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Zeros and fixed points of the linear combination of shifts of a meromorphic function

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Abstract

Let f be transcendental and meromorphic in the complex plane. In this article, we investigate the existences of zeros and fixed points of the linear combination and quotients of shifts of $f(z)$ when $f(z)$ is of order one. We also prove a result concerning the linear combination which extends a result of Bergweiler and Langley. Some results concerning the order of $f(z) < 1$ are also obtained.

MSC: 30D35; 39A05

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1 Introduction and main results

In this article, a function is called meromorphic if it is analytic in the whole complex plane except at possible isolated poles. We assume that readers are familiar with the basic results and notations of the Nevanlinna value distribution theory of meromorphic functions (see, *e.g.*, [1–4]).

Let f be a transcendental meromorphic function in the plane. The forward differences $\Delta^n(f)$ are defined in the standard way ([5], p.52) by $\Delta f = f(z + 1) - f(z)$, $\Delta^{n+1}(f) = \Delta^n f(z + 1) - \Delta^n f(z)$, $n = 0, 1, 2, \ldots$.

Recently, many excellent results concerning the Nevanlinna theory for difference operators have been obtained (see, *e.g.*, [6–13]). For example, Halburd and Korhonen [11], respectively Chiang and Feng [7] obtained the difference analogue of logarithmic derivatives lemma. Their results became a starting point of investigating Nevanlinna theory on difference operators and difference equations. Another important result belongs to Bergweiler and Langley. In [6], Bergweiler and Langley firstly investigate the existence of zeros of Δf and $\frac{\Delta f}{f}$. The results may be viewed as discrete analogue of the following existence theorem on the zeros of f'.

Theorem A ([14–16]) *Let $f(z)$ be transcendental and meromorphic in the plane with*

$$\varliminf_{r \to +\infty} \frac{T(r,f)}{r} = 0,$$

then f' has infinitely many zeros.

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Hurwitz theorem implies that under the hypotheses of Theorem A, $f(z + c) - f(z)$ has a zero near z_1 for all sufficiently small $c \in C \setminus \{0\}$ if z_1 is a zero of f'. When f is a transcendental entire function of order less than one, then the first difference Δf, and by repetition of this argument each difference $\Delta^n f$, for $n \geq 1$, is transcendental entire of order less than one and hence has infinitely many zeros. Bergweiler and Langley considered the divided difference case and obtained the following theorem.

Theorem B ([6]) *There exists $\delta_0 \in (0, \frac{1}{2})$ with the following property. Let f be a transcendental entire function with order*

$$\sigma(f) \leq \sigma \leq \frac{1}{2} + \delta_0 < 1,$$

then

$$G(z) = \frac{\Delta f(z)}{f(z)} = \frac{f(z + 1) - f(z)}{f(z)}$$

has infinitely many zeros.

From the proof of Theorem B, it can be seen that δ_0 is extremely small, so they conjectured that the conclusion of Theorem B still holds for $\delta(f) < 1$. Chen and Shon partly answered this question and obtained the following.

Theorem C ([17]) *Let $n \in N$ and f be a transcendental entire function of order $\sigma(f) < 1$. Let $c \in C \setminus \{0\}$ and a set $H = \{z_j\}$ consist of all different zeros of $f(z)$, satisfying any one of the following two conditions*:
  (i) *at most finitely many zeros z_j, z_k satisfy $z_j - z_k = c$;*
  (ii) $\underline{\lim}_{j \to +\infty} |\frac{z_{j+1}}{z_j}| = l > 1.$
Then

$$G(z) = \frac{\Delta f(z)}{f(z)} = \frac{f(z + c) - f(z)}{f(z)}$$

has infinitely many zeros and infinitely many fixed points.

Bergweiler and Langley [6] also obtained the following results.

Theorem D *Let f be a function transcendental and meromorphic of lower order $\lambda(f) < \lambda < 1$ in the plane. Let $c \in C \setminus \{0\}$ be such that at most finitely many poles z_j, z_k of f satisfy $z_j - z_k = c$. Then $g(z) = f(z + c) - f(z)$ has infinitely many zeros.*

In Theorem D, Bergweiler and Langley considered the existence of zeros of first difference operator when the transcendental meromorphic function is of lower order less than one. Chen and Shon [17] considered the case when the order of f is equal to one. They proved the following results.

