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Affine-periodic solutions for nonlinear dynamic equations on time scales

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Abstract
The existence of affine-periodic solutions for dynamic equations on time scales is studied. Mainly, via the topological degree theory, a general existence theorem is proved, which provides an effective method in the qualitative theory for nonlinear dynamic equations on time scales.

Keywords: affine-periodic solution; time scales; topological degree article

1 Introduction

The periodicity problem is a very important topic in the study of differential equations, but not all the natural phenomena can be described by periodicity only. We found that some differential equations exhibit a certain symmetry rather than periodicity [1–3]. For example, we denote by $GL_n(R)$ the n-dimensional general linear group over R^1 and consider the system

$$x' = f(t,x), \qquad ' = \frac{d}{dt}, \tag{1}$$

where $f : R^1 \times R^n \to R^n$ is continuous, and for some $Q \in GL_n(R)$, the following affine symmetry holds:

$$f(t+T,x) = Qf\left(t, Q^{-1}x\right). \tag{2}$$

In the sense of (2), we have the concept of an affine-periodic system (APS for short).

Definition 1.1 The system (1) is said to be a (Q,T)-affine-periodic system, if there exists $Q \in GL_n(R)$ and $T > 0$ such that

$$f(t+T,x) = Qf\left(t, Q^{-1}x\right)$$

holds for all $(t,x) \in R^1 \times R^n$.

For APS (1), we define its affine-periodic solution as follows.

Definition 1.2 If $x(t)$ is a solution of APS (1) on R^1 and

$$x(t+T) = Qx(t) \quad \forall t, \tag{3}$$

then $x(t)$ is said to be a (Q, T)-affine-periodic solution.

As a structural property of functions, the affine-periodicity is a generalization of pure periodicity. Recently, some existence theorems as regards affine-periodic solutions have been proved for APSs. We refer to [1–4]. Particularly, in [1], the existence of affine-periodic solutions for APS (1) was established via topological degree theory. However, the existence of affine-periodic solutions for APSs when 'time' is not continuous has not been discussed yet. The aim of this paper is to touch on such a topic for APSs on time scales.

A time scale is an arbitrary non-empty closed subset of R^1, often denoted by $\mathbb{T}$. The theory of time scales was first introduced by Hilger [5] in 1990 in order to study differences between discrete and continuous analysis. The time scale calculus offers great help on unifying discrete and continuous dynamic systems and presents a powerful tool for applications to economics and biology models, among others. Hence it has been attracting more and more attention during the past years and the existence of solutions for systems on time scales has become an interesting and popular topic. In this respect, Amster *et al.* [6] studied boundary value problems for dynamic systems and proved the existence of solutions via topological degree theory; Lizama and Mesquita [7] considered nonautonomous dynamic equations and proved the existence of almost automorphic solutions by assuming that the associated homogeneous equation of the system admits an exponential dichotomy. In the field of second- and higher-order periodic problems, the lower and upper solutions technique is a powerful tool and has been used in most of the studies, for example, the work of Akin [8] and Stehlik [9]. In this paper, we will establish a topological degree theory to prove the existence of affine-periodic solutions for APSs on time scales, which is not very common in this area.

To discuss the APSs on time scales, some basic notations and definitions are needed, and most of them can be found in [10]. The first basic and important concept in the theory of time scales is the forward (backward) jump operator.

Definition 1.3 Let $\mathbb{T}$ be a time scale. For $t \in \mathbb{T}$ we define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\},$$

while the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\}.$$

In Definition 1.3, we put $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$.

Definition 1.4 The graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by

$$\mu(t) = \sigma(t) - t.$$

A point $t \in \mathbb{T}$ is called right-scattered if $\sigma(t) > t$, while if $\rho(t) < t$ we say that t is left-scattered. Points that are right-scattered and left-scattered at the same time are called isolated. Also, if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then t is called right-dense, and if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is called left-dense. Points that are right-dense and left-dense at the same time are called dense. If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^\kappa = \mathbb{T} \setminus \{m\}$, otherwise $\mathbb{T}^\kappa = \mathbb{T}$. For a time scale $\mathbb{T}$, we denote by $[a,b]_\mathbb{T}$ ($(a,b)_\mathbb{T}$) the set $[a,b] \cap \mathbb{T}$ ($(a,b) \cap \mathbb{T}$), where $a, b \in R^1$. Then we have the definition of the so-called delta derivative.

