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A delayed SIR model with general nonlinear incidence rate

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Abstract

An SIR epidemic model is investigated and analyzed based on incorporating an incubation time delay and a general nonlinear incidence rate, where the growth of susceptible individuals is governed by the logistic equation. The threshold parameter σ_0 is defined to determine whether the disease dies out in the population. The model always has the trivial equilibrium and the disease-free equilibrium whereas it admits the endemic equilibrium if σ_0 exceeds one. The disease-free equilibrium is globally asymptotically stable if σ_0 is less than one, while it is unstable if σ_0 is greater than one. By applying the time delay as a bifurcation parameter, the local stability of the endemic equilibrium is studied and the condition which is absolutely stable or conditionally stable is established. Furthermore, a Hopf bifurcation occurs under certain conditions. Numerical simulations are carried out to illustrate the main results.

Keywords: delayed SIR model; general nonlinear incidence rate; asymptotic stability; Hopf bifurcation

1 Introduction

Mathematical models have been become the important tools in investigating transmission and control of infectious diseases. To better understand the transmission pattern of infectious disease, a great many epidemic models have been formulated (see [1–21] and references therein). Recently, Takeuchi *et al.* [12] developed a delayed SIR epidemic model with bilinear incidence rate in order to investigate the spread of vector diseases, and McCluskey [9] discussed the global stability of equilibria for the system. In 2009, Wang *et al.* [13] analyzed the following SIR vector disease model with incubation time delay and logistic growth rate with carrying capacity K:

$$\begin{aligned}
&\frac{dS(t)}{dt} = rS(t)\left(1-\frac{S(t)}{K}\right) - \beta S(t)I(t-\tau),\\
&\frac{dI(t)}{dt} = \beta S(t)I(t-\tau) - (\mu_1+\gamma)I(t),\\
&\frac{dR(t)}{dt} = \gamma I(t) - \mu_2 R(t).
\end{aligned} \tag{1}$$

$S(t)$, $I(t)$, and $R(t)$ are the numbers of susceptible, infective and recovered host individuals at time t, respectively. r denotes the intrinsic birth rate. β denotes the average number of contacts per infective per unit time. τ is the incubation time. μ_1 and μ_2 represent the

death rate of infective and recovered, respectively. γ is the recovered rate of infective individuals. It is reasonable to assume that all the parameters are positive constants. Wang *et al.* [13] presented the dynamic properties of system (1). The global stability of the disease-free equilibrium is derived when the basic reproduction number R_0 is less than unity. The unique endemic equilibrium is absolutely stable when $1 < R_0 < 3$, and it is conditionally stable when $R_0 > 3$. Moreover, the existence of a Hopf bifurcation is given.

Because of considering the behavioral changes of susceptible individuals, Zhang *et al.* [19] extended system (1) and proposed the following vector disease model with saturated incidence rate:

$$\begin{aligned}
&\frac{dS(t)}{dt} = rS(t)\left(1 - \frac{S(t)}{K}\right) - \beta\frac{S(t)}{1+\alpha S(t)}I(t-\tau),\\
&\frac{dI(t)}{dt} = \beta\frac{S(t)}{1+\alpha S(t)}I(t-\tau) - (\mu_1+\gamma)I(t),\\
&\frac{dR(t)}{dt} = \gamma I(t) - \mu_2 R(t),
\end{aligned} \tag{2}$$

where the parameters r, β, τ, μ_1, μ_2, and γ are the same meanings as that defined in model (1), and $\alpha \geq 0$ is a constant in order to represent the saturation effect. The global dynamics for model (2) was investigated. If R_0 is less than one, the disease-free equilibrium is globally asymptotically stable; while the unique endemic equilibrium may be stable or unstable under some conditions if R_0 is greater than one. Furthermore, the Hopf bifurcation emerges if other conditions are satisfied when R_0 is greater than one.

