stability of the linear differential equations is invariant under any monotone one-to-one correspondence which is n times continuously differentiable. In other words, if the differential equation (1) has the (generalized) Hyers-Ulam stability, then the reduced differential equation (4) also has the (generalized) Hyers-Ulam stability, and *vice versa*.

Moreover, we investigate the generalized Hyers-Ulam stability of the linear differential equation of second order and the Cauchy-Euler equation.

2 Hyers-Ulam stability is invariant

In the following main theorem, we prove that the (generalized) Hyers-Ulam stability of the linear differential equation of nth order is invariant.

Theorem 2.1 *Assume that the linear differential equation* (1) *defined on I can be reduced to another differential equation* (4) *defined on J via a monotone one-to-one correspondence $\tau : I \to J$ which is n times continuously differentiable. If the differential equation* (1) *has the* (*generalized*) *Hyers-Ulam stability, so does the reduced differential equation* (4).

Proof If the differential equation (1) has the Hyers-Ulam stability and if an n times continuously differentiable function $y : I \to \mathbf{R}$ satisfies the inequality (2) for all $x \in I$ and for some $\varepsilon > 0$, then there exists a solution $y_0 : I \to \mathbf{R}$ of the differential equation (1) such that the inequality (3) holds for any $x \in I$, where $K(\varepsilon)$ depends on ε only and satisfies $\lim_{\varepsilon \to 0} K(\varepsilon) = 0$.

Since the differential equation (4) can be reduced from (1) by using a monotone one-to-one correspondence $\tau : I \to J$ and there exists the inverse $\sigma : J \to I$ of τ, if we define a function $z : J \to \mathbf{R}$ by $z(t) = y(\sigma(t))$, then we can reduce the inequality (2) to a new inequality

$$\left|\mathcal{G}\left(z^{(m)}, z^{(m-1)}, \ldots, z', z, t\right)\right| \leq \varepsilon \tag{5}$$

for all $t \in J$. Moreover, if we set $z_0(t) = y_0(\sigma(t))$, then the inequality (3) is reduced to

$$\left|z(t) - z_0(t)\right| \leq K(\varepsilon) \tag{6}$$

for all $t \in J$.

Finally, it is obvious that z_0 is a solution of the differential equation (4) by considering the last part of the Introduction.

To prove this theorem for the case of generalized Hyers-Ulam stability, we consider the inequalities

$$\left|\mathcal{F}\left(y^{(n)}, y^{(n-1)}, \ldots, y', y, x\right)\right| \leq \varphi(x)$$

and

$$\left|y(x) - y_0(x)\right| \leq \Phi(x)$$

instead of (2) and (3), respectively, where $\varphi, \Phi : I \to [0, \infty)$ are continuous functions. Then the inequalities (5) and (6) are replaced by

$$\left|\mathcal{G}\left(z^{(m)}, z^{(m-1)}, \ldots, z', z, t\right)\right| \leq \psi(t)$$

and

$$\left|z(t) - z_0(t)\right| \leq \Psi(t),$$

respectively, where $\psi := \varphi \circ \sigma$ and $\Psi := \Phi \circ \sigma$.

The rest of the proof runs analogously to the first part of this proof. □

By exchanging the roles of the monotone one-to-one correspondence $\tau : I \to J$ and its inverse $\sigma : J \to I$, we can prove a corollary to Theorem 2.1.

Corollary 2.2 *If the differential equation* (4) *has the* (*generalized*) *Hyers-Ulam stability, so does the original differential equation* (1).

3 Stability of linear differential equation of second order

Throughout this section, we assume that I is a non-degenerate interval of $\mathbf{R}$. We now consider the linear inhomogeneous differential equation of the second order

$$y''(x) + f(x)y'(x) + g(x)y(x) = r(x), \tag{7}$$

where $f, g, r : I \to \mathbf{R}$ are given continuous functions. The Hyers-Ulam stability of the differential equation (7) has been proved under various additional conditions (see [20–24]). We will now investigate the generalized Hyers-Ulam stability of the linear differential equation (7) under weaker conditions in comparison with those of [20–24].

The proof of the following lemma can be found in [25], Section 2.16.

Lemma 3.1 *Assume that the homogeneous differential equation corresponding to* (7),

$$y''(x) + f(x)y'(x) + g(x)y(x) = 0, \tag{8}$$

has a general solution $y_h : I \to \mathbf{R}$ *of the form*

$$y_h(x) = c_1 y_1(x) + c_2 y_2(x),$$

where c_1 *and* c_2 *are arbitrary real-valued constants. Then the inhomogeneous linear differential equation* (7) *has a general solution* $y : I \to \mathbf{R}$ *of the form*

$$y(x) = c_1 y_1(x) + c_2 y_2(x) - y_1(x) \int_{a_1}^{x} \frac{y_2(t) r(t)}{W(y_1, y_2)(t)}\, dt + y_2(x) \int_{a_2}^{x} \frac{y_1(t) r(t)}{W(y_1, y_2)(t)}\, dt,$$

where a_1 *and* a_2 *are arbitrarily chosen points of* I *and*

$$W(y_1, y_2)(x) := y_1(x) y_2'(x) - y_1'(x) y_2(x)$$

is the Wronskian of y_1 *and* y_2.

