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Existence results for mixed Hadamard and Riemann-Liouville fractional integro-differential equations

Bashir Ahmad[1*], Sotiris K Ntouyas[1,2] and Jessada Tariboon[3]

*Correspondence:
bashirahmad_qau@yahoo.com
[1]Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah, 21589, Saudi Arabia
Full list of author information is available at the end of the article

Abstract
In this paper, we prove existence and uniqueness results for a mixed initial value problem which contains Hadamard derivative and Riemann-Liouville fractional integrals by using standard fixed point theorems. Examples illustrating the results are also presented.

MSC: 34A08; 34A12

Keywords: fractional differential equations; Hadamard derivative; Riemann-Liouville derivative; fixed point theorem

1 Introduction

Fractional differential equations involving Riemann-Liouville, Caputo and Hadamard type derivatives have been extensively studied by several researchers. The existing literature on the topic is quite enriched now and covers its theoretical aspects and wide range of applications. The tools of fractional calculus are effectively applied in modeling many engineering and scientific processes and phenomena. Examples include physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, economics, control theory, signal and image processing, biophysics, blood flow phenomena, *etc.* [1–3]. For some recent developments on the topic, see [4–7] and the references therein.

Recently, Caputo and Fabrizio [8] introduced a new definition of fractional derivative without singular kernel. For details and applications of this concept, see [9–11].

The Hadamard fractional derivative, introduced by Hadamard in 1892 [12], differs from the Riemann-Liouville and Caputo derivatives in the sense that the kernel of the integral (in the definition of Hadamard derivative) contains a logarithmic function of arbitrary exponent. Details and properties of the Hadamard fractional derivative and integral can be found in [2, 13–15]. For some recent works on the subject, we refer the reader to the works [16–24].

In this paper, we consider a new class of mixed initial value problems involving Hadamard derivative and Riemann-Liouville fractional integrals given by

$$\begin{cases} D^{\alpha}\left(x(t) - \sum_{i=1}^{m} I^{\beta_i} h_i(t, x(t))\right) = g(t, x(t), Kx(t)), & t \in J := [1, T], \\ x(1) = 0, \end{cases} \tag{1.1}$$

where D^{α} denotes the Hadamard fractional derivative of order α, $0 < \alpha \leq 1$, I^{ϕ} is the Riemann-Liouville fractional integral of order $\phi > 0$, $\phi \in \{\beta_1, \beta_2, \dots, \beta_m\}$, $g \in C(J \times \mathbb{R}^2, \mathbb{R})$, $h_i \in C(J \times \mathbb{R}, \mathbb{R})$ with $h(1,0) = 0$, $i = 1,2,\dots,m$, and $Kx(t) = \int_1^t \varphi(t,s)x(s)\,ds$, $\varphi(t,s) \in C(J^2, \mathbb{R})$ with $\varphi_0 = \sup\{|\varphi(t,s)|; (t,s) \in J \times J\}$.

Existence as well existence and uniqueness results are proved for the initial value problem (1.1) by using Krasnoselskii's fixed point theorem, Banach's fixed point theorem, and Leray-Schauder nonlinear alternative. We emphasize that the results obtained in the given setting are new and contribute significantly to the existing literature on Hadamard type fractional differential equations. The paper concludes with illustrative examples.

2 Preliminaries

In this section, we introduce some notations and definitions of fractional calculus [1, 2] and present a lemma needed for the proof of our main results.

Definition 2.1 ([2]) The Hadamard derivative of fractional order q for a function $g : [1,\infty) \to \mathbb{R}$ is defined as

$$D^q g(t) = \frac{1}{\Gamma(n-q)} \left(t\frac{d}{dt}\right)^n \int_1^t \left(\log\frac{t}{s}\right)^{n-q-1} \frac{g(s)}{s}\,ds, \quad n-1 < q < n, n = [q]+1,$$

where $[q]$ denotes the integer part of the real number q and $\log(\cdot) = \log_e(\cdot)$.

Definition 2.2 ([2]) The Hadamard fractional integral of order q for a function g is defined as

$$I^q g(t) = \frac{1}{\Gamma(q)} \int_1^t \left(\log\frac{t}{s}\right)^{q-1} \frac{g(s)}{s}\,ds, \quad q > 0,$$

provided the integral exists.