Although the bilinear incidence rate was frequently used in the literature of mathematical modeling, there are plenty of reasons why this bilinear incidence rate may require modification [5, 22]. For example, the saturated incidence rate of the form $\frac{\beta S(t)I(t)}{1+\alpha I(t)}$ or $\frac{\beta S(t)I(t)}{1+\alpha S(t)}$ was formulated as crowding effects of infective or behavioral changes of susceptible individuals were considered [2, 10, 18, 19, 23–25]. Moreover, other forms of nonlinear incidence rates are often developed in many papers (for details one can refer to [4, 5, 14–17, 22]). Motivated by those works, in the present paper, we attempt to extend system (1) or (2) to a more general incidence rate of the form $\beta F(S(t))I(t-\tau)$. It is assumed that function F is continuous on $[0,\infty)$ and continuously differentiable on $(0,\infty)$, which satisfies the following hypothesis. Furthermore, it is assumed $F(S)$ is strictly monotonically increasing on $[0,+\infty)$ with $F(0) = 0$.

Then the delayed SIR vector disease model can be written as

$$\begin{aligned}
&\frac{dS(t)}{dt} = rS(t)\left(1 - \frac{S(t)}{K}\right) - \beta F\big(S(t)\big)I(t-\tau),\\
&\frac{dI(t)}{dt} = \beta F\big(S(t)\big)I(t-\tau) - (\mu_1+\gamma)I(t),\\
&\frac{dR(t)}{dt} = \gamma I(t) - \mu_2 R(t).
\end{aligned} \tag{3}$$

It should be noted that the general nonlinear incidence rate in system (3) includes some special cases. If $F(S) = S$, then it becomes the classical bilinear incidence rate, which has been investigated by Wang *et al.* [13]. If $F(S) = S^q$ $(q > 0)$, then the incidence rate is used in [20]. If $F(S) = \frac{S}{1+\alpha S}$, it becomes the saturated one, which has been discussed in [18, 19].

For simplicity, we make system (3) non-dimensional by writing

$$\tilde{S}(\tilde{t}) = \frac{S(t)}{K}, \qquad \tilde{I}(\tilde{t}) = \frac{I(t)}{K}, \qquad \tilde{R}(\tilde{t}) = \frac{R(t)}{K}$$

and

$$\tilde{t} = \beta K t, \qquad \tilde{F}\big(\tilde{S}(\tilde{t})\big) = \frac{F(S(t))}{K}, \qquad \tilde{\mu}_1 = \frac{\mu_1}{\beta K},$$
$$\tilde{\gamma} = \frac{\gamma}{\beta K}, \qquad \tilde{\mu}_2 = \frac{\mu_2}{\beta K}, \qquad \tilde{\tau} = \frac{\tau}{\beta K}.$$

It is clear that $\tilde{F}$ is also strictly monotonically increasing on $[0, +\infty)$ with $F(0) = 0$. For notational simplicity, dropping the $\tilde{}$, system (3) can be turned into

$$\begin{aligned} &\frac{dS(t)}{dt} = rS(t)\big(1 - S(t)\big) - F\big(S(t)\big)I(t-\tau), \\ &\frac{dI(t)}{dt} = F\big(S(t)\big)I(t-\tau) - (\mu_1 + \gamma)I(t), \\ &\frac{dR(t)}{dt} = \gamma I(t) - \mu_2 R(t). \end{aligned} \tag{4}$$

The rest of the paper is structured as follows. In Section 2, the nonnegativity and boundedness of the solutions are discussed. In Section 3, the stabilities of the trivial equilibrium and the disease-free equilibrium are described. Section 4 deals with the existence and stability of the endemic equilibrium and the existence of a Hopf bifurcation. In Section 5, the numerical simulations are performed, followed by a brief conclusion in Section 6.

2 Nonnegativity and boundedness of solutions

The initial conditions for system (4) take the form

$$\begin{aligned} &S(\theta) = \phi_1(\theta), \qquad I(\theta) = \phi_2(\theta), \qquad R(\theta) = \phi_3(\theta), \\ &\phi_i(\theta) \geq 0, \qquad \phi_i(0) > 0, \quad i = 1, 2, 3, \end{aligned} \tag{5}$$

where $(\phi_1(\theta), \phi_2(\theta), \phi_3(\theta)) \in C([-\tau, 0], \mathbb{R}^3_+)$, here $\mathbb{R}^3_+ = \{(x_1, x_2, x_3); x_i \geq 0, i = 1, 2, 3\}$. The fundamental theory of functional differential equations [26] implies for any initial conditions (5), system (4) has a unique solution $(S(t), I(t), R(t))$. The following theorem shows that the solution is nonnegative and bounded for a positive initial value (5).