We now investigate the generalized Hyers-Ulam stability of the linear inhomogeneous differential equation of the second order (7) in the class of twice continuously differentiable functions.

Theorem 3.2 *Let* $f, g, r : I \to \mathbf{R}$ *be continuous functions. Assume that the homogeneous differential equation* (8) *has a general solution* $y_h : I \to \mathbf{R}$ *of the form* $y_h(x) = c_1 y_1(x) + c_2 y_2(x)$, *where* c_1 *and* c_2 *are arbitrary real-valued constants. If a twice continuously differentiable function* $y : I \to \mathbf{R}$ *satisfies the inequality*

$$\bigl| y''(x) + f(x)y'(x) + g(x)y(x) - r(x) \bigr| \le \varphi(x) \tag{9}$$

for all $x \in I$, *where* $\varphi : I \to [0, \infty)$ *is given such that each of the following integrals exists, then there exists a solution* $y_0 : I \to \mathbf{R}$ *of* (7) *such that*

$$\bigl| y(x) - y_0(x) \bigr| \le \bigl| y_1(x) \bigr| \left| \int_{a_1}^{x} \left| \frac{y_2(t)}{W(y_1, y_2)(t)} \right| \varphi(t)\, dt \right| + \bigl| y_2(x) \bigr| \left| \int_{a_2}^{x} \left| \frac{y_1(t)}{W(y_1, y_2)(t)} \right| \varphi(t)\, dt \right|$$

for all $x \in I$, *where* a_1, a_2 *are arbitrarily chosen points of* I.

Proof If we define a continuous function $s : I \to \mathbf{R}$ by

$$s(x) := y''(x) + f(x)y'(x) + g(x)y(x) \tag{10}$$

for all $x \in I$, then it follows from (9) that

$$\bigl|s(x) - r(x)\bigr| \le \varphi(x) \tag{11}$$

for all $x \in I$. In view of Lemma 3.1 and (10), there exist real-valued constants α_1 and α_2 such that

$$\begin{aligned} y(x) &= \alpha_1 y_1(x) + \alpha_2 y_2(x) \\ &\quad - y_1(x) \int_{a_1}^{x} \frac{y_2(t)s(t)}{W(y_1, y_2)(t)}\,dt + y_2(x) \int_{a_2}^{x} \frac{y_1(t)s(t)}{W(y_1, y_2)(t)}\,dt, \end{aligned} \tag{12}$$

where $a_1, a_2 \in I$ are arbitrarily chosen and $W(y_1, y_2)(t) \ne 0$ for all $t \in I$ because y_1 and y_2 are linearly independent.

We now define a function $y_0 : I \to \mathbf{R}$ by

$$\begin{aligned} y_0(x) &:= \alpha_1 y_1(x) + \alpha_2 y_2(x) \\ &\quad - y_1(x) \int_{a_1}^{x} \frac{y_2(t)r(t)}{W(y_1, y_2)(t)}\,dt + y_2(x) \int_{a_2}^{x} \frac{y_1(t)r(t)}{W(y_1, y_2)(t)}\,dt \end{aligned} \tag{13}$$

for each $x \in I$. According to Lemma 3.1, it is obvious that y_0 is a solution of (7). Moreover, it follows from (11), (12), and (13) that

$$\begin{aligned} &\bigl|y(x) - y_0(x)\bigr| \\ &\quad = \left| y_1(x) \int_{a_1}^{x} \frac{y_2(t)}{W(y_1, y_2)(t)} \bigl(r(t) - s(t)\bigr)\,dt + y_2(x) \int_{a_2}^{x} \frac{y_1(t)}{W(y_1, y_2)(t)} \bigl(s(t) - r(t)\bigr)\,dt \right| \\ &\quad \le \bigl|y_1(x)\bigr| \left| \int_{a_1}^{x} \left| \frac{y_2(t)}{W(y_1, y_2)(t)} \right| \varphi(t)\,dt \right| + \bigl|y_2(x)\bigr| \left| \int_{a_2}^{x} \left| \frac{y_1(t)}{W(y_1, y_2)(t)} \right| \varphi(t)\,dt \right| \end{aligned} \tag{14}$$

for any $x \in I$. □

If we set $c := a_1 = a_2$ in Theorem 3.2 and use the equality (14), then we obtain the following corollary.

Corollary 3.3 *Let $f, g, r : I \to \mathbf{R}$ be continuous functions. Assume that the homogeneous differential equation* (8) *has a general solution $y_h : I \to \mathbf{R}$ of the form $y_h(x) = c_1 y_1(x) + c_2 y_2(x)$, where c_1 and c_2 are arbitrary real-valued constants. If a twice continuously differentiable function $y : I \to \mathbf{R}$ satisfies the inequality* (9) *for all $x \in I$, where $\varphi : I \to [0, \infty)$ is given such that the following integral exists, then there exists a solution $y_0 : I \to \mathbf{R}$ of* (7) *such that*

$$\bigl|y(x) - y_0(x)\bigr| \le \left| \int_{c}^{x} \left| \frac{y_1(x)y_2(t) - y_1(t)y_2(x)}{W(y_1, y_2)(t)} \right| \varphi(t)\,dt \right|$$

for all $x \in I$, where c is an arbitrarily chosen point of I.