Definition 2.3 The Riemann-Liouville fractional integral of order $p > 0$ of a continuous function $f : (0,\infty) \to \mathbb{R}$ is defined by

$$I^p f(t) = \frac{1}{\Gamma(p)} \int_1^t (t-s)^{p-1} f(s)\,ds,$$

provided the right-hand side is point-wise defined on $(1,\infty)$.

The following lemma is obvious [2].

Lemma 2.1 *Suppose that* $0 < \alpha \leq 1$, *and functions* g, h_i, $i = 1,2,\dots,m$ *satisfy problem* (1.1). *Then the unique solution of the fractional integro-differential problem* (1.1) *is given by*

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} g\bigl(s, x(s), Kx(s)\bigr)\frac{ds}{s} + \sum_{i=1}^m I^{\beta_i} h_i\bigl(t, x(t)\bigr), \quad t \in J. \tag{2.1}$$

Let $E = C(J,\mathbb{R})$ be the space of continuous real-valued functions defined on $J = [1,T]$ endowed with the norm $\|x\| = \sup_{t\in J} |x(t)|$.

3 Main results

The first existence result for the problem (1.1) will be proved by using the Krasnoselskii's fixed point theorem [25].

Lemma 3.1 (Krasnoselskii's fixed point theorem) [25] *Let M be a closed, bounded, convex, and nonempty subset of a Banach space X. Let A, B be the operators such that* (a) $Ax + By \in M$ *whenever* $x, y \in M$; (b) *A is compact and continuous*; (c) *B is a contraction mapping. Then there exists $z \in M$ such that $z = Az + Bz$.*

Theorem 3.1 *Assume that*

(A_1) *There exist a constant $L_0 > 0$, such that*

$$\bigl|h_i\bigl(t, x(t)\bigr) - h_i\bigl(t, y(t)\bigr)\bigr| \leq L_0\bigl|x(t) - y(t)\bigr|, \quad \text{for } t \in J \text{ and } x, y \in \mathbb{R}, i = 1, 2, \dots, m.$$

(A_2) $|g(t, x, y)| \leq \nu(t) + \mu(t)|y|$, $\forall (t, x, y) \in J \times \mathbb{R}^2$, $\nu, \mu \in C(J, \mathbb{R}^+)$, *and* $|h_i(t, x)| \leq \theta_i(t)$, $\forall (t, x) \in J \times \mathbb{R}$, $\theta_i \in C(J, \mathbb{R}^+)$, $i = 1, 2, \dots, m$.

Then the problem (1.1) *has at least one solution on J provided that*

$$L_0 \sum_{i=1}^{m} \frac{(T-1)^{\beta_i}}{\Gamma(\beta_i + 1)} < 1.$$

Proof Setting $\sup_{t \in J} |\nu(t)| = \|\nu\|$, $\sup_{t \in J} |\mu(t)| = \|\mu\|$, $\sup_{t \in J} |\theta_i(t)| = \|\theta_i\|$, $i = 1, 2, \dots, m$, we consider $B_R = \{x \in C(J, \mathbb{R}) : \|x\| \leq R\}$, where

$$R \geq \left(\sum_{i=1}^{m} \frac{(T-1)^{\beta_i}}{\Gamma(\beta_i + 1)}\|\theta_i\| + \frac{(\log T)^{\alpha}}{\Gamma(\alpha + 1)}\|\nu\|\right) \bigg/ \left(1 - \varphi_0\|\mu\|\left[\frac{\gamma}{\Gamma(\alpha)} + \frac{(\log T)^{\alpha}}{\Gamma(\alpha + 1)}\right]\right),$$

$\gamma = T\int_0^{\log T} u^{\alpha-1}e^{-u}\,du$ and $\varphi_0\|\mu\|[\frac{\gamma}{\Gamma(\alpha)} + \frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)}] < 1$.

