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On Burgers equation on a time-space scale

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Abstract
The Lax equation is introduced on a time-space scale. The viscous Burgers and the nonlinear Schrodinger dynamic equations on a time-space scale are deduced from the Lax equation by using the Ablowitz-Kaup-Newel-Segur-Ladik method. It is shown that the Burgers equation turns to the heat equation on a time-space scale by the Cole-Hopf transformation. Further, using the separation of variables, we deduce the formula for solutions of the boundary value problem for the heat and Burgers equation on a time-space scale in terms of Fourier series by Hilger exponential functions.

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1 Introduction

In 1968 Lax [1] introduced the linear operator equation equivalent to the nonlinear Korteweg-de Vries (KdV) equation that describes the traveling solitary waves. The importance of Lax's observation is that any equation that can be cast into such a framework may have remarkable properties of the KdV equation, including the integrability and an infinite number of local conservation laws. Lax equation may be used to generate other nonlinear dynamic equations with the properties mentioned above.

There are several other methods to generate the integrable hierarchy of nonlinear dynamic equations: Ablowitz-Kaup-Newel-Segur (AKNS) method [2], Gelfand-Dickey method [3], Ablowitz-Kaup-Newel-Segur-Ladik (AKNSL) method [4], which is the extension of AKNS method on difference equations. Other nonlinear dynamic equations are studied in [5–7].

In [8] Hilger introduced the time scale calculus that unifies continuous and discrete analysis.

The papers [9–12] are the first articles dedicated to KdV-like dynamic equations on time scales. In [9] the notion of regular-discrete time scale was introduced (see Section 6). Also in [9] some KdV-like equations on a regular-discrete space scale were deduced by using the Gelfand-Dickey method.

Let $\mathbb{T}$ and $\mathbb{X}$ be arbitrary nonempty closed subsets of real numbers. The sets $\mathbb{T}$, $\mathbb{X}$ are called the time and space scales correspondingly. The set $\mathbb{T} \times \mathbb{X} = \{(t,x), t \in \mathbb{T}, x \in \mathbb{X}\}$ we call the time-space scale.

In [13] soliton-like equations on a regular-discrete time-space scale were obtained by using the AKNSL method.

In this paper we extend the Lax matrix equation on a time-space scale dynamic system. From this equation we deduce the viscous Burgers (see (4.4) below) and the nonlinear Schrodinger dynamic equations on a time-space scale by using the AKNSL method.

We expect that this extension will give a wider range of integrable nonlinear dynamic equations that could be used in modeling.

We show that the Burgers equation on a time-space scale can be linearized by using the Cole-Hopf transformation. We also derive the formulas for solutions of the boundary value problem for the Burgers equation (see (4.26) and the heat equation on a time-space scale. These formulas are pretty simple and may be used to study the wave motion on a time-space scale.

2 Basic notations from the time scale calculus

For $t \in \mathbb{T}$ and $x \in \mathbb{X}$, we define backward jump operators $\sigma(t) : \mathbb{T} \to \mathbb{T}$, $\rho(x) : \mathbb{X} \to \mathbb{X}$

$$\sigma(t) := \sup\{s \in \mathbb{T} : s < t\}, \qquad \rho(x) := \sup\{y \in \mathbb{X} : y < x\}. \tag{2.1}$$

For $x \in \mathbb{X}$, we define the forward jump operator $\beta(x) : \mathbb{X} \to \mathbb{X}$ by

$$\beta(x) := \inf\{y \in \mathbb{X} : y > x\}. \tag{2.2}$$

The graininess functions $\mu(t) : \mathbb{T} \to [0,\infty)$, $\nu(x), \alpha(x) : \mathbb{X} \to [0,\infty)$ are defined as

$$\mu(t) = t - \sigma(t), \qquad \nu(x) = x - \rho(x), \qquad \alpha(x) = \beta(x) - x. \tag{2.3}$$

We are considering nabla time and space derivatives [14] instead of delta derivatives [8] since in physics the applications of nabla derivatives by time variable are casual.

If $\mathbb{T}$ has a right-scattered minimum m, define $T_\kappa := \mathbb{T} - \{m\}$; otherwise, set $T_\kappa = \mathbb{T}$.

For $v : \mathbb{T} \to \mathbb{R}$ and $t \in T_\kappa$, define the nabla derivative [14] of v at t, denoted by $v^{\nabla_t}(t)$, to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood U of t such that $|v(\rho(t)) - v(s) - v^{\nabla_t}(t)[\rho(t) - s]| \le \varepsilon|\rho(t) - s|$ for all $s \in U$. For $\mathbb{T} = \mathbb{R}$, we have $v^{\nabla_t}(t) = v_t(t)$, the usual derivative, and for $\mathbb{T} = \mathbb{Z}$, we have the backward difference operator $v^{\nabla_t}(t) := v(t) - v(t-1)$.

In the same way one can define the nabla derivative $v^{\nabla_x}(x)$ by space (x) variable.

We are going to use the following denotations:

$$w^{(t)}(t,x) := w^{\nabla_t}(t,x) = \lim_{s \to t} \frac{w(s) - w(\sigma(t))}{s - \sigma(t)}, \tag{2.4}$$

$$w^{(x)}(t,x) = w^{\nabla_x}(t,x) := w^{\nabla_x}(x) = \lim_{y \to x} \frac{w(y) - w(\rho(x))}{y - \rho(x)}, \tag{2.5}$$

where

$$f^\rho(t,x) := f\bigl(t, \rho(x)\bigr), \qquad f^\sigma(t,x) := f\bigl(\sigma(t), x\bigr).$$

In view of the assumption that $\mu(t)$ depends only on time variable and $\nu(x)$ depends only on space variable, we have

$$\mu^{(x)}(t) = \nu^{(t)}(x) = 0, \qquad f^{\sigma\rho}(t,x) = f^{\rho\sigma}(t,x) := f\big(\sigma(t),\rho(x)\big).$$

Lemma 2.1 *If the functions $f(t,x)$, $\mu(t)$, $\nu(x)$ are twice nabla differentiable, then*

$$f^{(tx)}(t,x) = f^{(xt)}(t,x), \qquad f^{\rho t}(t,x) = f^{t\rho}(t,x). \tag{2.6}$$

For the proof of this lemma, see [13].

Frequently we are going to use the product and the quotient rules (see [14])

$$\begin{aligned}
&\big(f(t,x)g(t,x)\big)^{(t)} = f^{(t)}(t,x)g(t,x) + f^{\sigma}(t,x)g^{(t)}(t,x),\\
&\left(\frac{f(t,x)}{g(t,x)}\right)^{(t)} = \frac{f^{(t)}(t,x)g(t,x) - f(t,x)g^{(t)}(t,x)}{g^{\sigma}(t,x)g(t,x)},\\
&\big(f(t,x)g(t,x)\big)^{(x)} = f^{(x)}(t,x)g(t,x) + f^{\rho}(t,x)g^{(x)}(t,x),\\
&\left(\frac{f(t,x)}{g(t,x)}\right)^{(x)} = \frac{f^{(x)}(t,x)g(t,x) - f(t,x)g^{(x)}(t,x)}{g^{\rho}(t,x)g(t,x)}.
\end{aligned} \tag{2.7}$$

We say that a function $\theta(\cdot,x) : \mathbb{T} \to \mathbb{R}$ is t-regressive provided $\theta(\cdot,x)$ is ld-continuous and $1 - \mu(t)\theta(t,x) \neq 0$ holds for all $t \in \mathbb{T}$ and all $x \in \mathbb{X}$. We say that a function $\theta(t,\cdot) : \mathbb{X} \to \mathbb{R}$ is x-regressive provided $\theta(t,\cdot)$ is ld-continuous and $1 - \nu(x)\theta(t,x) \neq 0$ holds for all $t \in \mathbb{T}$ and all $x \in \mathbb{X}$.