4 Hyers-Ulam stability of Cauchy-Euler equation

In this section, we consider the (inhomogeneous) Cauchy-Euler (differential) equation

$$x^2y''(x) + \alpha xy'(x) + \beta y(x) = r(x), \tag{15}$$

where α and β are real-valued coefficients and $r : (0,\infty) \to \mathbf{R}$ is a differentiable function, and we will investigate the generalized Hyers-Ulam stability of this differential equation. Indeed, the generalized Hyers-Ulam stability of the Cauchy-Euler equation (15) has been proved under some additional conditions (see [26, 27]).

By using Theorem 2.1 and Corollary 3.3, we prove the generalized Hyers-Ulam stability of the Cauchy-Euler equation (15) for the case of $(\alpha - 1)^2 - 4\beta > 0$.

Theorem 4.1 *If the real-valued constants* α *and* β *are given with* $(\alpha - 1)^2 - 4\beta > 0$, *then the Cauchy-Euler equation* (15) *has the generalized Hyers-Ulam stability. In particular, let c be a positive real-valued constant and let* m_1, m_2 *be the distinct roots of the indicial equation* $m^2 + (\alpha - 1)m + \beta = 0$, *i.e.*,

$$m_1 = \frac{-(\alpha-1) - \sqrt{(\alpha-1)^2 - 4\beta}}{2}, \qquad m_2 = \frac{-(\alpha-1) + \sqrt{(\alpha-1)^2 - 4\beta}}{2}. \tag{16}$$

If $r : (0,\infty) \to \mathbf{R}$ *is a differentiable function and* $y : (0,\infty) \to \mathbf{R}$ *is a twice continuously differentiable function such that the inequality*

$$\left|x^2y''(x) + \alpha xy'(x) + \beta y(x) - r(x)\right| \le \varphi(x) \tag{17}$$

holds for any $x \in (0,\infty)$, *where* $\varphi : (0,\infty) \to [0,\infty)$ *is a given function such that the following integral exists, then there exists a solution* $y_0 : (0,\infty) \to \mathbf{R}$ *of the inhomogeneous Cauchy-Euler equation* (15) *such that*

$$\left|y(x) - y_0(x)\right| \le \frac{1}{m_2 - m_1}\left|\int_c^x \left|\left(\frac{x}{\zeta}\right)^{m_1} - \left(\frac{x}{\zeta}\right)^{m_2}\right| \frac{\varphi(\zeta)}{\zeta}\, d\zeta\right|$$

for all $x \in (0,\infty)$.

Proof If we define a monotone one-to-one correspondence $\tau : (0,\infty) \to \mathbf{R}$ by

$$\tau(x) := \ln x = t,$$

then $x = e^t$ for each $t \in \mathbf{R}$. We now define a twice continuously differentiable function $z : \mathbf{R} \to \mathbf{R}$ by

$$z(t) := y(x) = y\left(e^t\right)$$

and we get

$$xy'(x) = x\frac{d}{dx}y(x) = x\frac{d}{dt}y\left(e^t\right)\frac{dt}{dx} = z'(t),$$
$$x^2y''(x) = x^2\frac{d}{dx}y'(x) = x^2\frac{d}{dt}\left(e^{-t}z'(t)\right)\frac{dt}{dx} = z''(t) - z'(t).$$

Using these relations, we can reduce the Cauchy-Euler equation (15) to the linear differential equation

$$z''(t) + (\alpha - 1)z'(t) + \beta z(t) = r(e^t). \tag{18}$$

Similarly, the inverse $\sigma : \mathbf{R} \to (0, \infty)$ of τ given by $\sigma(t) := e^t = x$ reduces the linear differential equation (18) to the Cauchy-Euler equation (15).

Furthermore, by Corollary 3.3, the linear differential equation (18) has the generalized Hyers-Ulam stability. Therefore, due to Theorem 2.1, the Cauchy-Euler equation (15) has the generalized Hyers-Ulam stability.

In fact, as we did for (18), we can apply the monotone one-to-one correspondence τ to reduce the inequality (17) to

$$\left|z''(t) + (\alpha - 1)z'(t) + \beta z(t) - r(e^t)\right| \le \varphi(e^t) \tag{19}$$

for all $t \in \mathbf{R}$. According to (13) and Corollary 3.3 with $z(t)$, $\alpha - 1$, β, $r(e^t)$, $\varphi(e^t)$, and $(\ln c)$ instead of $y(x)$, $f(x)$, $g(x)$, $r(x)$, $\varphi(x)$, and a, respectively, there exist real-valued constants c_1 and c_2 such that

$$\begin{aligned} z_0(t) &= c_1 e^{m_1 t} + c_2 e^{m_2 t} \\ &\quad - \frac{e^{m_1 t}}{m_2 - m_1} \int_{\ln c}^{t} e^{-m_1 \eta} r(e^\eta)\, d\eta + \frac{e^{m_2 t}}{m_2 - m_1} \int_{\ln c}^{t} e^{-m_2 \eta} r(e^\eta)\, d\eta \end{aligned}$$

and

$$\left|z(t) - z_0(t)\right| \le \frac{1}{m_2 - m_1} \left| \int_{\ln c}^{t} \left|e^{m_1(t-\eta)} - e^{m_2(t-\eta)}\right| \varphi(e^\eta)\, d\eta \right|$$

for any $t \in \mathbf{R}$.