Definition 1.5 Assume $f : \mathbb{T} \to R^n$ is a function and let $t \in \mathbb{T}^\kappa$. Then we define $f^\Delta(t)$ to be the vector (provided it exists) with the property that given any $\epsilon > 0$, there is a neighborhood U of t (*i.e.* $U = (t-\delta, t+\delta)_\mathbb{T}$ for some $\delta > 0$) such that

$$\bigl|\bigl[f\bigl(\sigma(t)\bigr) - f(s)\bigr] - f^\Delta(t)\bigl[\sigma(t) - s\bigr]\bigr| \le \epsilon \bigl|\sigma(t) - s\bigr| \quad \text{for all } s \in U.$$

We call $f^\Delta(t)$ the delta derivative of f at t.

In order to describe classes of functions that are integrable, we need the following definition.

Definition 1.6 A function $f : \mathbb{T} \to R^n$ is called rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. The set of rd-continuous functions $f : \mathbb{T} \to R^n$ will be denoted by

$$C_{\mathrm{rd}} = C_{\mathrm{rd}}(\mathbb{T}) = C_{\mathrm{rd}}\bigl(\mathbb{T}, R^n\bigr).$$

Let $f \in C_{\mathrm{rd}}$. If $F^\Delta(t) = f(t)$, we have

$$F(t) - F(a) = \int_a^t f(s)\Delta s.$$

Remark 1.1 $F(t)$ is continuous on $\mathbb{T}$ when $f \in C_{\mathrm{rd}}(\mathbb{T})$ (see Theorem 1.16 in [10]).

We also need the notion of periodic time scales which was introduced by Atici *et al.* [11, 12]. The following definition is borrowed from [11–13].

Definition 1.7 We say that a time scale $\mathbb{T}$ is periodic if there exists $T > 0$ such that if $t \in \mathbb{T}$ then $t \pm T \in \mathbb{T}$. For $\mathbb{T} \neq R^1$, the smallest positive T is called the period of the time scale.

Now, we can define the APSs on time scales.

Definition 1.8 Let $\mathbb{T}$ be a T-periodic time scale, $f : \mathbb{T} \times R^n \to R^n$ a rd-continuous function. The system

$$x^\Delta = f(t,x) \tag{4}$$

is said to be a (Q,T)-affine-periodic system, if there exists $Q \in GL_n(R)$ such that

$$f(t+T, x) = Qf\bigl(t, Q^{-1}x\bigr)$$

holds for all $(t,x) \in \mathbb{T} \times R^n$.

The solutions of dynamic equations on time scales are defined as follows.

Definition 1.9 Consider the equation

$$x^{\Delta} = f(t,x), \tag{5}$$

where $f : \mathbb{T} \times R^n \to R^n$. A function $x : \mathbb{T} \to R^n$ is called a solution of (5) if

$$x \in \{y : y \in C(\mathbb{T}, R^n), y^{\Delta} \in C_{\mathrm{rd}}(\mathbb{T}, R^n)\}$$

and $x(t)$ satisfies (5) for all $t \in \mathbb{T}$. Given $t_0 \in \mathbb{T}$ and $x_0 \in R^n$, the problem

$$x^{\Delta} = f(t,x), \qquad x(t_0) = x_0$$

is called an initial value problem, and a solution of (5) with $x(t_0) = x_0$ is called a solution to this initial value problem.

Hence we have the definition of affine-periodic solutions.

Definition 1.10 A function $x : \mathbb{T} \to R^n$ is said to be an affine-periodic solution of (4) if $x(t)$ is a solution of (4) and for any $t \in \mathbb{T}$,

$$x(t + T) = Qx(t).$$

According to the definitions of APS on time scales and its solutions, we have the following existence theorem. The proof can be found in Section 3.

Consider the APS

$$x^{\Delta} = f(t,x), \tag{6}$$

where $f : \mathbb{T} \times R^n \to R^n$ is rd-continuous and ensures the uniqueness of solutions with respect to initial value, $Q \in O(n)$, $\mathbb{T}$ is a T-periodic time scale.

Theorem 1.1 *Let $D \subset R^n$ be a bounded open set. Assume the following hypotheses hold for the system* (6).