Theorem E *Let $c \in C \setminus \{0\}$ and f be a transcendental entire function of order $\sigma(f) = \sigma = 1$. If $f(z)$ have infinitely many zeros with the exponent of convergence of zeros $\lambda(f) < 1$, then $g(z) = \Delta f(z) = f(z + c) - f(z)$ has infinitely many zeros and infinitely many fixed points.*

In particular, if a set $H = \{z_j\}$ consists of all different zeros of $f(z)$, satisfying any one of the following two conditions:

(i) *at most finitely many zeros z_j, z_k satisfy $z_j - z_k = c$;*

(ii) $\underline{\lim}_{j \to +\infty} |\frac{z_{j+1}}{z_j}| = l > 1$,

then

$$G(z) = \frac{\Delta f(z)}{f(z)} = \frac{f(z+c) - f(z)}{f(z)}$$

has infinitely many zeros and infinitely many fixed points.

In order to continue to investigate the existence of zeros and fixed points of difference polynomials and their associated difference quotients, we give the linear combination of difference shifts of a meromorphic function as follows:

$$g(z) = \sum_{j=1}^{n} g_j(z) f(z + c_j), \qquad G(z) = \left[\sum_{j=1}^{n} g_j(z) f(z + c_j) \right] / f(z).$$

In fact, $\Delta^n f(z)$ is a special case of $g(z)$ since it can be rewritten as the following form:

$$\Delta^n f(z) = \sum_{j=0}^{n} C_n^j (-1)^{n-j} f(z + j).$$

By letting the polynomial coefficients of $g(z)$ be constants and c_j nonnegative integers, *i.e.*,

$$g_j(z) = C_n^{j-1} (-1)^{n-j+1}, \qquad c_j = j - 1,$$

for $j = 1, \dots, n+1$, we could find that $g(z)$ and $G(z)$ are general coefficient cases of $\Delta^n f(z)$ and $\frac{\Delta^n f(z)}{f(z)}$, respectively.

Hence, a natural question arises from Theorem E: What can be said about the existence of zeros and fixed points of Δf and $\frac{\Delta f}{f}$, when $f(z)$ is a meromorphic function with order $\sigma(f) = 1$?

In [18], Cui and Yang considered and gave answers to this question. Now, since we introduce the general form of Δf and $\frac{\Delta f}{f}$, the above question acquires its new forms:

Do the conclusions of Theorems B-E still hold for $g(z)$ and $g(z)/f(z)$, when $f(z)$ is a meromorphic function with order $\sigma(f) < 1$ or even $\sigma(f) \leq 1$? In the present article, we answer this question and obtain the following results.

Theorem 1.1 *Let f be a transcendental meromorphic function in the plane with the order $\sigma(f) = \sigma = 1$. If f has finitely many poles and infinitely many zeros with the exponent of convergence of zeros $\lambda(f) = \lambda \neq 1$. If the polynomial coefficients $\{g_j(z)\}_{j=1}^{n}$ of $g(z)$ satisfy*

$$\deg(g_l(z)) > \max_{1 \leq j \leq n, j \neq l} \{\deg(g_j(z))\},$$

then $g(z) = \sum_{j=1}^{j=n} g_j(z) f(z + c_j)$ has infinitely many zeros and fixed points.

Remark 1.1 By Lemma 2.8 in the following part, we can easily have the following result concerning $\sigma(f) < 1$.

Theorem 1.2 *Let f be a transcendental meromorphic function in the plane with the order $\sigma(f) = \sigma < 1$. If f has finitely many poles and the polynomial coefficients $\{g_j(z)\}_{j=1}^n$ of g(z) satisfy*

$$\deg\big(g_l(z)\big) > \max_{1 \le j \le n, j \ne l} \big\{\deg\big(g_j(z)\big)\big\},$$

then $g(z) = \sum_{j=1}^{j=n} g_j(z) f(z + c_j)$ has infinitely many zeros and fixed points.

Now we consider the existence of zeros and fixed points of $G(z)$. From the proofs of Theorem 1.1 and Theorem 1.2, we notice that the property of $g(z)$, *i.e.,* whether it is transcendental or not, actually determines our conclusions. Therefore, in the following we still first consider the property of $G(z)$ and then investigate zeros and fixed points of $G(z)$ when f is of order equal to or less than one.