If we set $t = \ln x$ and $z_0(t) = y_0(x)$ in the previous equality for $z_0(t)$ and if we substitute ζ for e^η in the integrals, then we get

$$y_0(x) = c_1 x^{m_1} + c_2 x^{m_2} - \frac{x^{m_1}}{m_2 - m_1} \int_c^x \frac{r(\zeta)}{\zeta^{m_1+1}}\, d\zeta + \frac{x^{m_2}}{m_2 - m_1} \int_c^x \frac{r(\zeta)}{\zeta^{m_2+1}}\, d\zeta,$$

which is a solution of the inhomogeneous Cauchy-Euler equation (15). Moreover, if we set $t = \ln x$, $z(t) = y(x)$, $z_0(t) = y_0(x)$, and if we substitute ζ for e^η in the integral of the last inequality for $|z(t) - z_0(t)|$, then we obtain the inequality for $|y(x) - y_0(x)|$ given in the statement of this theorem. □

If we set $\varphi(x) = \varepsilon$ in Theorem 4.1, then we get the following corollary.

Corollary 4.2 *Assume that the real-valued constants α, β are given with $(\alpha - 1)^2 - 4\beta > 0$ and ε is an arbitrarily given positive constant. Let c be a positive real-valued constant and let m_1, m_2 be given as* (16). *If $r : (0, \infty) \to \mathbf{R}$ is a differentiable function and $y : (0, \infty) \to \mathbf{R}$ is a twice continuously differentiable function such that the inequality*

$$\left|x^2 y''(x) + \alpha x y'(x) + \beta y(x) - r(x)\right| \le \varepsilon \tag{20}$$

holds for any $x \in (0,\infty)$, *then there exists a solution* $y_0 : (0,\infty) \to \mathbf{R}$ *of the inhomogeneous Cauchy-Euler equation* (15) *such that*

$$\left|y(x) - y_0(x)\right| \le \begin{cases} \frac{\varepsilon}{m_1 m_2} + \frac{\varepsilon}{m_2-m_1}(\frac{1}{m_2}(\frac{x}{c})^{m_2} - \frac{1}{m_1}(\frac{x}{c})^{m_1}) & (\text{for } m_1 \ne 0 \ne m_2), \\ \frac{\varepsilon}{m_2^2}((\frac{x}{c})^{m_2} - 1) - \frac{\varepsilon}{m_2}\ln\frac{x}{c} & (\text{for } m_1 = 0), \\ \frac{\varepsilon}{m_1^2}((\frac{x}{c})^{m_1} - 1) - \frac{\varepsilon}{m_1}\ln\frac{x}{c} & (\text{for } m_2 = 0) \end{cases}$$

for all $x \in (0,\infty)$.

Proof According to Theorem 4.1, there exists a solution $y_0 : (0,\infty) \to \mathbf{R}$ of the inhomogeneous Cauchy-Euler equation (15) such that

$$\begin{aligned} \left|y(x) - y_0(x)\right| &\le \frac{1}{m_2 - m_1}\left|\int_c^x \left|\left(\frac{x}{\zeta}\right)^{m_1} - \left(\frac{x}{\zeta}\right)^{m_2}\right| \frac{\varepsilon}{\zeta}\,d\zeta\right| \\ &= \begin{cases} \frac{\varepsilon}{m_2-m_1}\int_c^x (\frac{x^{m_2}}{\zeta^{m_2+1}} - \frac{x^{m_1}}{\zeta^{m_1+1}})\,d\zeta & (\text{for } c \le x), \\ \frac{\varepsilon}{m_2-m_1}\int_x^c (\frac{x^{m_1}}{\zeta^{m_1+1}} - \frac{x^{m_2}}{\zeta^{m_2+1}})\,d\zeta & (\text{for } x < c) \end{cases} \\ &= \frac{\varepsilon}{m_2 - m_1}\int_c^x \left(\frac{x^{m_2}}{\zeta^{m_2+1}} - \frac{x^{m_1}}{\zeta^{m_1+1}}\right) d\zeta \end{aligned}$$

for all $x \in (0,\infty)$. We can integrate the last inequality case by case and obtain the inequality for $|y(x) - y_0(x)|$. □

We now consider the case when $(\alpha - 1)^2 - 4\beta = 0$ and use Theorem 2.1 and Corollary 3.3 to prove the generalized Hyers-Ulam stability of the inhomogeneous Cauchy-Euler equation (15).