(H_1) *For each $\lambda \in (0,1]$, every possible affine-periodic solution $x(t)$ of the auxiliary equation $x^{\Delta} = \lambda f(t,x)$ satisfies the following: if $x(t) \in \bar{D}$, then*

$$x(t) \notin \partial D \quad \forall t \in [0,T]_{\mathbb{T}}.$$

(H_2) *The Brouwer degree*

$$\deg(g, D \cap \operatorname{Ker}(I - Q), 0) \neq 0, \quad \textit{if } \operatorname{Ker}(I - Q) \neq \{0\},$$

where

$$g(a) = \frac{1}{T}\int_0^T Pf(s,a)\Delta s,$$

$P : R^n \to \mathrm{Ker}(I - Q)$ *is an orthogonal projection.*

Then (6) *has at least one* (Q, T)*-affine-periodic solution* $x_*(t) \in D$ *for all* $t \in [0, T]_{\mathbb{T}}$.

As an application of Theorem 1.1, we have the following corollary on basis of Lyapunov functions. The proof and two examples can be found in Section 4.

Corollary 1.1 *Consider the system* (6). *Assume that there exist constants* $M > 0$ *and* $\delta > 0$, *such that*

$$\langle x(t), f(t, x(t))\rangle + \frac{1}{2}\mu(t)\left|f(t, x(t))\right|^2 \geq \delta > 0 \quad \forall \left|x(t)\right| \geq M, t \in \mathbb{T}. \tag{7}$$

If $\mathrm{Ker}(I - Q) \neq \{0\}$, *for all* $x \in \mathrm{Ker}(I - Q)$ *and* $|x(t)| \geq M$, $t \in \mathbb{T}$,

$$\left|\langle x, Pf(t, x)\rangle\right| \geq \delta > 0, \tag{8}$$

where $P : R^n \to \mathrm{Ker}(I - Q)$ *is an orthogonal projection.*

Then the system (6) *has at least one* (Q, T)*-affine-periodic solution* $x_*(t)$.

Before starting our proof, we first introduce some basic definitions and theorems in Section 2.

2 Preliminaries

In this section, we shall recall some useful theorems and prove a lemma which will be used in Section 3. The following theorem shows some easy and useful relationships concerning the delta derivative.

Theorem 2.1 *Assume* $f : \mathbb{T} \to R^n$ *is a function and let* $t \in \mathbb{T}^\kappa$. *Then we have the following*:

(i) *If* f *is differentiable at* t, *then* f *is continuous at* t.

(ii) *If* f *is continuous at* t *and* t *is right-scattered, then* f *is differentiable at* t *with*

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

(iii) *If* t *is right-dense, then* f *is differentiable at* t *iff the limit*

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

exists as a finite number. In this case

$$f^\Delta(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

(iv) *If* f *is differentiable at* t, *then*

$$f(\sigma(t)) = f(t) + \mu(t) f^\Delta(t).$$

To prove Corollary 1.1, a chain rule is obviously necessary. The following chain rule is due to Pötzsche [14] and Keller [15].

Theorem 2.2 *Let $f: R^1 \to R^1$ be continuously differentiable and suppose that $g: \mathbb{T} \to R$ is delta differentiable. Then $f \circ g: \mathbb{T} \to R^1$ is delta differentiable and the formula*

$$(f \circ g)^{\Delta}(t) = \int_0^1 \{f'(g(t) + h\mu(t)g^{\Delta}(t))\} \, \mathrm{d}hg^{\Delta}(t) \tag{9}$$

holds.

Consider the system (6). Now a basic topic is to investigate the existence of (Q, T)-affine-periodic solutions $x(t)$. The following lemma shows that this problem is equivalent to proving the existence of solutions of a boundary value problem (BVP for short).