Theorem 1.3 *Let $G(z) = \frac{g(z)}{f(z)}$ and the polynomial coefficients $\{g_j(z)\}_{j=1}^n$ of g(z) satisfy*

$$\deg\big(g_l(z)\big) > \max_{1 \le l \le n, j \ne l} \big\{\deg\big(g_j(z)\big)\big\}, \qquad c_l \ne 0.$$

Then G(z) is transcendental provided f satisfies any one of the following two conditions:
  (i) *f is transcendental entire and the order of growth of f satisfies*

$$\underline{\lim}_{r \to +\infty} \frac{T(r,f)}{r} = 0;$$

  (ii) *f is transcendental meromorphic and the order of growth of f satisfies*

$$\underline{\lim}_{r \to +\infty} \frac{T(r,f)}{r^{\frac{1}{2}}} = 0.$$

Remark 1.2 In fact, Theorem 1.3 is a generalization of the following last part of Theorem F.

Theorem F ([6]) *Let $n \in N$ and let f be a transcendental entire function of order $< \frac{1}{2}$, and set*

$$G(z) = \frac{\Delta^n f(z)}{f(z)}.$$

If G is transcendental, then G has infinitely many zeros. In particular, if f has order less than $\min\{\frac{1}{n}, \frac{1}{2}\}$, then G is transcendental and has infinitely many zeros.

In the following, we will investigate the existence of zeros and fixed points of $G(z)$ and obtain some results related to this.

Theorem 1.4 *Let f be a function transcendental and meromorphic in the plane with the order $\sigma(f) = \sigma = 1$. If f has finitely many poles and infinitely many zeros with the exponent*

of convergence of zeros $\lambda(f) = \lambda \neq 1$. If the polynomial coefficients $\{g_j(z)\}_{j=1}^n$ of $g(z)$ satisfy

$$\deg\big(g_l(z)\big) > \max_{1 \le j \le n, j \neq l} \big\{\deg\big(g_j(z)\big)\big\}, \quad c_l \neq 0,$$

then $G(z) = \frac{g(z)}{f(z)}$ has infinitely many zeros and fixed points.

The following theorem can be acquired by using Theorem 1.3 and using the same method as in the proof of Theorem 1.2.

Theorem 1.5 *Let f be a function transcendental and meromorphic in the plane with the order $\sigma(f) = \sigma < \frac{1}{2}$. If $f(z)$ has only finitely many poles and if the polynomial coefficients $\{g_j(z)\}_{j=1}^n$ of $g(z)$ satisfy*

$$\deg\big(g_l(z)\big) > \max_{1 \le j \le n, j \neq l} \big\{\deg\big(g_j(z)\big)\big\}, \quad c_l \neq 0,$$

then $G(z) = \frac{g(z)}{f(z)}$ has infinitely many zeros and fixed points.

In particular, if f is transcendental entire with $\sigma(f) < 1$, and the polynomial coefficients $\{g_j(z)\}_{j=1}^n$ of $g(z)$ satisfy

$$\deg\big(g_l(z)\big) > \max_{1 \le j \le n, j \neq l} \big\{\deg\big(g_j(z)\big)\big\}, \quad c_l \neq 0,$$

then $G(z) = \frac{g(z)}{f(z)}$ has infinitely many zeros and fixed points.

2 Some lemmas

Lemma 2.1 ([15]) *Let g be a function transcendental and meromorphic in the plane of order less than one. Let $h > 0$. Then there exists an ε-set E such that*

$$\frac{g'(z+c)}{g(z+c)} \to 0, \qquad \frac{g(z+c)}{g(z)} \to 1,$$

as $z \to \infty$ in $C\backslash E$, uniformly in c for $|c| \le h$. Further, E may be chosen so that for large z not in E, the function g has no zeros or poles in $|\zeta - z| \le h$.

Remark 2.1 ([6]) Following Hayman ([19], pp.75-76), define an ε-set E to be a countable union of discs

$$E = \bigcup_{j=1}^{\infty} B(b_j, r_j)$$

such that

$$\lim_{j \to \infty} |b_j| = \infty, \qquad \sum_{j=1}^{\infty} \frac{r_j}{|b_j|} < \infty.$$

Here and henceforth, $B(a, r)$ denotes the open disc of center a and radius r, and $S(a, r)$ will denote the corresponding boundary circle. Note that if E is an ε-set, then the set of $r \ge 1$, for which the circle $S(0, r)$ meets E, has finite logarithmic measure and hence zero logarithmic density.