If $\theta(t,x)$ is t-regressive, the nabla t-exponential function $\hat{e}_\theta(t,t_0,x)$ on a time scale $\mathbb{T}$ can be defined as the unique solution of the initial value problem (see [14, 15])

$$\hat{e}^{(t)}_\theta(t,t_0,x) = \theta(t,x)\hat{e}_\theta(t,t_0,x), \qquad \hat{e}_\theta(t,x_0,x_0) = 1. \tag{2.8}$$

If $\theta(t,x)$ is x-regressive, the nabla x-exponential function $\hat{e}_\theta(t,x,x_0)$ on a space scale $\mathbb{X}$ is defined similarly as the unique solution of the initial value problem

$$\hat{e}^{(x)}_\theta(t,x,x_0) = \theta(t,x)\hat{e}_\theta(t,x,x_0), \qquad \hat{e}_\theta(t,x_0,x_0) = 1. \tag{2.9}$$

Note that

$$f^{\rho}(t,x) = f(t,x) - \nu(x)f^{(x)}(t,x), \qquad f^{\sigma}(t,x) = f(t,x) - \mu(t)\nu^{(t)}(t,x). \tag{2.10}$$

3 Lax equation

Consider the nabla dynamic systems

$$\nu^{(x)}(t,x) = M(t,x)\nu(t,x), \qquad M(t,x) = \begin{pmatrix} M_{11}(t,x,z) & M_{12}(t,x) \\ M_{21}(t,x) & M_{22}(t,x,z) \end{pmatrix}, \tag{3.1}$$

$$\nu^{(t)}(t,x) = N(t,x)\nu(t,x), \qquad N(t,x) = \begin{pmatrix} A(t,x) & B(t,x) \\ C(t,x) & D(t,x) \end{pmatrix}, \tag{3.2}$$

where $v(t,x) = \binom{v_1(t,x)}{v_2(t,x)}$ and $M_{kj}(t,x,z)$, $k,j = 1,2$, are the functions that may depend on the spectral parameter z as well.

We derive the Lax equation as the compatibility condition of two linear dynamic systems (3.1), (3.2).

From (3.1), (3.2) by differentiation and usage of the product rule we get

$$\begin{aligned} v^{(xt)} &= M^{(t)}v + M^{\sigma}v^{(t)} = \bigl(M^{(t)}v + M^{\sigma}N\bigr)v, \\ v^{(tx)} &= N^{(x)}v + N^{\rho}v^{(x)} = \bigl(N^{(x)}v + N^{\rho}M\bigr)v. \end{aligned}$$

Here and further we often suppress (t,x) or (t,x,z) variables to shorten the formulas.

By equating the mixed derivatives $v^{(xt)} = v^{(tx)}$, we get the Lax matrix equation

$$N^{(x)}(t,x) + N^{\rho}(t,x)M(t,x,z) = M^{(t)}(t,x,z) + M^{\sigma}(t,x,z)N(t,x), \tag{3.3}$$

or in component form

$$\begin{aligned} A^{(x)} + A^{\rho}M_{11} + B^{\rho}M_{21} &= M_{11}^{(t)} + M_{11}^{\sigma}A + M_{12}^{\sigma}C, \\ C^{(x)} + C^{\rho}M_{11} + D^{\rho}M_{21} &= M_{21}^{(t)} + M_{21}^{\sigma}A + M_{22}^{\sigma}C, \\ B^{(x)} + A^{\rho}M_{12} + B^{\rho}M_{22} &= M_{12}^{(t)} + M_{11}^{\sigma}B + M_{12}^{\sigma}D, \\ D^{(x)} + C^{\rho}M_{12} + D^{\rho}M_{22} &= M_{22}^{(t)} + M_{21}^{\sigma}B + M_{22}^{\sigma}D. \end{aligned} \tag{3.4}$$

In view of $A^{\rho} = A - \nu(x)A^{(x)}$, $A^{\sigma} = A - \mu(t)A^{(t)}$, we get

$$\begin{aligned} &A^{(x)}\bigl(1 - \nu(x)M_{11}\bigr) + B^{\rho}M_{21} - M_{12}^{\sigma}C = M_{11}^{(t)}\bigl(1 - \mu(t)A\bigr), \\ &D^{(x)}\bigl(1 - \nu(x)M_{22}\bigr) + C^{\rho}M_{12} - M_{21}^{\sigma}B = M_{22}^{(t)}\bigl(1 - \mu(t)D\bigr), \\ &C^{(x)}\bigl(1 - \nu(x)M_{11}\bigr) + C\bigl(M_{11} - M_{22}^{\sigma}\bigr) + \bigl(D^{\rho} - A\bigr)M_{21} = M_{21}^{(t)}\bigl(1 - \mu(t)A\bigr), \\ &B^{(x)}\bigl(1 - \nu(x)M_{22}\bigr) + B\bigl(M_{22} - M_{11}^{\sigma}\bigr) + \bigl(A^{\rho} - D\bigr)M_{12} = M_{12}^{(t)}\bigl(1 - \mu(t)D\bigr). \end{aligned}$$

Choosing

$$M_{12}(t,x,z) = Q(t,x), \qquad M_{21}(t,x,z) = R(t,x),$$

we have

$$\begin{aligned} &A^{(x)}\bigl(1 - \nu(x)M_{11}\bigr) + B^{\rho}R - Q^{\sigma}C = M_{11}^{(t)}\bigl(1 - \mu(t)A\bigr), \\ &D^{(x)}\bigl(1 - \nu(x)M_{22}\bigr) + C^{\rho}Q - R^{\sigma}B = M_{22}^{(t)}\bigl(1 - \mu(t)D\bigr), \\ &C^{(x)}\bigl(1 - \nu(x)M_{11}\bigr) + C\bigl(M_{11} - M_{22}^{\sigma}\bigr) + \bigl(D^{\rho} - A\bigr)R = R^{(t)}\bigl(1 - \mu(t)A\bigr), \\ &B^{(x)}\bigl(1 - \nu(x)M_{22}\bigr) + B\bigl(M_{22} - M_{11}^{\sigma}\bigr) + \bigl(A^{\rho} - D\bigr)Q = Q^{(t)}\bigl(1 - \mu(t)D\bigr). \end{aligned} \tag{3.5}$$

From this system one can derive numerous nonlinear equations with respect to the functions $Q(t,x)$, $R(t,x)$ that may have the properties of Korteweg-de Vries equation (independence of the spectral parameter on time, infinite number of conservation laws, integrability) by taking the spectral expansions $N = \sum_{k=-m}^{m} N_k z^k$.

In a continuous time scale $\mu(t) = 0$, $M^\sigma(t,x) = M(t,x)$ system (3.5) is simplified as follows:

$$\begin{aligned} &A^{(x)}(1-\nu M_{11}) = M_{11t} + QC - B^\rho R, \\ &D^{(x)}(1-\nu M_{22}) = M_{22t} + RB - QC^\rho, \\ &C(M_{11}-M_{22}) = R\bigl(A-D^\rho\bigr) - C^{(x)}(1-\nu M_{11}) + R^{(t)}, \\ &B(M_{11}-M_{22}) = B^{(x)}(1-\nu M_{22}) - Q\bigl(D-A^\rho\bigr) - Q^{(t)}. \end{aligned} \tag{3.6}$$

If both time scale and space scale are continuous, that is, $\mathbb{T} = \mathbb{R}$, $\mathbb{X} = \mathbb{R}$, $\nu(x) = \mu(t) \equiv 0$, system (3.6) is further simplified, and we get the system of Lax equations introduced in [2]:

$$\begin{aligned} &A_x = M_{11t} + QC - BR, \qquad D_x = M_{22t} + RB - QC, \\ &C(M_{11}-M_{22}) = R(A-D) - C_x + R_t, \qquad B(M_{11}-M_{22}) = Q(A-D) + B_x - Q_t. \end{aligned}$$

4 Burgers equation on a time-space scale

The viscous Burgers equation occurs in mathematical models of gas dynamics, traffic flow, the flow through a shock wave traveling in a viscous fluid, and some probabilistic models [16, 17].