We define the operators $\mathcal{Q} : B_R \to E$ and $\mathcal{T} : B_R \to E$ by

$$\mathcal{Q}x(t) = \sum_{i=1}^{m} \frac{1}{\Gamma(\beta_i)} \int_1^t (t-s)^{\beta_i - 1} h_i\bigl(s, x(s)\bigr)\,ds, \quad t \in J, \tag{3.1}$$

$$\mathcal{T}x(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} g\bigl(s, x(s), Kx(s)\bigr)\frac{ds}{s}, \quad t \in J. \tag{3.2}$$

For any $x, y \in B_R$, we have

$$\begin{aligned}
\bigl|\mathcal{Q}x(t) + \mathcal{T}y(t)\bigr| &\leq \sum_{i=1}^{m} \frac{1}{\Gamma(\beta_i)} \int_1^t (t-s)^{\beta_i - 1}\bigl|h_i\bigl(s, x(s)\bigr)\bigr|\,ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} \bigl|g\bigl(s, x(s), Kx(s)\bigr)\bigr|\frac{ds}{s} \\
&\leq \sum_{i=1}^{m} \frac{1}{\Gamma(\beta_i)} \int_1^t (t-s)^{\beta_i - 1}\bigl|\theta_i(s)\bigr|\,ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} \bigl(\bigl|\nu(s)\bigr| + \bigl|\mu(s)\bigr|\bigl|Kx(s)\bigr|\bigr)\frac{ds}{s}
\end{aligned}$$

$$\leq \sum_{i=1}^{m} \frac{(T-1)^{\beta_i}}{\Gamma(\beta_i+1)} \|\theta_i\| + \frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)} \|\nu\| + \varphi_0 \|\mu\| R \left[\frac{\gamma}{\Gamma(\alpha)} + \frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)} \right]$$
$$\leq R.$$

Next we will show that $\mathcal{T}$ is continuous and compact. The operator $\mathcal{T}$ is obviously continuous. Also, $\mathcal{T}$ is uniformly bounded on B_R as

$$\|\mathcal{T}x\| \leq \frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)} \|\nu\| + \|\mu\| \varphi_0 R \left[\frac{\gamma}{\Gamma(\alpha)} + \frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)} \right].$$

Let $\tau_1, \tau_2 \in J$ with $\tau_1 < \tau_2$ and $x \in B_R$. We define $\sup_{(t,x_i) \in J \times B_R \times B_R} |g(t,x,y)| = \bar{g} < \infty$. Then we have

$$\begin{aligned} \left|\mathcal{T}x(\tau_2) - \mathcal{T}x(\tau_1)\right| &= \left| \frac{1}{\Gamma(\alpha)} \int_1^{\tau_2} \left(\log \frac{\tau_2}{s}\right)^{\alpha-1} g\bigl(s, x(s), Kx(s)\bigr) \frac{ds}{s} \right. \\ &\quad \left. - \frac{1}{\Gamma(\alpha)} \int_1^{\tau_1} \left(\log \frac{\tau_2}{s}\right)^{\alpha-1} g\bigl(s, x(s), Kx(s)\bigr) \frac{ds}{s} \right| \\ &\leq \frac{\|\nu\| + \|\mu\| \varphi_0 R(T-1)}{\Gamma(\alpha+1)} \bigl[(\log \tau_1)^{\alpha} - (\log \tau_2)^{\alpha}\bigr], \end{aligned}$$

which is independent of x and tends to zero as $\tau_2 - \tau_1 \to 0$. Thus, $\mathcal{T}$ is equicontinuous. So $\mathcal{T}$ is relatively compact on B_R. Hence, by the Arzelá-Ascoli theorem, $\mathcal{T}$ is compact on B_R.

Now we show that $\mathcal{Q}$ is a contraction mapping. Let $x, y \in B_R$. Then, for $t \in J$, we have

$$\begin{aligned} \left|\mathcal{Q}x(t) - \mathcal{Q}y(t)\right| &\leq \sum_{i=1}^{m} \frac{1}{\Gamma(\beta_i)} \int_1^t (t-s)^{\beta_i - 1} \left|h_i\bigl(s, x(s)\bigr) - h_i\bigl(s, y(s)\bigr)\right| ds \\ &\leq L_0 \|x - y\| \sum_{i=1}^{m} \frac{1}{\Gamma(\beta_i)} \int_1^t (t-s)^{\beta_i - 1} ds \\ &\leq L_0 \|x - y\| \sum_{i=1}^{m} \frac{(T-1)^{\beta_i}}{\Gamma(\beta_i + 1)}. \end{aligned}$$

Hence, by the given assumption, $\mathcal{T}$ is a contraction mapping.