Theorem 2.1 *System* (4) *has a nonnegative and bounded solution with the initial value* $(\phi_1(\theta), \phi_2(\theta), \phi_3(\theta)) \in C([-\tau, 0], \mathbb{R}^3_+)$ *and* $\phi_i(\theta) \geq 0$, $\phi_i(0) > 0$, $i = 1, 2, 3$.

Proof First we show that $S(t)$ is nonnegative for all $t \geq 0$. On the contrary, it is assumed that there exists $t_1 > 0$ such that $S(t_1) = 0$ and $S'(t_1) < 0$. Then the first equation of system (4) implies $S'(t_1) = 0$, which is a contradiction. Therefore, it follows that $S(t) \geq 0$ for all $t \geq 0$.

By using the variation-of-constant formula and the step-by-step integration method, integrating the second equation of system (4) from 0 to t for $0 < t \leq \tau$, we obtain

$$I(t) = e^{-(\mu_1+\gamma)t}\left(\phi_2(0) + \int_0^t F\big(S(\xi)\phi_2(\xi - \tau)e^{(\mu_1+\gamma)\xi}\, d\xi\big)\right).$$

It is easy to see that $I(t) > 0$ for all $0 \le t \le \tau$. Then integrating the second equation of system (4) from τ to t for $\tau < t \le 2\tau$ gives

$$I(t) = e^{-(\mu_1+\gamma)t}\left(I(\tau) + \int_\tau^t F\big(S(\xi)I(\xi-\tau)e^{(\mu_1+\gamma)\xi}\,d\xi\big)\right).$$

Note that $I(t) > 0$ for all $\tau \le t \le 2\tau$ and this process can easily be carried on. It implies that for all $t > 0$, we have $I(t) > 0$.

From the third equation of system (4), we obtain

$$R(t) = e^{-\mu_2 t}\left(\phi_3(0) + \int_0^t \gamma I(\xi)e^{\mu_2\xi}\,d\xi\right),$$

which shows $R(t)$ is nonnegative for all $t > 0$.

Next we prove that the solutions of system (4) are ultimately uniformly bounded for all $t \ge 0$. It follows from the first equation of system (4) that $S'(t) \le rS(t)(1 - S(t))$, which implies $\limsup_{t\to\infty} S(t) \le 1$. Then for sufficiently large t, adding the equations of system (4) yields

$$\begin{aligned}\frac{d(S(t)+I(t)+R(t))}{dt} &= rS(t)\big(1-S(t)\big) - \mu_1 I(t) - \mu_2 R(t)\\ &\le rS(t) - \mu_1 I(t) - \mu_2 R(t)\\ &= (r+1)S(t) - S(t) - \mu_1 I(t) - \mu_2 R(t)\\ &\le (r+1)S(t) - \mu_m\big(S(t)+I(t)+R(t)\big)\\ &\le (r+1) - \mu_m\big(S(t)+I(t)+R(t)\big),\end{aligned}$$

where $\mu_m = \min\{1, \mu_1, \mu_2\}$. Then we have $\limsup_{t\to\infty}(S(t) + I(t) + R(t)) \le \frac{r+1}{\mu_m}$. Therefore, $S(t)$, $I(t)$, $R(t)$ are ultimately uniformly bounded. The proof is completed. □

3 Stabilities of the trivial equilibrium and the disease-free equilibrium

In this section, we restrict our attention to the stability of the trivial equilibrium and the disease-free equilibrium. Let

$$\sigma_0 = \frac{F(1)}{\mu_1 + \gamma}. \tag{6}$$

It will be a threshold parameter.

Before the main results are established, the following lemma will be given first.