To derive a time-space scale version of the Burgers equation, consider the Lax equation (3.5) in the scalar case ($A = A_0$, $B = C = D = R = Q = M_{22} \equiv 0$)

$$A_0^{(x)}(t,x)\bigl(1-\nu(x)M_{11}(t,x,z)\bigr) = M_{11}^{(t)}(t,x,z)\bigl(1-\mu(t)A_0(t,x)\bigr). \tag{4.1}$$

By choosing

$$M_{11}(t,x,z) = F_1(t,x)z + F_0(t,x),$$

we get

$$\begin{aligned} &A_0^{(x)}(t,x)\bigl(1-\nu(x)F_1(t,x)z - \nu(x)F_0(t,x)\bigr) \\ &\quad = \bigl(F_1^{(t)}(t,x)z + F_0^{(t)}(t,x)\bigr)\bigl(1-\mu(t)A_0(t,x)\bigr), \end{aligned} \tag{4.2}$$

or, assuming that z is an arbitrary spectral parameter that does not depend on time t, we get

$$A_0^{(x)}\bigl(1-\nu(x)F_0\bigr) = F_0^{(t)}\bigl(1-\mu(t)A_0\bigr), \qquad -A_0^{(x)}\nu(x)F_1 = F_1^{(t)}\bigl(1-\mu(t)A_0\bigr),$$

or

$$\frac{\nu(x)A_0^{(x)}}{(1-\mu(t)A_0)} = \frac{\nu(x)F_0^{(t)}}{1-\nu F_0} = -\frac{F_1^{(t)}}{F_1}, \qquad F_1^{(t)}(1-\nu F_0) + \nu F_0^{(t)}F_1 = 0.$$

Using the quotient differentiation rule on a time scale

$$\left(\frac{F_0}{F_1}\right)^{(t)} = \frac{F_0^{(t)}F_1 - F_1^{(t)}F_0}{F_1^\sigma F_1},$$

we get

$$F_1^{(t)} + \nu(x)\big(F_0^{(t)}F_1 - F_1^{(t)}F_0\big) = 0, \qquad \frac{F_1^{(t)}}{F_1^\sigma F_1} + \nu(x)\left(\frac{F_0}{F_1}\right)^{(t)} = 0,$$
$$\left(\frac{\nu(x)F_0}{F_1} - \frac{1}{F_1}\right)^{(t)} = 0, \qquad \frac{\nu(x)F_0 - 1}{F_1} = C = \text{constant},$$
$$F_0^{(t)} = \frac{1-\nu(x)F_0}{1-\mu(t)A_0}A_0^{(x)}.$$

Choosing

$$F_0(t,x) := \frac{F(t,x)}{p^2}, \qquad A_0 = \frac{A}{p^2},$$
$$A(t,x) := F^2(t,x) + \big(p^2 - \nu(x)F(t,x)\big)F^{(x)}(t,x), \tag{4.3}$$

we get the Burgers equation on a time-space scale

$$F^{(t)}(t,x) = \frac{p^2 - \nu(x)F(t,x)}{p^2 - \mu(t)A(t,x)}A^{(x)}(t,x), \tag{4.4}$$

where $A(t,x)$ is given in (4.3) and p^2 is a viscosity constant coefficient.

In a continuous time scale $\mathbb{T} = \mathbb{R}$, $\mu(t) \equiv 0$, we get the equation

$$F_t(t,x) = \big(1 - \nu F(t,x)/p^2\big)\big(F^2(t,x) + \big(p^2 - \nu(x)F(t,x)\big)F^{(x)}(t,x)\big)^{(x)}. \tag{4.5}$$

Note that the Burgers equation (4.5) on a space scale first was introduced in [9].

If both time scale and space scale are continuous, that is, $\mathbb{T} = \mathbb{R}$, $\mathbb{X} = \mathbb{R}$, $\nu(x) = \mu(t) \equiv 0$, (4.4) turns to the classical Burgers equation [16]

$$F_t(t,x) = \big(F^2(t,x) + p^2F_x(t,x)\big)_x. \tag{4.6}$$

To linearize (4.4), consider the Cole-Hopf transformation

$$F(t,x) = \frac{p^2 v^{(x)}(t,x)}{v(t,x)}. \tag{4.7}$$

By using the quotient rule we have

$$F^{(x)} = \frac{v^{(xx)}v - v^{(x)}v^{(x)}}{v^\rho v}p^2, \qquad F^{(t)} = \frac{v^{(xt)}v - v^{(x)}v^{(t)}}{v^\sigma v}p^2,$$
$$1 - \frac{\nu(x)}{p^2}F = 1 - \nu(x)\frac{v^x}{v} = \frac{v^\rho}{v},$$
$$A = F^2 + \big(p^2 - \nu(x)F\big)F^{(x)} = \frac{(p^2v^{(x)})^2}{v^2} + \frac{p^4v^\rho}{v}\cdot\frac{v^{(xx)}v - v^{(x)}v^{(x)}}{v^\rho v} = \frac{p^4v^{(xx)}}{v},$$

and if

$$v^t(t,x) = p^2v^{(xx)}(t,x), \tag{4.8}$$

then

$$A = \frac{p^4 v^{(xx)}}{v} = \frac{p^2 v^{(t)}}{v}, \qquad 1 - \frac{\mu}{p^2}A = 1 - \mu\frac{v^{(t)}}{v} = \frac{v^\sigma}{v},$$

and in view of

$$v^\sigma\left(\frac{v^{(x)}}{v}\right)^{(t)} = v^\rho\left(\frac{v^{(t)}}{v}\right)^{(x)},$$

equation (4.4)

$$F^{(t)} = \frac{v^\rho}{v^\sigma}A^{(x)}$$

turns to the identity

$$\left(\frac{p^2 v^{(x)}}{v}\right)^{(t)} = \frac{v^\rho}{v^\sigma}\left(\frac{p^2 v^{(t)}}{v}\right)^{(x)}.$$

So if $v(t,x)$ satisfies the heat equation (4.8), then $F(t,x) = \frac{p^2 v^{(x)}(t,x)}{v(t,x)}$ satisfies the Burgers equation (4.4).

Consider the initial value problem for the heat equation on a time-space scale

$$v^{(t)}(t,x) = p^2 v^{(xx)}(t,x), \qquad v(t_0,x) = \varphi(x), \quad x \in \mathbb{X}, t, t_0 \in \mathbb{T}. \tag{4.9}$$

One can solve (4.9) by looking for a solution in a product form (the separation or Fourier method [18])

$$v(t,x) = T(t)X(x).$$

By the substitution in (4.8) and the separation of variables, we get

$$\frac{T^{(t)}(t)}{T(t)} = \frac{p^2 X^{(xx)}(x)}{X(x)} = -\lambda^2 = \text{const},$$
$$T(t) = \hat{e}_{-\lambda^2}(t,t_0), \qquad X(x) = K(\lambda)\hat{e}_{i\lambda/p}(x,x_0) + K_1(\lambda)\hat{e}_{-i\lambda/p}(x,x_0),$$

where $K(\lambda)$, $K_1(\lambda)$ are independent of (t,x), $\hat{e}_{-\lambda^2}(t,t_0)$ and $\hat{e}_{\pm i\lambda/p}(x,x_0)$ are the nabla exponential functions on $\mathbb{T}$ and $\mathbb{X}$ correspondingly.

Thus $v(t,x,\lambda) = \hat{e}_{-\lambda^2}(t,t_0)K(\lambda)\hat{e}_{i\lambda/p}(x,x_0)$ satisfies (4.8).

Since equation (4.8) is linear, it satisfies the superposition principle, that is, a linear combination of solutions is a solution (see [14]).

By the superposition principle the solution of (4.8) may be written in the form (see [18])

$$v(t,x) = \int_{-\infty}^{\infty} K(\lambda)\hat{e}_{-\lambda^2}(t,t_0)\hat{e}_{i\lambda/p}(x,x_0)\,d\lambda. \tag{4.10}$$

From the initial condition $v(t_0,x) = \varphi(x)$ we get

$$\varphi(x) = \int_{-\infty}^{\infty} K(\lambda)\hat{e}_{i\lambda/p}(x,x_0)\,d\lambda. \tag{4.11}$$

So the formal solution of the initial value problem (4.9) is given by (4.10), where $K(\lambda)$ is the solution of the integral equation (4.11).

Consider the initial value problem for Burgers equation

$$F^{(t)}(t,x) = \frac{p^2 - v(x)F(t,x)}{p^2 - \mu(t)A(t,x)} A^{(x)}(t,x), \qquad F(t_0,x) = f(x), \tag{4.12}$$

where $A(t,x) = F^2(t,x) + (p^2 - v(x)F(t,x))F^{(x)}(t,x)$.

From the initial condition

$$f(x) = F(t_0,x) = \frac{p^2 v^{(x)}(t_0,x)}{v(t_0,x)} = \frac{p^2 \varphi^{(x)}(x)}{\varphi(x)}$$

we get

$$\varphi(x) = \hat{e}_{f/p^2}(x,x_1), \quad x_1 \in \mathbb{X}.$$

By substitution (4.10) into (4.7) we obtain the representation of solutions of the Burgers equation on a time-space scale

$$F(t,x) = \frac{\int_{-\infty}^{\infty} i\lambda p K(\lambda)\hat{e}_{-\lambda^2}(t,t_0)\hat{e}_{i\lambda/p}(x,x_0)\,d\lambda}{\int_{-\infty}^{\infty} K(\lambda)\hat{e}_{-\lambda^2}(t,t_0)\hat{e}_{i\lambda}(x,x_0)\,d\lambda}, \tag{4.13}$$

where $K(\lambda)$ may be found by inversion of the Fourier transformation

$$\hat{e}_{f/p^2}(x,x_1) = \int_{-\infty}^{\infty} K(\lambda)\hat{e}_{i\lambda/p}(x,x_0)\,d\lambda. \tag{4.14}$$

Note that the inversion of a Fourier transformation on some time scales was studied in [19, 20], but there is no inversion formula for an arbitrary time (space) scale.