Thus all the assumptions of Krasnoselskii fixed point theorem are satisfied, which implies that the problem (1.1) has at least one solution on J. □

Theorem 3.2 *Assume that* $h_i : [1,T] \times \mathbb{R} \to \mathbb{R}$, $i = 1, 2, \ldots, m$, *are continuous functions satisfying the condition* (A_1). *In addition we assume that*

(A_3) $|g(t,x,y) - g(t,\bar{x},\bar{y})| \leq L_1|x - \bar{x}| + L_2|y - \bar{y}|$, $\forall t \in J$, $L_1, L_2 > 0$, $x, \bar{x}, y, \bar{y} \in \mathbb{R}$.

Then the problem (1.1) *has a unique solution if*

$$\Lambda := L_0 \sum_{i=1}^{m} \frac{(T-1)^{\beta_i}}{\Gamma(\beta_i+1)} + L_1 \frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)} + L_2 \varphi_0 \left[\frac{\gamma}{\Gamma(\alpha)} + \frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)} \right] < 1.$$

Proof Let us fix $\sup_{t \in [1,T]} |g(t,0,0)| = N$, $\sup_{t \in [1,T]} |h_i(t,0)| = K_i$, $i = 1, 2, \ldots, m$ and choose $r \geq \frac{M}{1-\Lambda}$, where $M = \sum_{i=1}^{m} K_i \frac{(T-1)^{\beta_i}}{\Gamma(\beta_i+1)} + N \frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)}$. Then we show that $FB_r \subset B_r$, where $B_r =$

$\{x \in C(J,\mathbb{R}) : \|x\| \leq r\}$ with the operator $F : C([1,T],\mathbb{R}) \to C([1,T],\mathbb{R})$ defined by

$$(Fx)(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} g\big(s,x(s),Kx(s)\big)\frac{ds}{s} + \sum_{i=1}^m I^{\beta_i} h_i\big(t,x(t)\big), \quad t \in J. \tag{3.3}$$

For $x \in B_r$, we have

$$\begin{aligned}
&\big|(Fx)(t)\big| \\
&\quad \leq \sup_{t\in[1,T]} \Bigg\{ \sum_{i=1}^m \frac{1}{\Gamma(\beta_i)} \int_1^t (t-s)^{\beta_i-1} \big|h_i\big(s,x(s)\big)\big|\,ds \\
&\qquad + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \big|g\big(s,x(s),Kx(s)\big)\big| \frac{ds}{s} \Bigg\} \\
&\quad \leq \sup_{t\in[1,T]} \Bigg\{ \sum_{i=1}^m \frac{1}{\Gamma(\beta_i)} \int_1^t (t-s)^{\beta_i-1} \big(\big|h_i\big(s,x(s)\big) - h_i(s,0)\big| + \big|h_i(s,0)\big|\big)\,ds \\
&\qquad + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \big(\big|g\big(s,x(s),Kx(s)\big) - g(s,0,0)\big| + \big|g(s,0,0)\big|\big) \frac{ds}{s} \Bigg\} \\
&\quad \leq \sum_{i=1}^m \frac{(T-1)^{\beta_i}}{\Gamma(\beta_i+1)} (L_0 r + K_i) + (L_1 r + N) \frac{(\log T)^\alpha}{\Gamma(\alpha+1)} + L_2 \varphi_0 r \left[\frac{\gamma}{\Gamma(\alpha)} + \frac{(\log T)^\alpha}{\Gamma(\alpha+1)}\right] \\
&\quad = \Lambda r + M \leq r,
\end{aligned}$$

which implies that $FB_r \subset B_r$.

Now, for $x, y \in C([1,T],\mathbb{R})$ and for each $t \in [1,T]$, we obtain

$$\begin{aligned}
&\big|(Fx)(t) - (Fy)(t)\big| \\
&\quad \leq \sup_{t\in[1,T]} \Bigg\{ \sum_{i=1}^m \frac{1}{\Gamma(\beta_i)} \int_1^t (t-s)^{\beta_i-1} \big|h_i\big(s,x(s)\big) - h_i\big(s,y(s)\big)\big|\,ds \\
&\qquad + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \big|g\big(s,x(s),Kx(s)\big) - g\big(s,y(s),Ky(s)\big)\big| \frac{ds}{s} \Bigg\} \\
&\quad \leq \Bigg\{ L_0 \sum_{i=1}^m \frac{(T-1)^{\beta_i}}{\Gamma(\beta_i+1)} + L_1 \frac{(\log T)^\alpha}{\Gamma(\alpha+1)} + L_2 \varphi_0 \left[\frac{\gamma}{\Gamma(\alpha)} + \frac{(\log T)^\alpha}{\Gamma(\alpha+1)}\right] \Bigg\} \|x-y\|.
\end{aligned}$$

Therefore $\|Fx - Fy\| \leq \Lambda \|x-y\|$, and as $\Lambda < 1$, F is a contraction. Thus, the conclusion of the theorem follows by the contraction mapping principle (Banach fixed point theorem). □

Our final existence result is based on the Leray-Schauder nonlinear alternative [26].