Lemma 3.1 (see [27]) *Consider the equation*

$$u'(t) = au(t-\tau) - bu(t), \tag{7}$$

where $a, b, \tau > 0$, and $u(t) > 0$ for $-\tau \le t \le 0$. We have

(i) *if $a < b$, then $\lim_{t\to\infty} u(t) = 0$;*

(ii) *if $a > b$, then $\lim_{t\to\infty} u(t) = +\infty$.*

The characteristic equation at an arbitrary equilibrium $(\bar{S}, \bar{I}, \bar{R})$ is given by

$$(\lambda + \mu_2)\big[\big(\lambda + F'(\bar{S})\bar{I} - r(1 - 2\bar{S})\big)\big(\lambda + \mu_1 + \gamma - F(\bar{S})e^{-\lambda\tau}\big) + F(\bar{S})F'(\bar{S})\bar{I}e^{-\lambda\tau}\big] = 0. \tag{8}$$

Theorem 3.1 *The trivial equilibrium E_0 of system* (4) *is always unstable.*

Proof At the equilibrium $E_0(0, 0, 0)$, the characteristic equation (8) reduces to

$$(\lambda + \mu_2)(\lambda - r)(\lambda + \mu_1 + \gamma) = 0. \tag{9}$$

It is obvious that (9) has a positive root $\lambda = r$, therefore E_0 is unstable. □

Theorem 3.2 *If $\sigma_0 < 1$, the disease-free equilibrium E_1 for system* (4) *is globally asymptotically stable*; *and if $\sigma_0 > 1$, the disease-free equilibrium E_1 for system* (4) *is unstable.*

Proof The characteristic equation (8) at $E_1 = (1, 0, 0)$ becomes

$$(\lambda + \mu_2)(\lambda + r)\big(\lambda + \mu_1 + \gamma - F(1)e^{-\lambda\tau}\big) = 0. \tag{10}$$

Assume that $\sigma_0 < 1$. Equation (10) has roots $-\mu_2 < 0$, $-r < 0$, and the root of the equation $\lambda + \mu_1 + \gamma - F(1)e^{-\lambda\tau} = 0$. Let $G(\lambda) = \lambda + \mu_1 + \gamma - F(1)e^{-\lambda\tau}$. Suppose $\operatorname{Re}(\lambda) \geq 0$, then $G(\lambda) = 0$ implies

$$\begin{aligned}
\operatorname{Re}(\lambda) &= -(\mu_1 + \gamma) + F(1)e^{-\operatorname{Re}(\lambda)\tau}\cos \operatorname{Im}(\lambda)\tau \\
&= (\mu_1 + \gamma)\big[\sigma_0 e^{-\operatorname{Re}(\lambda)\tau}\cos \operatorname{Im}(\lambda)\tau - 1\big] \\
&\leq (\mu_1 + \gamma)\big(\sigma_0 e^{-\operatorname{Re}(\lambda)\tau} - 1\big) \\
&\leq (\mu_1 + \gamma)(\sigma_0 - 1) < 0,
\end{aligned}$$

which is a contradiction. Then it follows that E_1 is locally asymptotically stable.

Now it is sufficient to prove E_1 is globally attractive if $\sigma_0 < 1$. From the first equation for system (4), it follows that

$$\frac{dS(t)}{dt} \leq rS(t)\big(1 - S(t)\big), \tag{11}$$

which implies $\limsup_{t\to\infty} S(t) \leq 1$. It indicates that for sufficiently large t, there exists a small $\varepsilon > 0$ such that $S(t) < 1 + \varepsilon$ and $F(1 + \varepsilon) < \mu_1 + \gamma$ because of $\sigma_0 = \frac{F(1)}{\mu_1 + \gamma} < 1$. Then for sufficiently large t, because of the monotonicity of the function $F(S)$, the second equation for system (4) can be rewritten as

$$\frac{dI(t)}{dt} \leq F(1 + \varepsilon)I(t - \tau) - (\mu_1 + \gamma)I(t). \tag{12}$$

By $F(1 + \varepsilon) < \mu_1 + \gamma$ and Lemma 3.1, we get $\limsup_{t\to\infty} I(t) \leq 0$, which implies $I(t) \to 0$ as $t \to \infty$. By the theory of asymptotic autonomous systems [28], it then follows that $S(t) \to 1$ and $R(t) \to 0$ as $t \to \infty$. The first part of the proof is completed.

If $\sigma_0 > 1$, then $G(0) = (\mu_1 + \gamma)(1 - \sigma_0) < 0$. When $\lambda \to +\infty$, $G(\lambda) \to +\infty$. Then $G(\lambda) = 0$ has at least one positive root. Therefore E_1 is unstable. □

4 The stability of endemic equilibrium and Hopf bifurcation

In this section, we pay attention to the stability of the endemic equilibrium and Hopf bifurcation when $\sigma_0 > 1$.