Theorem 4.3 *If the real-valued constants* α *and* β *are given with* $\alpha \ne 1$ *and* $\beta = \frac{(\alpha-1)^2}{4}$, *then the Cauchy-Euler equation* (15) *has the generalized Hyers-Ulam stability. In particular, let* c *be a positive real-valued constant and let* $\lambda = -\frac{\alpha-1}{2}$. *If* $r : (0,\infty) \to \mathbf{R}$ *is a differentiable function and* $y : (0,\infty) \to \mathbf{R}$ *is a twice continuously differentiable function such that the inequality*

$$\left|x^2 y''(x) + \alpha x y'(x) + \frac{(\alpha - 1)^2}{4} y(x) - r(x)\right| \le \varphi(x) \tag{21}$$

holds for each $x \in (0,\infty)$, *where* $\varphi : (0,\infty) \to [0,\infty)$ *is a given function such that the following integral exists, then there exists a solution* $y_0 : (0,\infty) \to \mathbf{R}$ *of the inhomogeneous Cauchy-Euler equation* (15) *with* $\beta = \frac{(\alpha-1)^2}{4}$ *such that*

$$\left|y(x) - y_0(x)\right| \le \left|\int_c^x \left|\ln\frac{x}{\zeta}\right| \left(\frac{x}{\zeta}\right)^{\lambda} \frac{\varphi(\zeta)}{\zeta}\,d\zeta\right|$$

for all $x \in (0,\infty)$.

Proof Analogously to the proof of Theorem 4.1, we define a monotone one-to-one correspondence $\tau : (0,\infty) \to \mathbf{R}$ and a twice continuously differentiable function $z : \mathbf{R} \to \mathbf{R}$ by

$\tau(x) = \ln x = t$ and $z(t) = y(x) = y(e^t)$, respectively. In a similar way to the first part of the proof of Theorem 4.1, the Cauchy-Euler equation (15) has the generalized Hyers-Ulam stability.

In particular, we apply the monotone one-to-one correspondence τ to reduce the inequality (21) to

$$\left| z''(t) + (\alpha - 1)z'(t) + \frac{(\alpha - 1)^2}{4} z(t) - r\bigl(e^t\bigr) \right| \le \varphi\bigl(e^t\bigr)$$

for any $t \in \mathbf{R}$. According to (13) and Corollary 3.3 with $z(t)$, $\alpha - 1$, $\frac{(\alpha-1)^2}{4}$, $r(e^t)$, $\varphi(e^t)$, and $(\ln c)$ instead of $y(x), f(x), g(x), r(x), \varphi(x)$, and a, respectively, there exist real-valued constants c_1 and c_2 such that

$$z_0(t) = c_1 e^{\lambda t} + c_2 t e^{\lambda t} - e^{\lambda t} \int_{\ln c}^{t} \eta e^{-\lambda \eta} r\bigl(e^{\eta}\bigr)\, d\eta + t e^{\lambda t} \int_{\ln c}^{t} e^{-\lambda \eta} r\bigl(e^{\eta}\bigr)\, d\eta$$

and

$$\bigl|z(t) - z_0(t)\bigr| \le \left| \int_{\ln c}^{t} |t - \eta| e^{\lambda(t-\eta)} \varphi\bigl(e^{\eta}\bigr)\, d\eta \right|$$

for all $t \in \mathbf{R}$.

If we set $t = \ln x$ and $z_0(t) = y_0(x)$ in the previous equality for $z_0(t)$ and if we substitute ζ for e^{η} in the integrals, then we get

$$y_0(x) = c_1 x^{\lambda} + c_2 x^{\lambda} \ln x - x^{\lambda} \int_{c}^{x} (\ln \zeta) \frac{r(\zeta)}{\zeta^{\lambda+1}}\, d\zeta + x^{\lambda} (\ln x) \int_{c}^{x} \frac{r(\zeta)}{\zeta^{\lambda+1}}\, d\zeta,$$

which is obviously a solution of the inhomogeneous Cauchy-Euler equation (15) with $\beta = \frac{(\alpha-1)^2}{4}$. Furthermore, if we set $t = \ln x$, $z(t) = y(x)$, $z_0(t) = y_0(x)$, and if we substitute ζ for e^{η} in the integral of the previous inequality for $|z(t) - z_0(t)|$, then we get the inequality for $|y(x) - y_0(x)|$ described in the statement of the present theorem. □

If we set $\varphi(x) = \varepsilon$ in Theorem 4.3, then we obtain the following corollary.

Corollary 4.4 *Assume that the real-valued constants α and β are given with $\alpha \neq 1$, $\beta = \frac{(\alpha-1)^2}{4}$ and ε is an arbitrarily given positive constant. Let c be a positive real-valued constant and let $\lambda = -\frac{\alpha-1}{2}$. If $r : (0,\infty) \to \mathbf{R}$ is a differentiable function and $y : (0,\infty) \to \mathbf{R}$ is a twice continuously differentiable function such that the inequality*

$$\left| x^2 y''(x) + \alpha x y'(x) + \frac{(\alpha - 1)^2}{4} y(x) - r(x) \right| \le \varepsilon$$

holds for all $x \in (0,\infty)$, then there exists a solution $y_0 : (0,\infty) \to \mathbf{R}$ of the inhomogeneous Cauchy-Euler equation (15) *with $\beta = \frac{(\alpha-1)^2}{4}$ such that*

$$\bigl|y(x) - y_0(x)\bigr| \le \frac{\varepsilon}{\lambda^2} + \frac{\varepsilon}{\lambda} \left(\frac{x}{c}\right)^{\lambda} \left(\ln \frac{x}{c} - \frac{1}{\lambda}\right)$$

for all $x \in (0,\infty)$.