Lemma 2.2 ([4]) *Let $f_j(z)$ $(j = 1, \ldots, n)$ be meromorphic functions, $g_j(z)$ $(j = 1, \ldots, n)$ be entire functions and satisfy*:

    (i) $\sum_{j=1}^{n} f_j(z) e^{g_j(z)} \equiv 0$;

    (ii) *when $1 \le j < k \le n$, $g_j(z) - g_k(z)$ is not a constant*;

    (iii) *when $1 \le j \le n$, $1 \le h < k \le n$,*

$$T(r, f_j) = o\big(T\big(r, e^{g_h - g_k}\big)\big) \quad (r \to \infty, r \notin E),$$

    where $E \subset (1, \infty)$ is of finite linear measure or finite logarithmic measure.
Then $f_j(z) \equiv 0$ $(j = 1, \ldots, n)$.

Lemma 2.3 ([6]) *Let f be a transcendental meromorphic function in the plane of order less than one. Let $h > 0$. Then there exists an ε-set E such that*

$$f(z + c) - f(z) = cf'(z)\big(1 + o(1)\big)$$

as $z \to \infty$ in $C \setminus E$, uniformly in c for $|c| \le h$.

Lemma 2.4 ([6]) *If f is transcendental and meromorphic of order less than one in the plane. Let $n \in N$. Then there exists an ε-set E_n such that*

$$\Delta^n f(z) \sim f^{(n)}(z)$$

as $z \to \infty$ in $C \setminus E_n$.

Lemma 2.5 ([20]) *Let f be a transcendental meromorphic function with finite order $\sigma(f) = \sigma$, $H = \{(k_1, j_1), (k_2, j_2), \ldots, (k_q, j_q)\}$ be a finite set of distinct pairs of integers that satisfy $k_i > j_i \ge 0$ for $i = 1, \ldots, q$, and let $\varepsilon > 0$ be a given constant. Then there exists a set $E \subset (1, \infty)$ with finite logarithmic measure such that for all z satisfying $|z| \notin E \cup [0, 1]$ and for all $(k, j) \in H$, we have*

$$\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \le |z|^{(k-j)(\sigma - 1 + \varepsilon)}.$$

Lemma 2.6 ([7]) *Let η_1, η_2 be two arbitrary complex numbers and let $f(z)$ be a meromorphic function of finite order σ. Let $\varepsilon > 0$ be given, then there exists a subset $E \subset R$ with finite logarithmic measure such that for all $r \notin E \cup [0, 1]$, we have*

$$\exp\big(-r^{\sigma - 1 + \varepsilon}\big) \le \left| \frac{f(z + \eta_1)}{f(z + \eta_2)} \right| \le \exp\big(r^{\sigma - 1 + \varepsilon}\big).$$

The following Lemma 2.7 can be easily obtained from Chiang and Feng (Theorem 9.4 [7]), here we omit its proof.

Lemma 2.7 *Let $g_1(z), \ldots, g_n(z)$ be polynomials such that there exists an integer $k, 1 \le k \le n$, so that*

$$\deg\big(g_k(z)\big) > \max_{1 \le j \le n, j \ne k} \big\{\deg\big(g_j(z)\big)\big\}$$

holds. Suppose that $f(z)$ is a meromorphic solution to

$$g_n(z)y(z + c_n) + \cdots + g_1(z)y(z + c_1) = 0,$$

where $c_j, j = 1, \ldots, n$, are complex numbers of which at least two are distinct. Then $\sigma(f) \geq 1$.

Lemma 2.8 *Let $f(z)$ be a transcendental meromorphic function satisfying $\sigma(f) = \sigma < 1$, and $\{g_j(z)\}$ be polynomials. If there exists an integer l, $1 \leq l \leq n$, such that*

$$\deg(g_l(z)) > \max_{1 \leq j \leq n, j \neq l} \{\deg(g_j(z))\}$$

holds, then $g(z)$ is transcendental and meromorphic.

Proof Suppose that $g(z)$ is not transcendental. Then we divide our proof into two cases.