Theorem 4.1 *If $\sigma_0 > 1$, system* (4) *admits exactly one endemic equilibrium $E_* = (S^*, I^*, R^*)$, where*

$$F(S^*) = \mu_1 + \gamma, \quad 0 < S^* < 1, \qquad I^* = \frac{rS^*(1-S^*)}{F(S^*)}, \qquad R^* = \frac{\gamma}{\mu_2} I^*.$$

Proof At the endemic equilibrium E_*, it follows from the second equation of system (4) that $F(S^*) = \mu_1 + \gamma$. Let $H(S) = F(S) - (\mu_1 + \gamma)$. It is obvious that $H(0) = F(0) - (\mu_1 + \gamma) = -(\mu_1 + \gamma) < 0$. For all $S \geq 1$, $H(S) \geq H(1) = F(1) - (\mu_1 + \gamma) = (\mu_1 + \gamma)(\sigma_0 - 1) > 0$ because $H(S)$ is monotonically increasing on the interval $[0, +\infty)$ and $\sigma_0 > 1$. Therefore $H(S) = 0$ has exactly one root $S^* \in (0,1)$. It is not difficult to compute the expressions I^* and R^* from system (4) at the endemic equilibrium E_*. □

By using (8), the characteristic equation at endemic equilibrium $E_* = (S^*, I^*, R^*)$ can be turned into

$$(\lambda + \mu_2)\left[\lambda^2 + a\lambda + b - e^{-\lambda\tau}(c\lambda + d)\right] = 0, \tag{13}$$

where

$$\begin{aligned} &a = \mu_1 + \gamma + F'(S^*)I^* - r(1 - 2S^*), \\ &b = (\mu_1 + \gamma)\left[F'(S^*)I^* - r(1 - 2S^*)\right], \\ &c = F(S^*), \qquad d = -r(1 - 2S^*)F(S^*). \end{aligned} \tag{14}$$

Then the characteristic roots at E_* are $-\mu_2$ and the roots of the following equation:

$$\lambda^2 + a\lambda + b - e^{-\lambda\tau}(c\lambda + d) = 0. \tag{15}$$

Proposition 4.1 *Assume $\sigma_0 > 1$ and $I^*F'(S^*) > r(1 - 2S^*)$, then all the roots of* (15) *have a negative real part for $\tau = 0$.*

Proof If the incubation time delay $\tau = 0$, (15) yields

$$\lambda^2 + (a - c)\lambda + (b - d) = 0. \tag{16}$$

It follows from the fact $(\mu_1 + \gamma) = F(S^*)$ and from (14) that

$$\begin{aligned} &b - d = (\mu_1 + \gamma)F'(S^*)I^* > 0, \\ &a - c = F'(S^*)I^* - r(1 - 2S^*). \end{aligned}$$

Since $I^*F'(S^*) > r(1 - 2S^*)$, it is obvious that $a - c > 0$, which completes the theorem. □

Proposition 4.2 *Assume $\sigma_0 > 1$, then the following statements hold.*

(i) *If* $I^*F'(S^*) \geq 2r(1-2S^*)$, *then all the roots of* (19) *have a negative real part for* $\tau > 0$.
(ii) *If* $I^*F'(S^*) < 2r(1-2S^*)$, *then there exists a monotone increasing sequence* $\{\tau_n\}_{n=0}^{\infty}$ *with* $\tau_0 > 0$ *such that* (15) *has a pair of imaginary roots for* $\tau = \tau_n$ $(n = 0, 1, 2, \ldots)$.

Proof Suppose that $\lambda = i\omega$, $\omega > 0$ is a root of (15). We substitute $\lambda = i\omega$ into (15) to derive

$$-\omega^2 + ia\omega + b - (\cos\omega\tau - i\sin\omega\tau)(ic\omega + d) = 0. \tag{17}$$

Separating the real and imaginary parts gives

$$\begin{aligned} &-\omega^2 + b = d\cos\omega\tau + c\omega\sin\omega\tau, \\ &a\omega = c\omega\cos\omega\tau - d\sin\omega\tau. \end{aligned} \tag{18}$$