Lemma 3.2 (Nonlinear alternative for single valued maps [26]) *Let E be a Banach space, C a closed, convex subset of E, U an open subset of C and $0 \in U$. Suppose that $F : \overline{U} \to C$ is a continuous, compact (that is, $F(\overline{U})$ is a relatively compact subset of C) map. Then either*

(i) *F has a fixed point in $\overline{U}$, or*

(ii) *there is a $u \in \partial U$ (the boundary of U in C) and $\lambda \in (0,1)$ with $u = \lambda F(u)$.*

Theorem 3.3 *Assume that $g : [1,T] \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function and the following conditions hold*:

(H_1) *there exist functions $p_1, p_2 \in C([1,T],\mathbb{R}^+)$, and $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ nondecreasing such that $|g(t,x,y)| \leq p_1(t)\psi(|x|) + p_2(t)|y|$ for each $(t,x,y) \in [1,T] \times \mathbb{R}^2$*;

(H_2) *there exist functions $q_i \in C([1,T],\mathbb{R}^+)$, and $\Omega_i : \mathbb{R}^+ \to \mathbb{R}^+$ nondecreasing such that $|h_i(t,x)| \leq q_i(t)\Omega_i(|x|)$ for each $(t,x) \in [1,T] \times \mathbb{R}$, $i = 1,2,\ldots,m$*;

(H_3) *there exists a number $M_0 > 0$ such that*

$$\frac{(1 - \|p_2\|\varphi_0[\frac{\gamma}{\Gamma(\alpha)} + \frac{(\log T)^\alpha}{\Gamma(\alpha+1)}])M_0}{\sum_{i=1}^m \frac{(T-1)^{\beta_i}}{\Gamma(\beta_i+1)}\|q_i\|\Omega_i(M_0) + \|p_1\|\psi(M_0)\frac{(\log T)^\alpha}{\Gamma(\alpha+1)}} > 1,$$

with

$$\|p_2\|\varphi_0\left[\frac{\gamma}{\Gamma(\alpha)} + \frac{(\log T)^\alpha}{\Gamma(\alpha+1)}\right] < 1.$$

Then the boundary value problem (1.1) *has at least one solution on* $[1,T]$.

Proof Consider the operator F defined by (3.3). It is easy to prove that F is continuous. Next, we show that F maps bounded sets into bounded sets in $C([1,T],\mathbb{R})$. For a positive number ρ, let $B_\rho = \{x \in C([1,T],\mathbb{R}) : \|x\| \leq \rho\}$ be a bounded set in $C([1,T],\mathbb{R})$. Then, for each $x \in B_\rho$, we have

$$\begin{aligned} \bigl|(Fx)(t)\bigr| &\leq \sum_{i=1}^m \frac{1}{\Gamma(\beta_i)}\int_1^t (t-s)^{\beta_i-1}\bigl|h_i\bigl(s,x(s)\bigr)\bigr|\,ds \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\bigl|g\bigl(s,x(s),Kx(s)\bigr)\bigr|\frac{ds}{s} \\ &\leq \sum_{i=1}^m \frac{(T-1)^{\beta_i}}{\Gamma(\beta_i+1)}\|q_i\|\Omega_i(\rho) + \|p_1\|\psi(\rho)\frac{(\log T)^\alpha}{\Gamma(\alpha+1)} \\ &\quad + \|p_2\|\varphi_0\rho\left[\frac{\gamma}{\Gamma(\alpha)} + \frac{(\log T)^\alpha}{\Gamma(\alpha+1)}\right]. \end{aligned}$$

Thus,

$$\|Fx\| \leq \sum_{i=1}^m \frac{(T-1)^{\beta_i}}{\Gamma(\beta_i+1)}\|q_i\|\Omega_i(r) + \|p_1\|\psi(\rho)\frac{(\log T)^\alpha}{\Gamma(\alpha+1)} + \|p_2\|\varphi_0\rho\left[\frac{\gamma}{\Gamma(\alpha)} + \frac{(\log T)^\alpha}{\Gamma(\alpha+1)}\right].$$