Case 1. If $g(z) \equiv 0$, that is,

$$g_n(z)f(z + c_n) + \cdots + g_1(z)f(z + c_1) \equiv 0.$$

Since

$$\deg(g_l(z)) > \max_{1 \leq j \leq n, j \neq l} \{\deg(g_j(z))\} \tag{2.1}$$

holds, and by using Lemma 2.7, we can get that

$$\sigma(f) \geq 1.$$

Clearly, this contradicts with $\sigma(f) < 1$. Hence $g(z) \not\equiv 0$.

Case 2. If $g(z)$ is rational, without loss of generality, we may suppose that $g(z)$ is a nonzero polynomial.

By Lemma 2.1 and $\sigma(f) < 1$, we know that there exists an ε-set E such that

$$f(z + c_j) = f(z)(1 + o(1)) \tag{2.2}$$

as $z \to \infty$ in $C \backslash E$. Therefore,

$$g(z) = f(z)\left[g_1(z)(1 + o(1)) + \cdots + g_n(z)(1 + o(1))\right]. \tag{2.3}$$

Since (2.1) holds, we know that

$$g_1(z) + \cdots + g_n(z) \not\equiv 0. \tag{2.4}$$

So, by (2.3) and (2.4), we obtain that

$$f(z) = \frac{g(z)}{g_1(z)(1 + o(1)) + \cdots + g_n(z)(1 + o(1))} \tag{2.5}$$

as $z \to \infty$ in $C \backslash E$. The following proof is standard, but a proof will be given for completeness.

Set $H = \{r : |z| > 1, z \in C \backslash E\}$. Thus by Remark 2.1, we see that $H \subset (1, +\infty)$ is of finite logarithmic measure. Since $g(z), g_j(z), j = 1, \ldots, n$, are polynomials, by (2.5) we get that

$$T(r, f) \leq T(r, g) + T\left(r, g_1(z)\left(1 + o(1)\right) + \cdots + g_n(z)\left(1 + o(1)\right)\right) + O(1)$$

$$\leq T(r, g) + \sum_{j=1}^{n} T(r, g_j) + O(1), \quad r \notin H.$$

Thus, for all sufficient large r,

$$T(r, f) = O(\log r),$$

which contradicts with the fact that f is transcendental.

Hence, $g(z)$ is transcendental. $\qquad \square$

Lemma 2.9 ([16, 21]) *If f is transcendental entire satisfying $\underline{\lim}_{r \to +\infty} \frac{T(r, f)}{r} = 0$, or transcendental meromorphic with $\underline{\lim}_{r \to +\infty} \frac{T(r, f)}{r^{\frac{1}{2}}} = 0$, then $\frac{f'}{f}$ has infinitely many zeros.*

3 Proofs of Theorem 1.1 and Theorem 1.2

Proof of Theorem 1.1 According to the condition, suppose that $\{b_\nu\}$, $\nu = 1, \ldots, n$, are poles of $f(z)$ with multiplicity τ_ν, and we let

$$p(z) = \prod_{\nu=1}^{n} (z - b_\nu)^{\tau_\nu}.$$

Then $p(z)f(z)$ is entire and transcendental.

Set $F(z) = p(z)f(z)$. Then, by the Hadamard factorization theorem and $\lambda(f) < \sigma(f)$, we have $F(z) = h(z)e^{az+b}$, where $a \neq 0$ and b are constants, $h(z)$ is an entire function satisfying $\sigma(h) = \lambda(h) = \lambda(f) < \sigma(f) = 1$.

Since $g(z) = \sum_{j=1}^{n} g_j(z)f(z + c_j)$ and $F(z) = p(z)f(z) = h(z)e^{az+b}$, we know that

$$g(z) = \sum_{j=1}^{n} g_j(z) \frac{h(z + c_j)}{p(z + c_j)} e^{az+b+ac_j}$$

$$= \left(\sum_{j=1}^{n} g_j(z) \frac{h(z + c_j)}{p(z + c_j)} e^{ac_j}\right) e^{az+b}.$$

Note that $\sigma(h) < 1$, by Lemma 2.1, there exists an ε-set E such that

$$h(z + 1) \sim h(z)$$

as $z \to \infty$ in $C \backslash E$. Thus

$$g(z) \sim \left(\sum_{j=1}^{n} g_j(z) \frac{1}{p(z + c_j)} e^{ac_j}\right) h(z) e^{az+b}.$$