Consider the boundary value problem

$$v(t_0,x) = \varphi(x), \qquad v^{(x)}(t,x_0) = v^{(x)}(t,b) = 0, \quad x_0, b, x \in \mathbb{X}, t_0, t \in \mathbb{T}, \tag{4.15}$$

for heat equation (4.8) on a time-space scale.

Introducing Bohner-Peterson's trigonometric functions on a space scale $\mathbb{X}$ (see [14, 19, 21])

$$\sin_{\lambda/p}(x,x_0) = \frac{\hat{e}_{i\lambda/p}(x,x_0) - \hat{e}_{-i\lambda/p}(x,x_0)}{2i},$$
$$\cos_{\lambda/p}(x,x_0) = \frac{\hat{e}_{i\lambda/p}(x,x_0) + \hat{e}_{-i\lambda/p}(x,x_0)}{2},$$

and using the method of separation of variables $v(t,x) = T(t)X(x)$, one can rewrite the solutions of $p^2 X^{xx} + \lambda^2 X(x) = 0$ in the form

$$X(x) = C_3 \sin_{\lambda/p}(x,x_0) + C_4 \cos_{\lambda/p}(x,x_0), \quad p, \lambda \in \mathbb{R}. \tag{4.16}$$

From the boundary conditions (4.15) we get

$$C_3 = 0, \qquad \sin_{\lambda/p}(b,x_0) = 0. \tag{4.17}$$

In the case $\nu(x) = 0$ the boundary condition turns to

$$\sin\bigl(\lambda(b - x_0)/p\bigr) = 0,$$

which is satisfied if

$$\lambda = \Lambda_k := \frac{-\pi k}{b - x_0}, \quad k \in \mathbb{Z}, \nu(x) = 0, \tag{4.18}$$

and the solution of (4.8), (4.15) is given by the classical Fourier series formula

$$v(t,x) = \sum_{k=0}^{\infty} B_k \cos\bigl(\Lambda_k(x - x_0)/p\bigr)\hat{e}_{-\Lambda_k^2}(t,t_0), \tag{4.19}$$

$$\begin{aligned} B_0 &= \frac{1}{b - x_0}\int_{x_0}^{b} v(t_0,x)\,dx, \\ B_m &= \frac{2}{b - x_0}\int_{x_0}^{b} v(t_0,x)\cos\bigl(\Lambda_m(x - x_0)\bigr)\,dx, \quad m = 1,2,\ldots. \end{aligned} \tag{4.20}$$

Lemma 4.1 *If* $\nu(x) > 0$ *and*

$$\begin{aligned} &\lambda_k = -\frac{p\tan(K)}{\nu(x)} = \frac{ip(e^{2iK} - 1)}{\nu(e^{2iK} + 1)}, \\ &K = \frac{k\pi\nu}{b - x_0}, \qquad a = -\frac{2ie^{iK}}{\nu(x)}\sin(K), \quad k \in \mathbb{Z}, \end{aligned} \tag{4.21}$$

then

$$\hat{e}_a(b,x_0) = 1, \qquad \sin_{\lambda_k/p}(b,x_0) = 0, \qquad \int_{x_0}^{b} \hat{e}_a(x,x_0)\nabla x = 0. \tag{4.22}$$

Note that using L'Hospital's rule one can prove

$$\lim_{\nu\to 0} \lambda_k = \Lambda_k = \frac{-\pi k}{b - x_0}, \qquad \lim_{\nu\to 0} a = \frac{-2ik\pi}{b - x_0}, \qquad \lim_{\nu\to 0} \hat{e}_a(b,x_0) = e^{a(b-x_0)} = 1.$$

Proof Indeed, if $\nu > 0$, we have

$$\begin{aligned} a &= -\frac{2ie^{iK}}{\nu(x)}\sin(K) = \frac{1 - e^{2iK}}{\nu(x)}, \\ \hat{e}_a(b,x_0) &= \exp\int_{x_0}^{b} \lim_{q\searrow\nu(y)} \frac{\mathrm{Log}(1 - aq)}{-q}\nabla y = \exp\int_{x_0}^{b} \frac{\mathrm{Log}(1 - a\nu(y))}{-\nu(y)}\nabla y \\ &= \exp\int_{x_0}^{b} \frac{2iK\nabla y}{-\nu(y)} = \exp\biggl(\int_{x_0}^{b} \frac{-2ik\pi}{b - x_0}\nabla y\biggr)\nabla x = \exp(-2ki\pi) = 1, \end{aligned}$$

and since

$$\frac{2i\lambda_k}{p + i\lambda_k\nu} = \frac{1 - e^{2iK}}{\nu} = a,$$

we get

$$\frac{\hat{e}_{i\lambda_k/p}(b,x_0)}{\hat{e}_{-i\lambda_k/p}(b,x_0)} = \hat{e}_{2i\lambda_k/(p+i\lambda_k\nu)}(b,x_0) = \hat{e}_a(b,x_0) = 1,$$

and

$$\sin_{\lambda_k/p}(b,x_0) = \frac{\hat{e}_{i\lambda_k/p}(b,x_0) - \hat{e}_{-i\lambda_k/p}(b,x_0)}{2i} = 0,$$
$$\int_{x_0}^{b} \hat{e}_a(x,x_0)\nabla x = \frac{\hat{e}_a(x,x_0)}{a}\Big|_{x_0}^{b} = \frac{\hat{e}_a(b,x_0)-1}{a} = 0. \qquad \square$$

Remark 4.1 To obtain the formula for coefficients B_k on an arbitrary space scale similar to the remarkable formula (4.20) on a continuous space scale, it would be interesting to extend the orthogonality of the exponential functions on the unit circle to Hilger exponential functions, but we do not think it is possible.

Indeed, if $M = \frac{m\pi\nu}{b-x_0}$, we have on an arbitrary space scale ($\nu(x) > 0$)

$$\frac{1}{b-x_0}\int_{x_0}^{b} \hat{e}_{(1-e^{2i(K-M)})/\nu}(x,x_0)\nabla x = \delta_{km} = \begin{cases} 1, & k = m, \\ 0, & k \neq m, \end{cases}$$

but this property does not imply the orthogonality of Hilger exponents as we have for usual exponential functions $e^{2ik\pi(x-x_0)/(b-x_0)}$, $e^{-2im\pi(x-x_0)/(b-x_0)}$:

$$\frac{1}{b-x_0}\int_{x_0}^{b} e^{2ik\pi(x-x_0)/(b-x_0)} e^{-2im\pi(x-x_0)/(b-x_0)}\,dx = \delta_{km}, \quad k,m \in \mathbb{Z}.$$

So the second condition (4.17) is satisfied if the eigenvalues λ_k are chosen as in (4.21), and the formal solution of the boundary value problem (4.8), (4.15) is given by Fourier series formula with Bohner-Peterson's trigonometric functions

$$\nu(t,x) = \sum_{k=0}^{\infty} B_k \cos_{\lambda_k/p}(x,x_0)\hat{e}_{-\lambda_k^2}(t,t_0), \tag{4.23}$$

and B_k could be found from the initial condition

$$\varphi(x) = \sum_{k=0}^{\infty} B_k \cos_{\lambda_k/p}(x,x_0). \tag{4.24}$$

Consider the boundary value problem

$$F(t_0,x) = f(x), \qquad F(t,x_0) = F(t,b) = 0, \quad t,t_0 \in \mathbb{T}, x,x_0 \in \mathbb{X} \tag{4.25}$$

for the Burgers equation (4.4).

By substitution (4.23) into (4.7) we get the representation of solutions of boundary value problem (4.4), (4.25)

$$F(t,x) = -\frac{p\sum_{k=0}^{\infty}\lambda_k B_k \sin_{\lambda_k/p}(x,x_0)\hat{e}_{-\lambda_k^2}(t,t_0)}{\sum_{k=0}^{\infty} B_k \cos_{\lambda_k/p}(x,x_0)\hat{e}_{-\lambda_k^2}(t,t_0)}, \tag{4.26}$$

where the numbers B_k may be found from the initial condition

$$\hat{e}_{f/p^2}(x,x_0) = \sum_{k=0}^{\infty} B_k \cos_{\lambda_k/p}(x,x_0), \tag{4.27}$$

since by taking $t = t_0$ in (4.7), we get

$$f(x) = \frac{p^2\varphi^{(x)}(x)}{\varphi(x)}, \quad \varphi(x) = \hat{e}_{f/p^2}(x,x_0).$$

Example 4.1 Consider the case $\mathbb{X} = h\mathbb{Z}$. From (4.21) we get

$$\lambda = \lambda_k = -\frac{p}{h}\tan\left(\frac{kh\pi}{b-x_0}\right), \quad k = 0,1,2,\ldots,N.$$

Since $b-x_0 = Nh$, where N is some natural number, we have the finite numbers of different λ_k, that is, $0 \le k < N$.