Now we show that F maps bounded sets into equicontinuous sets of $C([1,T],\mathbb{R})$. Let $t_1, t_2 \in [1,T]$ with $t_1 < t_2$ and $x \in B_\rho$, where B_ρ is a bounded set of $C([1,T],\mathbb{R})$. Then we have

$$\begin{aligned} &\bigl|(Fx)(t_2) - (Fx)(t_1)\bigr| \\ &\quad \leq \left|\sum_{i=1}^m \frac{1}{\Gamma(\beta_i)}\int_1^{t_2}(t_2-s)^{\beta_1-1}h_i\bigl(s,x(s)\bigr)\,ds - \sum_{i=1}^m \frac{1}{\Gamma(\beta_i)}\int_1^{t_1}(t_1-s)^{\beta_i-1}h_i\bigl(s,x(s)\bigr)\,ds\right| \\ &\qquad + \left|\frac{1}{\Gamma(\alpha)}\int_1^{t_2}\left(\log\frac{t_2}{s}\right)^{\alpha-1}g\bigl(s,x(s),Kx(s)\bigr)\frac{ds}{s}\right. \end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{\Gamma(\alpha)} \int_1^{t_1} \left(\log \frac{t_1}{s}\right)^{\alpha-1} g\big(s, x(s), Kx(s)\big) \frac{ds}{s} \Bigg| \\
&\leq \sum_{i=1}^m \frac{\|q_i\| \Omega_i(\rho)}{\Gamma(\beta_i)} \left\{ \int_1^{t_1} \big[(t_2-s)^{\beta_i-1} - (t_1-s)^{\beta_i-1}\big]\, ds + \int_{t_1}^{t_2} (t_2-s)^{\beta-1}\, ds \right\} \\
&\quad + \frac{\|p_1\|\psi(\rho) + \|p_2\|\varphi_0 \rho(T-1)}{\Gamma(\alpha)} \left| \int_1^{t_1} \left[\left(\log \frac{t_2}{s}\right)^{\alpha-1} - \left(\log \frac{t_1}{s}\right)^{\alpha-1} \right] \frac{1}{s}\, ds \right. \\
&\quad \left. + \int_{t_1}^{t_2} \left(\log \frac{t_2}{s}\right)^{\alpha-1} \frac{1}{s}\, ds \right|.
\end{aligned}
$$

Obviously the right-hand side of the above inequality tends to zero independently of $x \in B_\rho$ as $t_2 - t_1 \to 0$. Therefore it follows by the Arzelá-Ascoli theorem that $F : C([1,T],\mathbb{R}) \to C([1,T],\mathbb{R})$ is completely continuous.

The result will follow from the Leray-Schauder nonlinear alternative once we have shown the boundedness of the set of all solutions to equations $x = \lambda Fx$ for $\lambda \in [0,1]$.

Let x be a solution. Then, for $t \in [1,T]$, following the computations used in proving that F is bounded, we obtain

$$
\begin{aligned}
\big|x(t)\big| &\leq \sum_{i=1}^m \frac{(T-1)^{\beta_i}}{\Gamma(\beta_i+1)} \|q_i\| \Omega_i\big(\|x\|\big) + \|p_1\| \psi\big(\|x\|\big) \frac{(\log T)^\alpha}{\Gamma(\alpha+1)} \\
&\quad + \|p_2\| \varphi_0 \|x\| \left[\frac{\gamma}{\Gamma(\alpha)} + \frac{(\log T)^\alpha}{\Gamma(\alpha+1)} \right].
\end{aligned}
$$

Consequently, we get

$$
\frac{(1 - \|p_2\|\varphi_0[\frac{\gamma}{\Gamma(\alpha)} + \frac{(\log T)^\alpha}{\Gamma(\alpha+1)}])\|x\|}{\sum_{i=1}^m \frac{(T-1)^{\beta_i}}{\Gamma(\beta_i+1)} \|q_i\| \Omega_i(\|x\|) + \|p_1\| \psi(\|x\|) \frac{(\log T)^\alpha}{\Gamma(\alpha+1)}} \leq 1.
$$

In view of (A_3), there exists M_0 such that $\|x\| \neq M_0$. Let us set

$$
U = \big\{ x \in C\big([1,T],\mathbb{R}\big) : \|x\| < M_0 + 1 \big\}.
$$

Note that the operator $F : \overline{U} \to C([1,T],\mathbb{R})$ is continuous and completely continuous. From the choice of U, there is no $x \in \partial U$ such that $x = \lambda Fx$ for some $\lambda \in (0,1)$. Hence, by the Leray-Schauder alternative, we deduce that F has a fixed point $x \in \overline{U}$ which is a solution of the problem (1.1). □

4 Examples

In this section, we illustrate the existence results obtained in Section 3 with the aid of examples.