Since

$$\deg\big(g_l(z)\big) > \max_{1 \leq j \leq n, j \neq l}\big\{\deg\big(g_j(z)\big)\big\}$$

holds, we know that

$$\sum_{j=1}^{n} g_j(z)\frac{1}{p(z+c_j)}e^{ac_j} \not\equiv 0.$$

If $g(z)$ has only finitely many zeros, then there exists a nonzero rational function $R(z)$ such that

$$\sum_{j=1}^{n} g_j(z)\frac{h(z+c_j)}{p(z+c_j)}e^{ac_j} = R(z). \tag{3.1}$$

By Lemma 2.8 and $\sigma(h) < 1$, we know that (3.1) is impossible. Hence $g(z)$ has infinitely many zeros.

Now we prove that $g(z) = \sum_{j=1}^{n} g_j(z)f(z+c_j)$ has infinitely many fixed points. Set

$$g^*(z) = g(z) - z.$$

Thus we only need to prove that $g^*(z)$ has infinitely many zeros.

If $g^*(z)$ has only finitely many zeros, then we have

$$g^*(z) = R^*(z)e^{dz+\alpha},$$

where $R^*(z)$ is a rational function, $d \neq 0$ and α are constants. Without loss of generality, we may suppose that $\alpha = b$. In fact, if $\alpha \neq b$, then $R^*(z)e^{dz+\alpha} = e^{\alpha-b}R^*(z)e^{dz+b}$, and $e^{\alpha-b}R^*(z)$ is also a rational function.

So,

$$R^*(z)e^{dz+b} = h_0(z)e^{az+b} - z,$$

where $h_0(z) = \sum_{j=1}^{n} g_j(z)\frac{h(z+c_j)}{p(z+c_j)}e^{ac_j}$. Since

$$\deg\big(g_l(z)\big) > \max_{1 \leq j \leq n, j \neq l}\big\{\deg\big(g_j(z)\big)\big\}$$

and $\sigma(h) < 1$, by Lemma 2.8, we know that $h_0(z)$ is transcendental, and $\sigma(h_0) < 1$.

We affirm that $a = d$.

If $a \neq d$, then

$$R^*(z)e^{dz+b} - h_0(z)e^{az+b} + ze^0 = 0.$$

This satisfies conditions of Lemma 2.2. Thus, by Lemma 2.2, we have that $R^*(z) \equiv h_0(z) \equiv z \equiv 0$, this is a contradiction. Hence $a = d$.

So,

$$\left[R^*(z) - h_0(z)\right]e^{az+b} = z,$$

again by Lemma 2.2, we get

$$R^*(z) - h_0(z) \equiv 0, \quad z \equiv 0.$$

This also is a contradiction.

Hence, $g(z)$ has infinitely many fixed points. □

Proof of Theorem 1.2 According to Lemma 2.8, we know that $g(z)$ is transcendental under conditions of Theorem 1.2 and the order $\sigma(g) < 1$, which means $g(z)$ has either infinitely many zeros or infinitely many poles. Since $f(z)$ has only finitely many poles, *i.e.*, $g(z)$ has finitely many poles. Hence $g(z)$ must have infinitely many zeros.

The same discussion also holds when we consider the fixed points of $g(z)$. Hence, $g(z)$ has infinitely many fixed points. □

We complete our proofs of Theorem 1.1 and Theorem 1.2.

4 Proofs of Theorem 1.3 and Theorem 1.4

Proof of Theorem 1.3 First we prove that $G(z)$ is transcendental. Suppose that G is rational. Then we obtain that

$$\sum_{j=1}^{n} g_j(z)\left[\frac{f(z+c_j) - f(z)}{f(z)}\right] = R_1(z) - \sum_{j=1}^{n} g_j(z),$$

where $R_1(z)$ is rational.