By taking $x = x_m$, $m = 0,1,\ldots,N-1$, we get the linear system with respect to unknown numbers B_k

$$\varphi(x_m) = \sum_{k=0}^{N-1} B_k \cos_{\lambda_k/p}(x_m,x_0), \quad m = 0,1,2,\ldots,N-1.$$

Remark 4.2 The formal formulas (4.10), (4.13), (4.22), (4.26) have sense if the Fourier integrals or series are convergent. In the case $\mathbb{X} = n\mathbb{Z}$ the sums in (4.23) and (4.26) are finite and there is no need to prove the convergence.

It would be interesting to figure out for which class of functions $\varphi(x)$ the series (4.24), (4.23) are convergent for an arbitrary space scale. This could be a topic of a separate paper.

Note that to prove the convergence similar to the classical harmonic analysis in an arbitrary space scale, one needs to use the following properties:

(1) boundedness of the functions $\cos(\lambda_k(x-x_0)/p)$, $\hat{e}_{-\lambda_k^2}(t,t_0)$.

(2) $\lim_{k\to\infty}\lambda_k = \infty$.

(3) The space scale analogue of formula (4.20).

Note that property (1) may be proved, but property (2) is not true since from (4.21) $\lim_{k\to\infty}\lambda_k$ does not exist in the case $\nu(x) > 0$.

To compare with the classical case, consider the continuous space scale $\mathbb{X} = \mathbb{R}$. In this case if $\varphi(x) \in L_1(\mathbb{R})$ and $\varphi(x)$ is continuous on $\mathbb{R}$, (4.11) turns to the usual Fourier transformation, and using the inverse transformation we get from (4.11)

$$K(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\varphi(y)e^{-\frac{i\lambda}{p}(y-x_0)}\,dy = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\frac{1}{p^2}\int_{x_1}^{y} f(x')\,dx' - \frac{i\lambda}{p}(y-x_0)}\,dy. \tag{4.28}$$

So in the continuous space scale by substitution (4.28) into (4.10) we get the following formula for the solution of (4.9):

$$v(t,x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\varphi(y)\int_{-\infty}^{\infty} e^{\frac{i\lambda}{p}(x-y)}\hat{e}_{-\lambda^2}(t,t_0)\,d\lambda\,dy, \quad t\in\mathbb{T}, x\in\mathbb{R}, \tag{4.29}$$

and the formula for a solution of (4.12)

$$F(t,x) = p^2 \frac{\partial}{\partial x} \ln\left(\int_{-\infty}^{\infty} \exp\left(\frac{1}{p^2}\int_{x_0}^{y} f(x')\,dx'\right)\left(\int_{-\infty}^{\infty} e^{\frac{i\lambda}{p}(x-y)}\hat{e}_{-\lambda^2}(t,t_0)\,d\lambda\right)dy\right), \quad (4.30)$$

where $t \in \mathbb{T}$, $x \in \mathbb{R}$.

If both time and space scales are continuous, that is, $\mathbb{T} = \mathbb{R}$, $\mathbb{X} = \mathbb{R}$, formula (4.29) may be further simplified (we choose here $t_0 = 0$) as follows:

$$v(t,x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \varphi(y)\left(\int_{-\infty}^{\infty} e^{\frac{i\lambda}{p}(x-y)-t\lambda^2}\,d\lambda\right)dy.$$

In view of

$$\int_{-\infty}^{\infty} e^{\frac{i\lambda}{p}(x-y)-t\lambda^2}\,d\lambda = \sqrt{\frac{\pi}{t}}\exp\left(-\frac{(x-y)^2}{4p^2t}\right),$$

we get the Poisson formula for a solution of the initial value problem (4.9)

$$v(t,x) = \frac{1}{2\sqrt{t\pi}}\int_{-\infty}^{\infty} \varphi(y)\exp\left(-\frac{(x-y)^2}{4p^2t}\right)dy.$$

Furthermore, one can get the well-known formula for the solution of initial value problem for the classical viscous Burgers equation

$$F(t,x) = p^2 \frac{\partial}{\partial x} \ln\left(\frac{1}{2\sqrt{t\pi}}\int_{-\infty}^{\infty} \exp\left(\frac{1}{p^2}\int_{x_0}^{y} f(x')\,dx' - \frac{(x-y)^2}{4p^2t}\right)dy\right),$$

or

$$F(t,x) = p^2 \frac{\partial}{\partial x} \ln\left(\int_{-\infty}^{\infty} \exp\left(\frac{1}{p^2}\int_{x_0}^{y} f(x')\,dx' - \frac{(x-y)^2}{4p^2t}\right)dy\right), \quad t \in \mathbb{R}, x \in \mathbb{R}. \quad (4.31)$$

Choosing $f(x) = \operatorname{sech}^2(x)$, $x_0 = 0$, in view of

$$\int_0^y \operatorname{sech}^2(x')\,dx' = \tanh(y),$$

we get

$$\begin{aligned} F(t,x) &= p^2 \frac{\partial}{\partial x} \ln\left(\int_{-\infty}^{\infty} \exp\left(\frac{1}{p^2}\tanh(y) - \frac{(x-y)^2}{4p^2t}\right)dy\right) \\ F(t,x) &= \frac{1}{2t}\left(\frac{\int_{-\infty}^{\infty} y\exp(\frac{1}{p^2}\tanh(y) - \frac{(x-y)^2}{4p^2t})\,dy}{\int_{-\infty}^{\infty} \exp(\frac{1}{p^2}\tanh(y) - \frac{(x-y)^2}{4p^2t})\,dy} - x\right). \end{aligned} \quad (4.32)$$

Note that to study the behavior of solutions of the Burgers equation, one may visualize the graphs of (4.32).

5 Nonlinear Schrodinger equation on a space scale

Choose in (3.3)

$$\begin{aligned} &M_{11}(t,x,z) = F_1(t,x)z + F_0(t,x), \qquad M_{22}(t,x,z) = G_1(t,x)z + G_0(t,x), \quad z^{(t)} = 0, \\ &M_{1t}(t,x,z) = F_{1t}(t,x)z + F_{0t}(t,x), \qquad M_{4t}(t,x,z) = G_{1t}(t,x) + zG_{0t}(t,x), \end{aligned} \tag{5.1}$$

and consider the spectral expansion

$$\begin{aligned} &A(t,x) = A_2(t,x)z^2 + A_1(t,x)z + A_0(t,x), \qquad B(t,x) = B_1(t,x)z + B_0(t,x), \\ &C(t,x) = C_1(t,x)z + C_0(t,x), \qquad D(t,x) = D_2(t,x)z^2 + D_1(t,x)z + D_0(t,x). \end{aligned} \tag{5.2}$$

Denoting

$$\begin{aligned} &\tilde{A}_j = A_j(t,x) - D_j^\rho(t,x), \qquad \tilde{D}_j = D_j(t,x) - A_j^\rho(t,x), \quad j = 0,1,2, \\ &\tilde{F}_j = F_j(t,x) - G_j(t,x), \quad j = 0,1, \end{aligned}$$

we get from (3.5)

$$\begin{aligned} &\bigl(A_2z^2 + A_1z + A_0\bigr)^{(x)}(1 - \nu F_1z - \nu F_0) \\ &\quad = \bigl(F_1^{(t)}z + F_0^{(t)}\bigr)\bigl(1 - \mu A_2z^2 - \mu A_1z - \mu A_0\bigr) + Q^\sigma(C_1z + C_0) - R(B_1z + B_0)^\rho, \\ &\bigl(D_2z^2 + D_1z + D_0\bigr)^{(x)}(1 - \nu G_1z - \nu G_0) \\ &\quad = \bigl(G_1^{(t)}z + G_0^{(t)}\bigr)\bigl(1 - \mu D_2z^2 - \mu D_1z - \mu D_0\bigr) + R^\sigma(B_1z + B_0) - Q(C_1z + C_0)^\rho, \\ &(C_1z + C_0)^{(x)}(1 - \nu F_1z - \nu F_0) + (C_1z + C_0)\bigl[\bigl(F_1 - G_1^\sigma\bigr)z + F_0 - G_0^\sigma\bigr] \\ &\quad = R^{(t)}\bigl(1 - \mu A_2z^2 - \mu A_1z - \mu A_0\bigr) - R\bigl[\bigl(D_2^\rho - A_2\bigr)z^2 + \bigl(D_1^\rho - A_1\bigr)z + D_0^\rho - A_0\bigr], \\ &(B_1z + B_0)^{(x)}(1 - \nu G_1z - \nu G_0) + (B_1z + B_0)\bigl[\bigl(G_1 - F_1^\sigma\bigr)z + G_0 - F_0^\sigma\bigr] \\ &\quad = Q^{(t)}\bigl(1 - \mu D_2z^2 - \mu D_1z - \mu D_0\bigr) - Q\bigl[\bigl(A_2^\rho - D_2\bigr)z^2 + \bigl(A_1^\rho - D_1\bigr)z + A_0^\rho - D_0\bigr]. \end{aligned}$$