Example 4.1 Consider the following mixed Hadamard and Riemann-Liouville fractional integro-differential equation with initial condition:

$$
\begin{cases}
D^{\frac{1}{4}}(x(t) - \sum_{i=1}^3 I^{\beta_i} h_i(t, x(t))) \\
\quad = \frac{1}{2} + \frac{(\sqrt{t} + \log t)|x(t)|}{|x(t)|+3} + (1 + \log t) \int_1^t \frac{\sin(\pi \log t) x(s)}{3(s^2+1)}\, ds, \quad t \in [1, e^{\frac{1}{2}}], \\
x(1) = 0.
\end{cases}
\tag{4.1}
$$

Here $\alpha = 1/4$, $\beta_1 = 2/5$, $\beta_2 = 3/5$, $\beta_3 = 4/5$, $m = 3$, $T = e^{\frac{1}{2}}$, and

$$h_1(t,x) = \frac{e^{-t}}{\sqrt{2}}\left(\frac{|x|}{1+|x|}\right), \qquad h_2(t,x) = \frac{1}{4t}\left(\frac{1}{|x|+3}+1\right)|x|,$$
$$h_3(t,x) = \frac{2(\log t)}{3}\sin|x|, \qquad g(t,x,y) = \frac{1}{2} + \frac{(\sqrt{t}+\log t)|x|}{|x|+3} + y(1+\log t),$$
$$\varphi(t,s) = \frac{\sin(\pi\log t)}{3(s^2+1)}.$$

With the given data, we find that $\varphi_0 = 1/6$, $|h_i(t,x) - h_i(t,y)| \le (1/3)|x-y|$, $i = 1,2,3$, $|g(t,x,y)| \le (1/2) + 2\sqrt{t} + (1+\log t)|y|$, which satisfy (A_1) and (A_2) with $L_0 = (1/3)$, $\nu(t) = (1/2) + 2\sqrt{t}$, and $\mu(t) = 1 + \log t$. Further, $\gamma = 5.059974208$,

$$L_0\sum_{i=1}^{m}\frac{(T-1)^{\beta_i}}{\Gamma(\beta_i+1)} = 0.856880869 < 1 \quad \text{and}$$
$$\varphi_0\|\mu\|\left[\frac{\gamma}{\Gamma(\alpha)} + \frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)}\right] = 0.580837503 < 1.$$

Therefore, by Theorem 3.1, the problem (4.1) has at least one solution on $[1, e^{\frac{1}{2}}]$.

Example 4.2 Consider the initial value problem given by

$$\begin{cases} D^{\frac{1}{2}}(x(t) - \sum_{i=1}^{4} I^{\beta_i}h_i(t,x(t))) \\ \quad = \frac{1}{10(1+t^2)}(\frac{x^2(t)+10|x(t)|}{5+|x(t)|}) + \frac{1}{6}\int_1^t \frac{e^{1-st}x(s)}{(1+t)(1+|\cos(\pi s/2)|)}\,ds + \frac{3}{4}, \quad t\in[1,e], \\ x(1) = 0. \end{cases} \tag{4.2}$$

Here $\alpha = 1/2$, $\beta_1 = 1/2$, $\beta_2 = 3/2$, $\beta_3 = 5/2$, $\beta_4 = 7/2$, $m = 4$, $T = e$,, and

$$h_1(t,x) = \frac{1}{20t}\left(\frac{1}{1+|x|}+1\right)|x|, \qquad h_2(t,x) = \frac{2e^{-2t}}{3}\sin|x|,$$
$$h_3(t,x) = \frac{1}{5(2+\ln t)}\tan^{-1}|x|, \qquad h_4(t,x) = \frac{1}{10}\left(\frac{|x|}{t+|x|}\right),$$
$$g(t,x,y) = \frac{1}{10(1+t^2)}\left(\frac{x^2+10|x|}{5+|x|}\right) + \frac{1}{6}y + \frac{3}{4},$$
$$\varphi(t,s) = \frac{e^{1-st}}{(1+t)(1+|\cos(\pi s/2)|)}.$$