Proof According to Theorem 4.3, there exists a solution $y_0 : (0,\infty) \to \mathbf{R}$ of the inhomogeneous Cauchy-Euler equation (15) with $\beta = \frac{(\alpha-1)^2}{4}$ such that

$$\begin{aligned} \left|y(x) - y_0(x)\right| &\leq \left| \int_c^x \left| \ln \frac{x}{\zeta} \right| \left(\frac{x}{\zeta}\right)^{\lambda} \frac{\varepsilon}{\zeta}\, d\zeta \right| \\ &= \begin{cases} \int_c^x (\frac{x}{\zeta})^{\lambda} (\ln \frac{x}{\zeta}) \frac{\varepsilon}{\zeta}\, d\zeta & (\text{for } c \leq x), \\ \int_x^c (\frac{x}{\zeta})^{\lambda} (\ln \frac{\zeta}{x}) \frac{\varepsilon}{\zeta}\, d\zeta & (\text{for } x < c) \end{cases} \\ &= \int_c^x \frac{\varepsilon x^{\lambda}}{\zeta^{\lambda+1}} \ln \frac{x}{\zeta}\, d\zeta \\ &= \frac{\varepsilon}{\lambda^2} + \frac{\varepsilon}{\lambda} \left(\frac{x}{c}\right)^{\lambda} \left(\ln \frac{x}{c} - \frac{1}{\lambda}\right) \end{aligned}$$

for all $x \in (0,\infty)$. □

We apply Theorem 2.1 and Corollary 3.3 to prove the generalized Hyers-Ulam stability of the Cauchy-Euler equation (15) for the case of $(\alpha - 1)^2 - 4\beta < 0$.

Theorem 4.5 *If the real-valued constants α and β are given with $(\alpha - 1)^2 - 4\beta < 0$, then the Cauchy-Euler equation* (15) *has the generalized Hyers-Ulam stability. In particular, let $c > 0$ be a given real-valued constant and let*

$$\lambda = -\frac{\alpha - 1}{2} \quad \textit{and} \quad \mu = \frac{1}{2}\sqrt{4\beta - (\alpha - 1)^2}.$$

If $r : (0,\infty) \to \mathbf{R}$ is a differentiable function and $y : (0,\infty) \to \mathbf{R}$ is a twice continuously differentiable function such that the inequality (17) *holds for all $x \in (0,\infty)$, where $\varphi : (0,\infty) \to [0,\infty)$ is a given function such that the following integral exists, then there exists a solution $y_0 : (0,\infty) \to \mathbf{R}$ of the inhomogeneous Cauchy-Euler equation* (15) *such that*

$$\left|y(x) - y_0(x)\right| \leq \frac{1}{\mu} \left| \int_c^x \frac{x^{\lambda}}{\zeta^{\lambda+1}} \left| \sin\left(\mu \ln \frac{x}{\zeta}\right) \right| \varphi(\zeta)\, d\zeta \right|$$

for all $x \in (0,\infty)$.

Proof In a similar way to the proofs of Theorems 4.1 and 4.3, we conclude that the Cauchy-Euler equation (15) has the generalized Hyers-Ulam stability.

Using the monotone one-to-one correspondence $\tau : (0,\infty) \to \mathbf{R}$ defined by $\tau(x) = \ln x$, we can reduce the inequality (17) to (19) and we apply (13) and Corollary 3.3 to verify the existence of real-valued constants c_1 and c_2 such that

$$\begin{aligned} z_0(t) &= c_1 e^{\lambda t} \cos \mu t + c_2 e^{\lambda t} \sin \mu t - \frac{e^{\lambda t} \cos \mu t}{\mu} \int_{\ln c}^t e^{-\lambda \eta} (\sin \mu \eta) r\left(e^{\eta}\right) d\eta \\ &\quad + \frac{e^{\lambda t} \sin \mu t}{\mu} \int_{\ln c}^t e^{-\lambda \eta} (\cos \mu \eta) r\left(e^{\eta}\right) d\eta \end{aligned}$$

and

$$\left|z(t) - z_0(t)\right| \leq \frac{1}{\mu} \left| \int_{\ln c}^t e^{\lambda(t-\eta)} \left|\sin \mu(t - \eta)\right| \varphi\left(e^{\eta}\right) d\eta \right|$$

for all $t \in \mathbf{R}$, where $\mu > 0$.