By equating the terms next to the powers of parameter z^k, $k = 3,2,1,0$, we get

$$\begin{aligned} &-\nu F_1A_2^{(x)} = -\mu A_2F_1^{(t)}, \\ &(1 - \nu F_0)A_2^{(x)} - \nu F_1A_1^{(x)} = -\mu A_2F_0^{(t)} - \mu A_1F_1^{(t)}, \\ &(1 - \nu G_0)D_2^{(x)} - \nu G_1D_1^{(x)} = -\mu D_2G_0^{(t)} - \mu D_1G_1^{(t)}, \\ &-\nu F_1C_1^{(x)} + C_1\bigl(F_1 - G_1^\sigma\bigr) + \bigl(D_2^\rho - A_2\bigr)R = -\mu A_2R^{(t)}, \\ &-\nu G_1B_1^{(x)} + B_1\bigl(G_1 - F_1^\sigma\bigr) + \bigl(A_2^\rho - D_2\bigr)Q = -\mu D_2Q^{(t)}, \\ &(1 - \nu F_0)A_1^{(x)} - \nu F_1A_0^{(x)} = (1 - \mu A_0)F_1^{(t)} - \mu A_1F_0^{(t)} + Q^\sigma C_1 - RB_1^\rho, \\ &(1 - \nu G_0)D_1^{(x)} - \nu G_1D_0^{(x)} = (1 - \mu D_0)G_1^{(t)} - \mu D_1G_0^{(t)} + R^\sigma B_1 - QC_1^\rho, \\ &(1 - \nu F_0)C_1^{(x)} - \nu F_1C_0^{(x)} + \bigl(F_1 - G_1^\sigma\bigr)C_0 + \bigl(F_0 - G_0^\sigma\bigr)C_1 = \bigl(A_1 - D_1^\rho\bigr)R - \mu A_1R^{(t)}, \\ &(1 - \nu G_0)B_1^{(x)} - \nu G_1B_0^{(x)} + \bigl(G_0 - F_0^\sigma\bigr)B_1 + \bigl(G_1 - F_1^\sigma\bigr)B_0 = \bigl(D_1 - A_1^\rho\bigr)Q - \mu D_1Q^{(t)}, \\ &(1 - \nu F_0)A_0^{(x)} = (1 - \mu A_0)F_0^{(t)} + Q^\sigma C_0 - RB_0^\rho, \end{aligned}$$

$$(1-\nu G_0)D_0^{(x)} = (1-\mu D_0)G_0^{(t)} + R^\sigma B_0 - QC_0^\rho,$$

$$(1-\mu A_0)R^{(t)} = (1-\nu F_0)C_0^{(x)} + C_0\big(F_0 - G_0^\sigma\big) + R\big(D_0^\rho - A_0\big), \tag{5.3}$$

$$(1-\mu D_0)Q^{(t)} = (1-\nu G_0)B_0^{(x)} + B_0\big(G_0 - F_0^\sigma\big) + Q\big(A_0^\rho - D_0\big). \tag{5.4}$$

Considering the continuous time scale case $\mathbb{T} = \mathbb{R}$, $\mu(t) \equiv 0$ and choosing

$$F_1(t,x) = 1, \qquad G_1(t,x) = 0, \qquad F_0(t,x) = G_0(t,x), \tag{5.5}$$

we get

$$A_2^{(x)} = D_2^{(x)} = A_1^{(x)} = D_1^{(x)} \equiv 0, \tag{5.6}$$

$$C_1^\rho = R\tilde{A}_2, \qquad B_1 = Q\tilde{A}_2, \qquad \nu A_0^{(x)} = RB_1^\rho - QC_1, \qquad RB_1 - QC_1^\rho = 0, \tag{5.7}$$

$$C_0^\rho = R\tilde{A}_1 - (1-\nu F_0)C_1^x, \qquad B_0 = Q\tilde{A}_1 + (1-\nu F_0)B_1^x = Q\tilde{A}_1 + \tilde{A}_2(1-\nu F_0)Q^x, \tag{5.8}$$

$$A_0^{(x)} = \frac{F_0^{(t)} + QC_0 - RB_0^\rho}{1-\nu F_0}, \qquad D_0^{(x)} = \frac{F_0^{(t)} + RB_0 - QC_0^\rho}{1-\nu F_0}, \tag{5.9}$$

$$R_t = (1-\nu F_0)C_0^{(x)} + R\big(D_0^\rho - A_0\big), \qquad Q_t = (1-\nu F_0)B_0^{(x)} + Q\big(A_0^\rho - D_0\big). \tag{5.10}$$

In the case $\nu(x) = 0$ from (5.6)-(5.9) we have

$$C_1 = R\tilde{A}_2, \qquad B_1 = Q\tilde{A}_2, \qquad C_0 = R\tilde{A}_1 - R_x\tilde{A}_2, \qquad B_0 = Q\tilde{A}_1 + Q_x\tilde{A}_2$$

$$A_0 = -RQ\tilde{A}_2, \qquad D_0 = -A_0 = RQ\tilde{A}_2$$

and the evolution equations (5.10) on a space scale

$$R_t = C_{0x} - 2RA_0, \qquad Q_t = B_{0x} + 2QA_0$$

turn to the nonlinear system for unknown potentials (see [2]) $R(t,x)$, $Q(t,x)$

$$R_t = R_x\tilde{A}_1 - \tilde{A}_2\big(R_{xx} - 2R^2Q\big), \qquad Q_t = Q_x\tilde{A}_1 + \tilde{A}_2\big(Q_{xx} - 2Q^2R\big), \quad t \in \mathbb{R}, x \in \mathbb{R}. \tag{5.11}$$

Considering the case $\nu(x) > 0$, from (5.7) we get

$$C_1^{(x)}(t,x) = \frac{C_1(t,x)}{\nu(x)} - \frac{R(t,x)\tilde{A}_2}{\nu(x)}, \qquad C_1(t,x) = -\tilde{A}_2 \int_{x_1}^x \hat{e}_{1/\nu}\big(x, \rho(y)\big)\frac{R(t,y)\nabla y}{\nu(y)}, \tag{5.12}$$

where the variation of a parameter formula (see [14]) is used for the solution of the first order dynamic equation on a space scale.