According to Lemma 2.3, we know that

$$f(z+c_j) - f(z) = c_j f'(z)\left(1 + o(1)\right)$$

as $z \to \infty$ in $C \setminus E$. So we get

$$\sum_{j=1}^{n} c_j g_j(z)\frac{f'(z)}{f(z)}\left(1 + o(1)\right) = R_1(z) - \sum_{j=1}^{n} g_j(z). \tag{4.1}$$

Since $c_l \neq 0$ and

$$\deg\left(g_l(z)\right) > \max_{1 \leq j \leq n, j \neq l}\left\{\deg\left(g_j(z)\right)\right\},$$

we know that

$$\sum_{j=1}^{n} c_j g_j(z) \not\equiv 0.$$

So,

$$\frac{f'(z)}{f(z)} = \frac{R_1(z) - \sum_{j=1}^{n} g_j(z)}{\sum_{j=1}^{n} c_j g_j(z)(1 + o(1))}. \tag{4.2}$$

Since $g_j(z)$ are polynomials. Hence the right-hand side of (4.2) is rational. We can easily get a contradiction from the two sides of (4.2) since condition (i) or (ii) separately guarantees that the left-hand side $\frac{f'(z)}{f(z)}$ has infinitely many zeros by Lemma 2.9. Therefore, $G(z)$ is transcendental. We complete the proof of Theorem 1.3. □

Proof of Theorem 1.4 Now we prove that $G(z)$ has infinitely many zeros. We continue to use symbols in the proofs of Theorem 1.1 and Theorem 1.2.

Since

$$
\begin{aligned}
G(z) &= \frac{g(z)}{f(z)} \\
&= \frac{\left(\sum_{j=1}^{n} g_j(z)\frac{h(z+c_j)}{p(z+c_j)}e^{ac_j}\right)e^{az+b}}{\frac{h(z)}{p(z)}e^{az+b}} \\
&= \sum_{j=1}^{n} g_j(z)\frac{p(z)e^{ac_j}}{p(z+c_j)}\left[\frac{h(z+c_j)}{h(z)}\right] \\
&= \sum_{j=1}^{n} g_j(z)\frac{p(z)e^{ac_j}}{p(z+c_j)}\frac{\Delta_j h(z)}{h(z)} + \sum_{j=1}^{n} g_j(z)\frac{p(z)e^{ac_j}}{p(z+c_j)} \\
&= G_0(z) + g_0(z),
\end{aligned}
$$

where

$$g_0(z) = \sum_{j=1}^{n} g_j(z)\frac{p(z)e^{ac_j}}{p(z+c_j)}$$

is rational, $\Delta_j h(z) = h(z + c_j) - h(z)$ and

$$G_0(z) = \sum_{j=1}^{n} g_j(z)\frac{p(z)e^{ac_j}}{p(z+c_j)}\frac{\Delta_j h(z)}{h(z)}.$$

From Theorem 1.3, since $h(z)$ is transcendental and entire with $\sigma(h) < 1$, we know that $G_0(z)$ is transcendental and $\sigma(G_0) \le \sigma(h) < 1$, and by Lemma 2.3 and Lemma 2.4, we also know that there exists an ε-set E such that

$$G_0(z) \sim \sum_{j=1}^{n} g_j(z)\frac{p(z)e^{ac_j}}{p(z+c_j)}\frac{c_j h'(z)}{h(z)} \sim g_0(z)\frac{c_l h'(z)}{h(z)}$$

as $z \to \infty$ in $C \backslash E$, where E contains all zeros and poles of $G_0(z)$. Hence, there exists a subset $F_1 \subset (1, \infty)$ of finite logarithmic measure such that for large $|z| = r$ not in F_1, $z \notin E$ and

$$G(z) \sim g_0(z)\frac{c_l h'(z)}{h(z)} + g_0(z). \tag{4.3}$$

We could find a meromorphic function $q(z)$ such that $c_l \frac{h'}{h} = \frac{q'}{q}$. In fact, by integrating both sides of $c_l \frac{h'}{h} = \frac{q'}{q}$, we obtain that $q(z) = Ah(z)^{c_l}$, where A is a nonzero constant. Since $\sigma(h) < 1$, we know that $q(z)$ is of finite order < 1 and transcendental. Hence, by Lemma 2.5, there exists a set $F_2 \subset (1, \infty)$ of finite logarithmic measure for any given ε $(0 < 2\varepsilon < 1 - \sigma(q))$, such that for large $|z| = r$ not in F_2,

$$\left| g_0(z) \frac{c_l h'(z)}{h(z)} \right| = \left| g_0(z) \frac{q'(z)}{q(z)} \right| \leq \left| g_0(z) \right| \cdot |z|^{\sigma(q)-1+\varepsilon}. \tag{4.4}$$

Define an ε-set E^* which consists of all zeros and poles of $G(z)$, then there exists a set $F_3 \subset (1, \infty)$ of finite logarithmic measure such that if $z \in E^*$, then $|z| = r \in F_3$.