Squaring and adding both equations in (18), we obtain

$$\omega^4 + \left(a^2 - 2b - c^2\right)\omega^2 + b^2 - d^2 = 0. \tag{19}$$

By applying (14), we get

$$a^2 - 2b - c^2 = \left[F'\left(S^*\right)I^* - r\left(1 - 2S^*\right)\right]^2 \geq 0, \quad b - d > 0$$

and

$$b + d = (\mu_1 + \gamma)\left[F'\left(S^*\right)I^* - 2r\left(1 - 2S^*\right)\right].$$

Firstly assume that $I^*F'(S^*) \geq 2r(1-2S^*)$. Then we arrive at $a^2 - 2b - c^2 > 0$ and $b + d \geq 0$. That is to say, (19) has no positive real root ω, which is a contradiction. Therefore, all the roots of (15) have negative real part for $\tau > 0$. The first part of the proof is completed.

Secondly suppose $I^*F'(S^*) < 2r(1-2S^*)$, which indicates $b + d < 0$. Therefore, there exists a unique positive real ω_0 satisfying (19), where

$$\omega_0 = \sqrt{\frac{\sqrt{(a^2 - 2b - c^2)^2 - 4(b-d)(b+d)} - (a^2 - 2b - c^2)}{2}}. \tag{20}$$

It should be noted that $\lambda = -i\omega_0$ is also a root of (15). Then (15) has a single pair of purely imaginary roots $\pm i\omega_0$. Then using (18), we obtain

$$(ac - d)\omega_0^2 + bd = \left(c^2\omega_0^2 + d^2\right)\cos\omega_0\tau,$$

and it follows that

$$\tau_n = \frac{1}{\omega_0}\arccos\frac{(ac-d)\omega_0^2 + bd}{c^2\omega_0^2 + d^2} + \frac{2n\pi}{\omega_0}, \quad n = 0, 1, 2, \ldots. \tag{21}$$

This completes the proof of the theorem. □

We give the following proposition without any proof, since the proof is similar to that of [6].

Proposition 4.3 *If* $\sigma_0 > 1$ *and* $I^*F'(S^*) < 2r(1-2S^*)$, *then we have the transversality condition*

$$\left.\frac{d\,\mathrm{Re}(\lambda(\tau))}{d\tau}\right|_{\lambda=i\omega_0} > 0.$$

Summarizing the above propositions, we obtain the following theorem.

Theorem 4.2 *Assume* $\sigma_0 > 1$, *then the following statements hold.*

(i) *If* $I^*F'(S^*) \geq 2r(1-2S^*)$, *then the endemic equilibrium of system* (4) *is locally asymptotically stable for* $\tau \geq 0$.

(ii) *If* $I^*F'(S^*) < 2r(1-2S^*)$, *then the endemic equilibrium of system* (4) *is locally asymptotically stable for* $0 \leq \tau < \tau_0$ *and it is unstable for* $\tau > \tau_0$.

Remark 4.1 If both $\sigma_0 > 1$ and $I^*F'(S^*) < 2r(1-2S^*)$ hold true, system (4) undergoes a Hopf bifurcation at the endemic equilibrium E_* when τ crosses τ_n ($n = 0, 1, \ldots$).

5 Numerical results

In this section, we consider the numerical results of system (4) with the saturated incidence rate of the form $F(S) = \frac{S}{1+\alpha S}$. That is to say, we give the numerical simulations of system (2). In system (2), we set $\beta = 0.01$, $K = 100$, $r = \mu_1 = \mu_2 = 0.1$, and $\alpha = 0.01$. Then we get the non-dimensional quantities $\tilde{r} = \tilde{\mu}_1 = \tilde{\mu}_2 = 0.1$, $\tilde{\alpha} = 0.01$, and $\tilde{t} = \beta K t = t$. Dropping the $\tilde{}$ for convenience, we obtain the following non-dimensional system corresponding to system (2):

$$\begin{aligned}
&\frac{dS(t)}{dt} = rS(t)\big(1-S(t)\big) - \frac{S(t)}{1+\alpha S(t)}I(t-\tau),\\
&\frac{dI(t)}{dt} = \frac{S(t)}{1+\alpha S(t)}I(t-\tau) - (\mu_1+\gamma)I(t),\\
&\frac{dR(t)}{dt} = \gamma I(t) - \mu_2 R(t).
\end{aligned} \tag{22}$$