Further from (5.8) we get

$$C_0^\rho(t,x) = C_0(t,x) - \nu(x)C_0^x(t,x) = R\tilde{A}_1 - (1-\nu F_0)\frac{C_1 - R\tilde{A}_2}{\nu}, \tag{5.13}$$

$$RB_0 - QC_0^\rho = R\big[Q\tilde{A}_1 + \tilde{A}_2(1-F_0\nu)Q^x\big] - Q\bigg[R\tilde{A}_1 - (1-\nu F_0)\frac{C_1 - R\tilde{A}_2}{\nu}\bigg]$$

$$= \tilde{A}_2(1-F_0\nu)R\big(Q^x - Q/\nu\big) + (1-F_0\nu)\frac{QC_1}{\nu} = \frac{(1-F_0\nu)}{\nu}\big[QC_1 - RQ^\rho\tilde{A}_2\big],$$

$$QC_0^\rho - RB_0 = \frac{1-\nu F_0}{\nu}\big[\tilde{A}_2RQ^\rho - QC_1\big]. \tag{5.14}$$

From (5.13)

$$C_0^{(x)}(t,x) = \frac{C_0}{\nu} + \frac{(1-\nu F_0)C_1}{\nu^2} - \frac{(1-\nu F_0)\tilde{A}_2 + \nu\tilde{A}_1}{\nu^2}R, \tag{5.15}$$

$$C_0(t,x) = \int_{x_1}^{x} \hat{e}_{1/\nu}\big(x,\rho(y)\big)\left(\frac{(1-\nu F_0)C_1(t,y)}{\nu^2(y)} - \frac{(1-\nu F_0)\tilde{A}_2 + \nu\tilde{A}_1}{\nu^2(y)}R(t,y)\right)\nabla y. \tag{5.16}$$

Further

$$\begin{aligned}
\nu A_0^{(x)} &= RB_1^\rho - QC_1 = R\big(B_1 - \nu B_1^x\big) - Q\big(C_1^\rho + \nu C_1^x\big) = -\nu RB_1^x - \nu QC_1^x \\
&= -\nu RQ^x\tilde{A}_2 - Q(C_1 - R\tilde{A}_2) = \tilde{A}_2R\big(Q - Q^x\nu\big) - QC_1, \\
\nu A_0^{(x)} &= \tilde{A}_2RQ^\rho - QC_1 = \tilde{A}_2RQ^\rho + Q\tilde{A}_2\int_{x_1}^{x} \hat{e}_{1/\nu}\big(x,\rho(y)\big)\frac{R(t,y)\nabla y}{\nu(y)}.
\end{aligned} \tag{5.17}$$

Otherwise

$$\begin{aligned}
QC_0 - RB_0^\rho &= Q\big(C_0^\rho + \nu C_0^x\big) - R\big(B_0 - \nu B_0^x\big) = \nu\big(QC_0^x + RB_0^x\big) + QC_0^\rho - RB_0, \\
QC_0 - RB_0^\rho &= \nu\big(QC_0^x + RB_0^x\big) + \frac{1-\nu F_0}{\nu}\big[\tilde{A}_2RQ^\rho - QC_1\big], \\
D_0^{(x)} &= \frac{F_0^{(t)} + RB_0 - QC_0^\rho}{1-\nu F_0} = \frac{F_0^{(t)}}{(1-\nu F_0)} - \frac{1}{\nu}\big[\tilde{A}_2RQ^\rho - QC_1\big], \\
A_0^{(x)} &= \frac{F_0^{(t)} + QC_0 - RB_0^\rho}{1-\nu F_0} = \frac{F_0^{(t)} + \nu(QC_0^x + RB_0^x)}{1-\nu F_0} + \frac{1}{\nu}\big[\tilde{A}_2RQ^\rho - QC_1\big], \\
\nu A_0^{(x)} &= \frac{\nu F_0^{(t)} + \nu^2(QC_0^x + RB_0^x)}{1-\nu F_0} + \tilde{A}_2RQ^\rho - QC_1.
\end{aligned} \tag{5.18}$$

For the consistency of this expression and (5.17), we assume that F_0 satisfies

$$F_0^{(t)}(t,x) = -\nu(x)\big(Q(t,x)C_0^{(x)}(t,x) + RB_0^{(x)}(t,x)\big). \tag{5.19}$$

Then

$$A_0^{(x)}(t,x) + D_0^{(x)}(t,x) = \frac{F_0^{(t)}(t,x)}{1-\nu(x)F_0(t,x)},$$

and

$$D_0^{(x)}(t,x) = \int_{x_1}^{x} \frac{F_{0t}(t,y)\nabla y}{1-\nu(y)F_0(t,y)} - A_0(t,x). \tag{5.20}$$

By direct calculations from (5.19)

$$\begin{aligned}
F_0^{(t)} &= -QC_0 + QR\tilde{A}_1 + \frac{1}{\nu}(1-\nu F_0)RQ\tilde{A}_2 - (1-\nu F_0)\frac{QC_1}{\nu} \\
&\quad - \nu RQ^x\tilde{A}_1 - \nu R\tilde{A}_2\big[(1-\nu F_0)Q^x\big]^x \\
&= R\tilde{A}_1Q^\rho - QC_0 - (1-\nu F_0)\frac{QC_1}{\nu} + \tilde{A}_2R\left[\frac{1-F_0\nu}{\nu}Q - \nu\big((1-\nu F_0)Q^x\big)^x\right], \\
F_0^{(t)} &= R\tilde{A}_1Q^\rho - QC_0 - (1-\nu F_0)\frac{QC_1}{\nu} + \tilde{A}_2R\left[\frac{1-F_0\nu}{\nu}Q^\rho + (1-\nu F_0)^\rho Q^{(x)\rho}\right],
\end{aligned}$$

or

$$F_0^{(t)}(t,x) = \tilde{A}_1Q^\rho R + \tilde{A}_2R\left[\frac{1-\nu F_0}{\nu}Q^\rho + (1-\nu F_0)^\rho Q^{(x)\rho}\right] - \frac{C_0\nu + (1-\nu F_0)C_1}{\nu}Q. \quad (5.21)$$

Since (5.21) is the first order linear partial integro-differential equation with variable coefficients with respect to $F_0(t,x)$, one may prove the existence of a solution by using the linear theory.

Evolution nonlinear equations (5.10) on a space scale

$$\begin{aligned}
R_t(t,x) &= (1-\nu F_0)C_0^x - R\big(A_0 + A_0^\rho - (A_0+D_0)^\rho\big), \\
Q_t(t,x) &= (1-\nu F_0)B_0^x + Q\big(A_0^\rho - D_0\big),
\end{aligned}$$

under the assumption that the solution $F_0(t,x)$ of (5.21) exists, become

$$\begin{aligned}
R_t(t,x) &= (1-\nu F_0)C_0^{(x)} - 2RA_0 + \nu A_0^{(x)}R + R\int_{x_1}^{\rho(x)} \frac{F_{0t}\nabla x}{1-\nu F_0}, \\
Q_t(t,x) &= (1-\nu F_0)\big[Q\tilde{A}_1 + (1-\nu F_0)Q^{(x)}\tilde{A}_2\big]^x \\
&\quad + Q\left(2A_0 - \nu A_0^{(x)} - \int_{x_1}^{x} \frac{F_{0t}(t,y)\nabla x}{1-\nu(y)F_0(t,y)}\right).
\end{aligned} \quad (5.22)$$

In view of (5.15) we have

$$\begin{aligned}
R_t(t,x) &= \frac{(1-\nu F_0(t,x))}{\nu(x)}\left(C_0(t,x) - R(t,x)\tilde{A}_1\nu + \big(1-\nu F_0(t,x)\big)\frac{C_1(t,x) - R(t,x)\tilde{A}_2}{\nu(x)}\right) \\
&\quad - \left(2A_0(t,x) - \tilde{A}_2R(t,x)Q^\rho(t,x) + Q(t,x)C_1(t,x) - \int_{x_1}^{\rho(x)} \frac{F_{0t}(t,y)\nabla y}{1-\nu(y)F_0(t,y)}\right) \\
&\quad \times R(t,x),
\end{aligned} \quad (5.23)$$

where $C_1(t,x)$, $C_0(t,x)$ are given by (5.12), (5.16).

6 Nonlinear Schrodinger equation on a regular-discrete space scale

In this section we simplify equations (5.22), (5.23) under additional assumption that the backward jump operator $\rho(x)$ is invertible.

A space scale $\mathbb{X}$ is called regular-discrete [9] if the following two conditions are satisfied simultaneously:

$$\rho\bigl(\beta(x)\bigr) = x, \qquad \beta\bigl(\rho(x)\bigr) = x \quad \text{for all } x \in \mathbb{X},$$

where ρ and β are the backward and forward jump operators, respectively. From this definition ρ is invertible for the regular-discrete space scale $\mathbb{X}$ and $\rho^{-1} = \beta$.

Set $x_* = \min X$ if there exists a finite $\min X$, and $x_* = -\infty$ otherwise. Also set $x^* = \max X$ if there exists a finite $\max X$, and $x^* = -\infty$ otherwise.

Lemma 6.1 [9] *A space scale $\mathbb{X}$ is regular-discrete if and only if the following two conditions hold*:

(1) *The point x_* is right-dense and the point x^* is left-dense.*

(2) *Each point of $\mathbb{X} - \{x_*, x^*\}$ is either two-sided dense or two-sided scattered.*

In particular, $\mathbb{R}$, $h\mathbb{Z}$, and K_q are regular-discrete space scales.

Let us derive a nonlinear Schrodinger equation on a space scale assuming that the space scale $\mathbb{X}$ is regular-discrete.

Note that the system of evolution nonlinear equations obtained in [13] has sense only on the regular-discrete space scales.