Using the given values, we find that $\varphi_0 = 1/2$, $|h_i(t,x) - h_i(t,y)| \le (1/10)|x-y|$, $i = 1,2,3,4$, $|g(t,x,y) - g(t,\bar{x},\bar{y})| \le (1/10)|x-\bar{x}| + (1/6)|y-\bar{y}|$, which satisfy (A_1) and (A_3) with $L_0 = (1/10)$, $L_1 = (1/10)$, and $L_2 = (1/6)$. Since $\int_0^1 u^{-1/2}e^{-u}\,du = \sqrt{\pi}\,\mathrm{erf}(1)$, where $\mathrm{erf}(\cdot)$ is the Gauss error function, we have $\gamma = 4.06015694$. Hence, we obtain $\Lambda = 0.88873633 < 1$. Therefore, by Theorem 3.2, the problem (4.2) has a unique solution on $[1, e]$.

Example 4.3 Consider the following initial value problem of mixed Hadamard and Riemann-Liouville fractional integro-differential equations:

$$\begin{cases} D^{\frac{3}{4}}(x(t) - \sum_{i=1}^{5} I^{\beta_i} h_i(t, x(t))) \\ \quad = \frac{1}{1+2t^2}(\frac{\log t}{3}x(t) + \frac{1}{2}) + \frac{e^{1-t}}{3}\int_1^t \frac{1+|\sin \pi st|}{7+3st}x(s)\,ds, \quad t \in [1, e^{\frac{3}{2}}], \\ x(1) = 0. \end{cases} \tag{4.3}$$

Here $\alpha = 3/4$, $\beta_1 = 1/2$, $\beta_2 = 3/4$, $\beta_3 = 5/4$, $\beta_4 = 3/2$, $\beta_5 = 7/4$, $m = 5$, $T = e^{\frac{3}{2}}$, and

$$h_i(t, x) = \left(\frac{1}{i + 2\log t}\right)\left(\frac{x}{24 + i}\right), \quad i = 1, 2, 3, 4, 5,$$

$$g(t, x, y) = \frac{1}{1 + 2t^2}\left(\frac{\log t}{3}x + \frac{1}{2}\right) + \frac{ye^{1-t}}{3}, \qquad \varphi(t, s) = \frac{1 + |\sin \pi st|}{7 + 3st}.$$

With the given data, it is found that $\varphi_0 = 1/5$, $|g(t, x, y)| \leq (1/(1 + 2t^2))((1/2)|x| + (1/2)) + (e^{1-t}/3)|y|$, $|h_i(t, x)| \leq (1/(i + 2\log t))(|x|/(24 + i))$, $i = 1, 2, 3, 4, 5$. Clearly the conditions (H_1)-(H_3) are satisfied with $p_1(t) = 1/(1 + 2t^2)$, $\psi(|x|) = (1/2)(|x| + 1)$, $p_2(t) = e^{1-t}/3$, $q_i(t) = 1/(i + 2\log t)$, $\Omega_i(|x|) = |x|/(24 + i)$, $i = 1, 2, 3, 4, 5$. Moreover, $\gamma = 4.681329240$, $\|p_1\| = 1/3$, $\|p_2\| = 1/3$, $\|q_i\| = 1/i$, $i = 1, 2, 3, 4, 5$, $\|p_2\|\varphi_0[\frac{\gamma}{\Gamma(\alpha)} + \frac{(\log T)^\alpha}{\Gamma(\alpha+1)}] \approx 0.3529974 < 1$. Hence there exists a positive number $M_0 > \widetilde{M}_0 \approx 1.8885976$. Therefore, by Theorem 3.3, the problem (4.3) has at least one solution on $[1, e^{\frac{3}{2}}]$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Each of the authors, BA, SKN, and JT contributed to each part of this work equally and read and approved the final version of the manuscript.

Author details
[1]Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah, 21589, Saudi Arabia. [2]Department of Mathematics, University of Ioannina, Ioannina, 451 10, Greece. [3]Nonlinear Dynamic Analysis Research Center, Department of Mathematics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, Bangkok, 10800, Thailand.

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