Therefore, we have $r = \mu_1 = \mu_2 = 0.1$ and $\alpha = 0.01$. If we choose $\gamma = 0.1$, the endemic equilibrium of system (22) is $E_* = (0.2004, 0.0798, 0.0798)$, $\sigma_0 = 4.9505$, and $\tau_0 = 2.0288$ by applying (17). It should also be noted that $I^*F'(S^*) = 0.0795$ and $2r(1-2S^*) = 0.1198$, which imply the endemic equilibrium E_* is conditionally stable. Furthermore, we can see that the endemic equilibrium E_* is asymptotical stable if the time delay $\tau = 1 < \tau_0 = 2.0288$ (see Figure 1), while the endemic equilibrium E_* loses its stability, Hopf bifurcation occurs, and system (22) exhibits a stable period solution if $\tau = 2.5 > \tau_0$ (see Figure 2).

If γ is chosen as 0.35 and other parameters are set as in Figure 1, then the endemic equilibrium is $E_* = (0.4520, 0.0546, 0.1909)$, $\sigma_0 = 2.2002$, and $I^*F'(S^*) = 0.0541 > 2r(1-2S^*) = 0.0192$, which imply the condition (i) of Theorem 4.2 is satisfied. Moreover, from Figure 3, we can see the endemic equilibrium E_* is globally asymptotically stable although $\tau = 2.5 > \tau_0$.

6 Conclusion

In this paper, a delayed SIR vector disease model with incubation time delay is established, in which the growth of susceptible individuals follows the logistic function in the absence

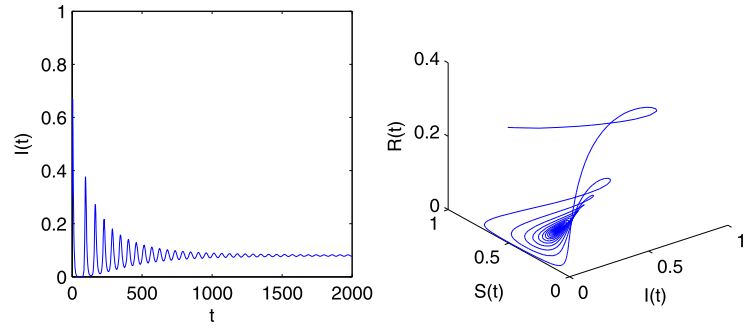

Figure 1 Temporal behavior of the infective individuals and corresponding three-dimensional phase for system (22) with $\sigma_0 = 4.9505$ and $I^*F'(S^*) = 0.0795 < 2r(1 - 2S^*) = 0.1198$. The initial value is set to be $(0.9, 0.3, 0.2)$ and $\tau = 1 < \tau_0 = 2.0288$.

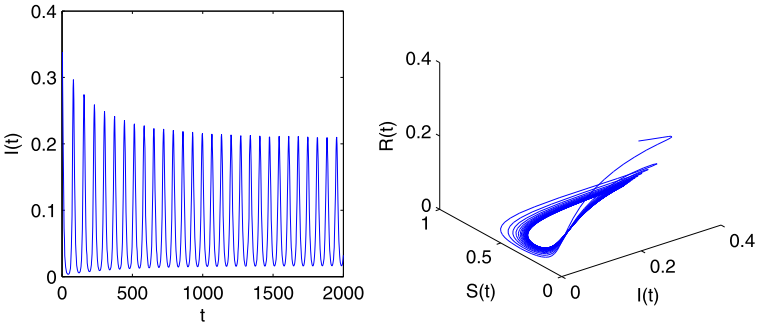

Figure 2 Temporal behavior of the infective individuals and corresponding three-dimensional phase for system (22) with $\sigma_0 = 4.9505$ and $I^*F'(S^*) = 0.0795 < 2r(1 - 2S^*) = 0.1198$. The initial value is set to be $(0.35, 0.3, 0.2)$ and $\tau = 2.5 > \tau_0 = 2.0288$.