Thus, by (4.3) and (4.4), we see that for large $|z| = r \notin [0,1] \cup F_1 \cup F_2 \cup F_3$, $G(z)$ has no zeros and poles on $|z| = r$ and

$$\begin{aligned} \left| G(z) - g_0(z) \right| &= \left| G_0(z) \right| \\ &= \left| g_0(z) \frac{c_l h'(z)}{h(z)} \left(1 + o(1) \right) \right| \leq \left| g_0(z) \right| \cdot |z|^{-\varepsilon} < \left| G(z) \right| + \left| -g_0(z) \right| \end{aligned} \tag{4.5}$$

holds on $|z| = r$.

According to the Rouché theorem, we obtain that

$$n\left(r, \frac{1}{G(z)} \right) - n\left(r, G(z) \right) = n\left(r, \frac{1}{g_0(z)} \right) - n\left(r, g_0(z) \right) = \deg(g_l) \geq 1. \tag{4.6}$$

Since $G(z)$ is transcendental and $\sigma(G(z)) < 1$, we know that at least one of $n(r, \frac{1}{G(z)}) \to \infty$ and $n(r, G(z)) \to \infty$ (as $r \to \infty$) holds. By (4.6), $G(z)$ has either finitely many zeros and poles or infinitely many zeros and poles. Thus, we obtain that

$$n\left(r, G(z) \right) \to \infty$$

and

$$n\left(r, \frac{1}{G(z)} \right) \to \infty.$$

That is to say, $G(z)$ must have infinitely many zeros.

Let $G^*(z) = G(z) - z$, we need to consider zeros of $G^*(z)$ in order to estimate the fixed points of $G(z)$. Since $G(z) = G_0(z) + g_0(z)$, we obtain the following inequality by using the above methods and notations:

$$G_*(z) - g_0(z) = G_0(z) - z \sim g_0(z) \frac{c_l h'(z)}{h(z)} - z = g_0(z) \frac{q'(z)}{q(z)} - z.$$

We could let $g_0(z) \frac{q'(z)}{q(z)} - z = g_0(z) \frac{m'(z)}{m(z)}$ since by integrating we obtain that $m(z) = \frac{q}{B \exp \int \frac{z}{g_0} dz}$, where B is a nonzero constant. $m(z)$ is of finite order because $q(z)$ is of finite order. And we let $r(z) = \int \frac{z}{g_0} dz$, then $r'(z) = \frac{z}{g_0}$, which means $\exp \int \frac{z}{g_0} dz$ is also of finite order. Therefore, we can obtain that

$$G_*(z) - g_0(z) \sim g_0(z) \frac{m'(z)}{m(z)}.$$

The following proof is similar to the above. We can get a similar equation to that of (4.6) by applying the Rouché theorem. At last, we get our conclusion that $G(z)$ has infinitely many fixed points.

Theorem 1.4 follows. □

5 Proof of Theorem 1.5

When $f(z)$ is meromorphic with $\sigma(f) < \frac{1}{2}$, since

$$\deg\big(g_l(z)\big) > \max_{1\le j\le n, j\ne l}\big\{\deg\big(g_j(z)\big)\big\}, \qquad c_l \ne 0$$

by Theorem 1.3 we could immediately find that $G(z)$ is transcendental and the order of $G(z)$ is less than $\frac{1}{2}$, which means $G(z)$ has either infinitely many zeros or infinitely many poles. In what follows, we could obtain

$$n\left(r, \frac{1}{m(z)}\right) - n\big(r, m(z)\big) = n\left(r, \frac{1}{g(z)}\right) - n\big(r, g(z)\big) = \deg(g_l) \ge 1$$

by using the same method as in the proof of Theorem 1.4, where $m(z)$ represents $G(z)$ or $G(z) - z$. Therefore, $G(z)$ must have infinitely many zeros and fixed points.

When $f(z)$ is transcendental and entire with $\sigma(f) < 1$, we can also obtain the same conclusion by the same discussion as above. Theorem 1.5 follows.

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