If we set $t = \ln x$ and $z_0(t) = y_0(x)$ in the previous equality for $z_0(t)$ and if we substitute ζ for e^η in the integrals, then we get

$$\begin{aligned} y_0(x) &= c_1 x^\lambda \cos(\mu \ln x) + c_2 x^\lambda \sin(\mu \ln x) \\ &\quad - \frac{x^\lambda \cos(\mu \ln x)}{\mu} \int_c^x \frac{\sin(\mu \ln \zeta)}{\zeta^{\lambda+1}} r(\zeta)\, d\zeta \\ &\quad + \frac{x^\lambda \sin(\mu \ln x)}{\mu} \int_c^x \frac{\cos(\mu \ln \zeta)}{\zeta^{\lambda+1}} r(\zeta)\, d\zeta, \end{aligned}$$

which is obviously a solution of the inhomogeneous Cauchy-Euler equation (15) with $(\alpha - 1)^2 - 4\beta < 0$. Finally, if we let $t = \ln x$, $z(t) = y(x)$, $z_0(t) = y_0(x)$, and if we substitute ζ for e^η in the integral of the inequality for $|z(t) - z_0(t)|$, then we obtain the inequality for $|y(x) - y_0(x)|$ given in the present theorem. □

If we set $\varphi(x) = \varepsilon$ in Theorem 4.5, then we can easily prove the following corollary.

Corollary 4.6 *Assume that the real-valued constants α and β are given with $(\alpha-1)^2 - 4\beta < 0$ and ε is an arbitrarily given positive constant. Let $c > 0$ be a given real-valued constant and let*

$$\lambda = -\frac{\alpha - 1}{2} \quad \textit{and} \quad \mu = \frac{1}{2}\sqrt{4\beta - (\alpha - 1)^2}.$$

If a differentiable function $r : (0, \infty) \to \mathbf{R}$ and a twice continuously differentiable function $y : (0, \infty) \to \mathbf{R}$ satisfy the inequality (20) *for all $x \in (0, \infty)$, then there exists a solution $y_0 : (0, \infty) \to \mathbf{R}$ of the inhomogeneous Cauchy-Euler equation* (15) *such that*

$$\begin{aligned} &\bigl|y(x) - y_0(x)\bigr| \\ &\le \begin{cases} \varepsilon \bigl| \frac{1}{\beta\mu} (\frac{x}{c})^\lambda (\lambda \sin(\mu \ln \frac{x}{c}) - \mu \cos(\mu \ln \frac{x}{c})) + \frac{(-1)^{m_x}}{\beta} e^{\lambda \frac{\pi}{\mu} m_x} \bigr| \\ \quad + \varepsilon \sum_{m=1}^{m_x} \frac{1}{|\beta|} e^{\lambda \frac{\pi}{\mu}(m-1)} \bigl| e^{\lambda \frac{\pi}{\mu}} + 1 \bigr| \quad (\textit{for } x \ge c), \\ \varepsilon \bigl| \frac{(-1)^{m_c}}{\beta\mu} (\frac{x}{c})^\lambda e^{\lambda \frac{\pi}{\mu} m_c} (\lambda \sin(\mu \ln \frac{x}{c}) - \mu \cos(\mu \ln \frac{x}{c})) + \frac{1}{\beta} \bigr| \\ \quad + \varepsilon \sum_{m=1}^{m_c} \bigl| \frac{1}{\beta\mu} (\frac{x}{c})^\lambda e^{\lambda \frac{\pi}{\mu}(m-1)} (\lambda \sin(\mu \ln \frac{x}{c}) - \mu \cos(\mu \ln \frac{x}{c})) \\ \quad \times (e^{\lambda \frac{\pi}{\mu}} + 1) \bigr| \quad (\textit{for } 0 < x < c), \end{cases} \end{aligned}$$

where m_x and m_c are defined in (23) *and* (25).

Proof According to Theorem 4.5, there exists a solution $y_0 : (0, \infty) \to \mathbf{R}$ of the inhomogeneous Cauchy-Euler equation (15) such that

$$\begin{aligned} \bigl|y(x) - y_0(x)\bigr| &\le \frac{\varepsilon}{\mu} \left| \int_c^x \frac{x^\lambda}{\zeta^{\lambda+1}} \left| \sin\left(\mu \ln \frac{x}{\zeta}\right) \right| d\zeta \right| \\ &= \begin{cases} \frac{\varepsilon}{\mu} \int_c^x \frac{x^\lambda}{\zeta^{\lambda+1}} |\sin(\mu \ln \frac{x}{\zeta})|\, d\zeta & (\text{for } c \le x), \\ \frac{\varepsilon}{\mu} \int_x^c \frac{x^\lambda}{\zeta^{\lambda+1}} |\sin(\mu \ln \frac{x}{\zeta})|\, d\zeta & (\text{for } x < c) \end{cases} \end{aligned} \tag{22}$$

for all $x \in (0, \infty)$.