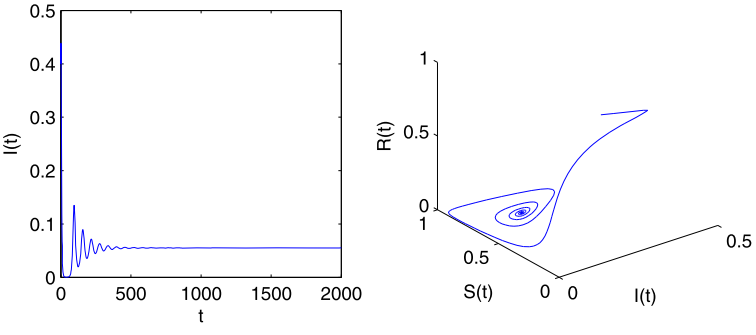

Figure 3 Plots of the infective individuals and corresponding three-dimensional phase for system (22) with $\sigma_0 = 2.2002$ and $I^*F'(S^*) = 0.0541 > 2r(1 - 2S^*) = 0.0192$. The initial value is set to be $(0.4, 0.3, 0.25)$ and $\tau = 2.5 > \tau_0 = 2.0288$. The values of parameters are as in Figure 1 but $\gamma = 0.35$.

of disease and the more general form of the nonlinear incidence rate is considered. The stability of the equilibria has been discussed by analyzing the roots of characteristic equations and applying the theory of asymptotic autonomous systems. It is shown that the trivial equilibrium is always unstable. The stability of the disease-free equilibrium is completely determined by the threshold parameter σ_0: the disease-free equilibrium is globally

asymptotically stable if $\sigma_0 < 1$ while it is unstable if $\sigma_0 > 1$. Moreover, if $\sigma_0 > 1$, there exists a unique endemic equilibrium. It is found that $I^*F'(S^*) = 2r(1 - 2S^*)$ is the condition which determines the absolute stability or conditional stability of the endemic equilibrium. To be specific, the endemic equilibrium is absolutely stable if $I^*F'(S^*) \geq 2r(1 - 2S^*)$ holds true, while it is conditionally stable if $I^*F'(S^*) < 2r(1 - 2S^*)$ is satisfied. Furthermore, there is a certain threshold time value τ_0 such that the endemic equilibrium is locally asymptotically stable when $0 < \tau < \tau_0$, whereas it is unstable when $\tau > \tau_0$. It is worth noting that, if $\sigma_0 > 1$ and $I^*F'(S^*) < 2r(1 - 2S^*)$, the system exhibits a Hopf bifurcation when the time delay τ crosses τ_n $(n = 0, 1, \ldots)$.

References [4, 5, 8, 10, 11] have discussed the delayed SIR vector disease models with nonlinear incidence functions. But the growth of the number of susceptible individuals is governed by a constant rate rather than the logistic function. They have proved that the endemic equilibrium is globally asymptotically stable for any delay and the model does not exhibit a Hopf bifurcation, which implies that the incubation delay does not cause any periodic oscillations. On the other hand, [6, 13, 19] have also investigated the delayed SIR vector disease models with the logistic growth of susceptible individuals. They have found that the endemic equilibrium is unstable and a Hopf bifurcation occurs under some conditions for some delays. For example, Wang *et al.* [13] investigated system (1) with the incidence function $F(S) = S$. They have proved if $R_0 > 3$, the endemic equilibrium is stable when the delay $\tau < \tau_0$ is satisfied, while the endemic equilibrium is unstable and the model undergoes Hopf bifurcation when $\tau = \tau_n$, $n = 0, 1, 2, \ldots$. Therefore, the logistic growth of susceptible individuals should be more responsible for the instability of the endemic equilibrium, and Hopf bifurcation may be the result of the logistic growth of susceptible individuals.

Wang *et al.* [13] analyzed system (1) for the incidence function $F(S) = S$. Zhang *et al.* [19] also formulated system (2) for the incidence function $F(S) = \frac{S}{1+\alpha S}$. As a matter of fact, two systems in the above-mentioned papers could be studied as special cases for system (3). It should be pointed out here that the threshold parameter σ_0 defined in the present paper is the same as R_0 derived in [13] and is equivalent to R_0 given in [19]. Furthermore, our results for the stability of equilibria extend the results in [13] and [19]. The numerical simulations performed further illustrate the theoretical results.

Competing interests
The author declares to have no competing interests.

Author's contributions
LL proposed the model and completed all the parts of this manuscript. LL read and approved the manuscript.

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