From (5.6)-(5.9) we have

$$\begin{aligned} &C_1 = R^\beta \tilde{A}_2, \qquad B_1 = Q\tilde{A}_2, \qquad \nu A_0^{(x)} = RB_1^\rho - QC_1 = -\nu\tilde{A}_2\bigl(R^\beta Q\bigr)^{(x)}, \\ &C_0^\rho = R\tilde{A}_1 - (1 - \nu F_0)C_1^x, \end{aligned} \tag{6.1}$$

or

$$C_0^\rho = R\tilde{A}_1 - \tilde{A}_2(1 - \nu F_0)R^{\beta(x)}, \qquad B_0 = Q\tilde{A}_1 + (1 - \nu F_0)Q^x\tilde{A}_2. \tag{6.2}$$

Since

$$\begin{aligned} &RB_0 - QC_0^\rho = (1 - \nu F_0)\tilde{A}_2\bigl(RQ^x + QR^{\beta(x)}\bigr) = (1 - \nu F_0)\tilde{A}_2\bigl(QR^\beta\bigr)^{(x)}, \\ &QC_0 - RB_0^\rho = Q\bigl(C_0^\rho + \nu C_0^x\bigr) - R\bigl(B_0 - \nu B_0^x\bigr) = \nu\bigl(QC_0^x + RB_0^x\bigr) - (1 - \nu F_0)\tilde{A}_2\bigl(QR^\beta\bigr)^{(x)}, \end{aligned}$$

from (5.9), assuming (5.19) is true, we get (5.20) and

$$A_0^{(x)} = \frac{F_0^{(t)} + \nu(QC_0^{(x)} + RB_0^{(x)})}{1 - \nu F_0} - \tilde{A}_2\bigl(QR^\beta\bigr)^{(x)}, \qquad A_0^{(x)} = -\tilde{A}_2\bigl(QR^\beta\bigr)^{(x)}.$$

Further from (5.19) we get the linear partial differential equation with respect to unknown function $F_0(t,x)$:

$$F_{0t}(t,x) = \nu\tilde{A}_2\bigl(Q\bigl[R^{\beta(x)\beta}(1 - \nu F_0)^\beta\bigr]^{(x)} - R\bigl[Q^{(x)}(1 - \nu F_0)\bigr]^{(x)}\bigr) - \nu\tilde{A}_1\bigl(R^{\beta(x)}Q + Q^{(x)}R\bigr),$$

or

$$F_{0t}(t,x) = \nu\bigl(\tilde{A}_2 QR^{\beta(x)\beta}(1 - \nu F_0)^\beta - \tilde{A}_2R^\beta Q^{(x)}(1 - \nu F_0) - \tilde{A}_1 QR^\beta\bigr)^{(x)}. \tag{6.3}$$

In view of (5.20)

$$
\begin{aligned}
&A_0(t,x) - D_0(t,x) = 2A_0(t,x) - \int_{x_1}^{x} \frac{F_{0t}(t,y)\nabla y}{1-\nu(y)F_0(t,y)}, \\
&A_0(t,x) = -\tilde{A}_2 R^{\beta}(t,x)Q(t,x), \\
&A_0(t,x) - D_0^{\rho}(t,x) = A_0 - D_0 + \nu D_0^{(x)} = A_0 - D_0 + \nu\left(\frac{F_{0t}}{1-\nu F_0} - A_0^{(x)}\right), \\
&A_0(t,x) - D_0^{\rho}(t,x) = 2A_0(t,x) - \nu A_0^{(x)}(t,x) - \int_{x_1}^{x} \frac{F_{0t}(t,y)\nabla y}{1-\nu(y)F_0(t,y)} + \frac{\nu F_{0t}(t,x)}{1-\nu F_0(t,x)}.
\end{aligned}
$$

Further

$$
\begin{aligned}
&R_t = (1-\nu F_0)C_0^{x} - R\big(A_0 - D_0^{\rho}\big), \qquad Q_t = (1-\nu F_0)B_0^{x} + Q\big(A_0^{\rho} - D_0\big), \\
&Q_t = (1-\nu F_0)\big[Q\tilde{A}_1 + (1-\nu F_0)Q^{(x)}\tilde{A}_2\big]^{x} + 2A_0 Q - Q\nu A_0^{(x)} - Q\int_{x_1}^{x} \frac{F_{0t}}{1-\nu F_0}\nabla x,
\end{aligned}
$$

and we get the following nonlinear system of dynamic equations on a space scale:

$$
\begin{aligned}
R_t(t,x) = {}& (1-\nu F_0)\big[R\tilde{A}_1 - (1-\nu F_0)R^{\beta(x)}\tilde{A}_2\big]^{\beta(x)} + 2\tilde{A}_2 Q R^{\beta} R \\
& - R\left(\nu\tilde{A}_2\big(R^{\beta(x)}Q\big)^{(x)} - \int_{x_1}^{x} \frac{F_{0t}\nabla y}{1-\nu F_0} + \frac{\nu F_{0t}}{1-\nu F_0}\right), \quad t\in\mathbb{R}, x\in\mathbb{X}, \qquad (6.4)\\
Q_t(t,x) = {}& (1-\nu F_0)\big[Q\tilde{A}_1 + (1-\nu F_0)Q^{(x)}\tilde{A}_2\big]^{x} \\
& + Q\left(\nu\tilde{A}_2\big(R^{\beta(x)}Q\big)^{(x)} - 2\tilde{A}_2 Q R^{\beta} - \int_{x_1}^{x} \frac{F_{0t}}{1-\nu F_0}\nabla x\right), \quad t\in\mathbb{R}, x\in\mathbb{X}. \qquad (6.5)
\end{aligned}
$$

In a special case $\tilde{A}_1 = 0$, we get more simple equations

$$
\begin{aligned}
R_t(t,x) = {}& -\tilde{A}_2(1-\nu F_0)\big[(1-\nu F_0)R^{\beta(x)}(t,x)\big]^{\beta(x)} - \nu\tilde{A}_2\big(R^{\beta}Q\big)^{(x)}R(t,x) \\
& + 2\tilde{A}_2 R^{\beta} R Q(t,x) \\
& + R(t,x)\int_{x_1}^{x} \frac{F_{0t}(t,y)\nabla y}{1-\nu F_0(t,y)} - \frac{\nu(x) R F_{0t}(t,x)}{1-\nu F_0(t,x)}, \quad t\in\mathbb{R}, x\in\mathbb{X}, \qquad (6.6)\\
Q_t(t,x) = {}& \tilde{A}_2(1-\nu F_0)\big[(1-\nu F_0)Q^{(x)}\big]^{x} \\
& + \tilde{A}_2\nu Q\big(R^{\beta}Q\big)^{(x)} - 2\tilde{A}_2 R^{\beta} Q^2 - Q\int_{x_1}^{x} \frac{F_{0t}(t,y)\nabla y}{1-\nu F_0}, \quad t\in\mathbb{R}, x\in\mathbb{X}, \qquad (6.7)
\end{aligned}
$$

where

$$
F_{0t}(t,x) = \tilde{A}_2\nu\big(Q R^{\beta(x)\beta}(1-\nu F_0)^{\beta} - R^{\beta} Q^{(x)}(1-\nu F_0)\big)^{(x)}. \qquad (6.8)
$$

Note that in the case $\mathbb{T} = \mathbb{R}$, $\mathbb{X} = \mathbb{R}$ we have $F_0(t,x) \equiv 0$, and both equations (5.11) turn to the same equation

$$
Q_t(t,x) = Q_x(t,x)\tilde{A}_1 + Q_{xx}(t,x)\tilde{A}_2 - 6\tilde{A}_2\big|Q(t,x)\big|^2 Q(t,x) \qquad (6.9)
$$

under the assumption

$$\mathrm{Re}[\tilde{A}_2] = \mathrm{Re}[\tilde{A}_0] = \mathrm{Im}[\tilde{A}_1] = 0, \qquad Q(t,x) = -\overline{R}(t,x).$$

Furthermore, if $\tilde{A}_1 = 0$, $\tilde{A}_2 = -i$, equation (6.9) turns to the nonlinear Schrodinger equation

$$iQ_t(t,x) = Q_{(xx)}(t,x) - \left|Q(t,x)\right|^2 Q(t,x). \tag{6.10}$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The idea to deduce and solve the Burgers equation on a time-space scale by using the AKNSL method, the Cole-Hopf transformation, and the separation of variables belongs to Gro Hovhannisyan. The proofs are verified by all authors. All authors read and approved the final manuscript.

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