If $0 < c \leq x$ then we set

$$\gamma_x(m) := xe^{-\frac{m\pi}{\mu}} \quad \text{and} \quad m_x := \left[\frac{\mu}{\pi} \ln \frac{x}{c}\right], \tag{23}$$

where $[z]$ denotes the greatest integer not exceeding the given real number z. Then we have

$$[c, x] = \left[c, \gamma_x(m_x)\right] \cup \bigcup_{m=1}^{m_x} \left[\gamma_x(m), \gamma_x(m-1)\right] \tag{24}$$

for each $x \geq c$. Hence, it follows from (22) and (24) that

$$\begin{aligned}
\left|y(x) - y_0(x)\right| &\leq \frac{\varepsilon}{\mu} \left| \int_c^{\gamma_x(m_x)} \frac{x^\lambda}{\zeta^{\lambda+1}} \sin\left(\mu \ln \frac{x}{\zeta}\right) d\zeta \right| \\
&\quad + \frac{\varepsilon}{\mu} \sum_{m=1}^{m_x} \left| \int_{\gamma_x(m)}^{\gamma_x(m-1)} \frac{x^\lambda}{\zeta^{\lambda+1}} \sin\left(\mu \ln \frac{x}{\zeta}\right) d\zeta \right|
\end{aligned}$$

for any $x \geq c$. Moreover, if we substitute $\eta = \ln \frac{x}{\zeta}$ in the above integrals, then we have

$$\begin{aligned}
&\left|y(x) - y_0(x)\right| \\
&\quad \leq \frac{\varepsilon}{\mu} \left| -\int_{\ln \frac{x}{c}}^{\frac{\pi}{\mu} m_x} e^{\lambda\eta} \sin(\mu\eta)\, d\eta \right| + \frac{\varepsilon}{\mu} \sum_{m=1}^{m_x} \left| -\int_{\frac{\pi}{\mu} m}^{\frac{\pi}{\mu}(m-1)} e^{\lambda\eta} \sin(\mu\eta)\, d\eta \right| \\
&\quad = \varepsilon \left| \frac{1}{\beta\mu} \left(\frac{x}{c}\right)^\lambda \left(\lambda \sin\left(\mu \ln \frac{x}{c}\right) - \mu \cos\left(\mu \ln \frac{x}{c}\right)\right) + \frac{(-1)^{m_x}}{\beta} e^{\lambda \frac{\pi}{\mu} m_x} \right| \\
&\qquad + \varepsilon \sum_{m=1}^{m_x} \frac{1}{|\beta|} e^{\lambda \frac{\pi}{\mu}(m-1)} \left| e^{\lambda \frac{\pi}{\mu}} + 1 \right|
\end{aligned}$$

for all $x \geq c$.

If $0 < x < c$ then we set

$$\gamma_c(m) := ce^{-\frac{m\pi}{\mu}} \quad \text{and} \quad m_c := \left[\frac{\mu}{\pi} \ln \frac{c}{x}\right]. \tag{25}$$

Then we obtain

$$[x, c] = \left[x, \gamma_c(m_c)\right] \cup \bigcup_{m=1}^{m_c} \left[\gamma_c(m), \gamma_c(m-1)\right] \tag{26}$$

for any $0 < x < c$. Thus, it follows from (22) and (26) that

$$\begin{aligned}
\left|y(x) - y_0(x)\right| &\leq \frac{\varepsilon}{\mu} \left| \int_x^{\gamma_c(m_c)} \frac{x^\lambda}{\zeta^{\lambda+1}} \sin\left(\mu \ln \frac{x}{\zeta}\right) d\zeta \right| \\
&\quad + \frac{\varepsilon}{\mu} \sum_{m=1}^{m_c} \left| \int_{\gamma_c(m)}^{\gamma_c(m-1)} \frac{x^\lambda}{\zeta^{\lambda+1}} \sin\left(\mu \ln \frac{x}{\zeta}\right) d\zeta \right|
\end{aligned}$$

for each $0 < x < c$. Furthermore, if we substitute $\eta = \ln \frac{x}{\zeta}$ in the last integrals, then we have

$$\begin{aligned}
&\left|y(x) - y_0(x)\right| \\
&\quad\le \frac{\varepsilon}{\mu}\left|-\int_0^{\frac{\pi}{\mu}m_c+\ln\frac{x}{c}} e^{\lambda\eta}\sin(\mu\eta)\,d\eta\right| + \frac{\varepsilon}{\mu}\sum_{m=1}^{m_c}\left|-\int_{\frac{\pi}{\mu}m+\ln\frac{x}{c}}^{\frac{\pi}{\mu}(m-1)+\ln\frac{x}{c}} e^{\lambda\eta}\sin(\mu\eta)\,d\eta\right| \\
&\quad= \varepsilon\left|\frac{(-1)^{m_c}}{\beta\mu}\left(\frac{x}{c}\right)^{\lambda} e^{\lambda\frac{\pi}{\mu}m_c}\left(\lambda\sin\left(\mu\ln\frac{x}{c}\right) - \mu\cos\left(\mu\ln\frac{x}{c}\right)\right) + \frac{1}{\beta}\right| \\
&\qquad+ \varepsilon\sum_{m=1}^{m_c}\left|\frac{1}{\beta\mu}\left(\frac{x}{c}\right)^{\lambda} e^{\lambda\frac{\pi}{\mu}(m-1)}\left(\lambda\sin\left(\mu\ln\frac{x}{c}\right) - \mu\cos\left(\mu\ln\frac{x}{c}\right)\right)\left(e^{\lambda\frac{\pi}{\mu}}+1\right)\right|
\end{aligned}$$

for all $0 < x < c$. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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