Lemma 2.1 *Consider the system* (6). *The existence of (Q, T)-affine-periodic solutions of* (6) *is equivalent to the existence of solutions of the BVP*

$$x^{\Delta} = f(t, x),$$
$$x(T) = Qx(0).$$

Proof For any solution $x(t)$ of (6), let $u(t) = Q^{-1}x(t + T)$. By Theorem 2.1:

(i) If t is a right-dense point, we have

$$\begin{aligned} u^{\Delta}(t) &= \lim_{\Delta t \to 0} \frac{u(t + \Delta t) - u(t)}{\Delta t} \\ &= \lim_{\Delta t \to 0} \frac{Q^{-1}x(t + \Delta t + T) - Q^{-1}x(t + T)}{\Delta t} \\ &= Q^{-1}x^{\Delta}(t + T) \\ &= Q^{-1}f(t + T, x(t + T)) \\ &= f(t, Q^{-1}x(t + T)) \\ &= f(t, u(t)). \end{aligned}$$

(ii) If t is a right-scattered point, we have

$$\begin{aligned} u^{\Delta}(t) &= \frac{u(\sigma(t)) - u(t)}{\mu(t)} \\ &= \frac{Q^{-1}x(\sigma(t) + T) - Q^{-1}x(t + T)}{\mu(t)} \\ &= Q^{-1}x^{\Delta}(t + T) \\ &= f(t, u(t)). \end{aligned}$$

By (i) and (ii), we see that $u(t)$ is a solution of (6). Since $f(t, x)$ ensures the uniqueness of solutions with respect to initial value and $u(0) = Q^{-1}x(T)$, we know that $u(t) \equiv x(t)$ if and only if $x(0) = Q^{-1}x(T)$. □

Finally, as a useful tool in our proof, we introduce the definition of a retraction map.

Definition 2.1 Let X be a topological space and A a subspace of X. Then a continuous map $r : X \to A$ is called a retraction if the restriction of r to A is the identity map on A.

3 Proof of Theorem 1.1

By a coordinate transformation, we can always make $0 \in \mathbb{T}$ without loss of generality.

Consider the BVP of the auxiliary equation

$$x^{\Delta} = \lambda f(t,x), \tag{10}$$

$$x(T) = Qx(0), \tag{11}$$

where $\lambda \in [0,1]$. Let $x(t)$ be any solution of (10)-(11). Then BVP (10)-(11) is equivalent to the integral equation

$$x(0) + \lambda \int_0^T f(\tau, x(\tau))\Delta\tau = Qx(0).$$

Denote $x(0)$ by x_0. Then

$$(I - Q)x_0 = -\lambda \int_0^T f(\tau, x(\tau))\Delta\tau, \tag{12}$$

where I is the identity matrix.

Consider (12) in two parts:

(I) If $\operatorname{Ker}(I - Q) \neq \{0\}$.

In this case, $(I - Q)^{-1}$ does not exist. By a coordinate transformation, we can just let

$$Q = \begin{pmatrix} I & 0 \\ 0 & Q_1 \end{pmatrix}$$

without loss of generality, where $(I - Q_1)^{-1}$ exists.

Let $P : R^n \to \operatorname{Ker}(I - Q)$ be the orthogonal projection. Then

$$\begin{aligned}(I - Q)x_0 &= (I - Q)\left(x^0_{\mathrm{Ker}} + x^0_{\perp}\right) \\ &= -\lambda \int_0^T f(\tau, x(\tau))\Delta\tau \\ &= -\lambda \int_0^T Pf(\tau, x(\tau))\Delta\tau - \lambda \int_0^T (I - P)f(\tau, x(\tau))\Delta\tau, \end{aligned} \tag{13}$$

where $x^0_{\mathrm{Ker}} \in \operatorname{Ker}(I - Q)$, $x^0_{\perp} \in \operatorname{Im}(I - Q)$ and $x_0 = x^0_{\mathrm{Ker}} + x^0_{\perp}$.

Let $\mathcal{J} = (I - Q)|_{\operatorname{Im}(I-Q)}$. It is easy to see that $\mathcal{J}^{-1}$ exists. Thus (13) is equivalent to

$$(I - Q)x^0_{\mathrm{Ker}} = -\lambda \int_0^T Pf(\tau, x(\tau))\Delta\tau = 0,$$

$$(I - Q)x^0_{\perp} = -\lambda \int_0^T (I - P)f(\tau, x(\tau))\Delta\tau.$$

Thus we have

$$x_{\perp}^{0} = -\lambda \mathcal{J}^{-1}(I-P)\int_{0}^{T} f\bigl(\tau, x(\tau)\bigr)\Delta\tau.$$

Let

$$X = \bigl\{x : [0,T]_{\mathbb{T}} \to R^{n} : x(t) \text{ is continuous on } [0,T]_{\mathbb{T}}\bigr\},$$

and define the norm as $\|x\| = \sup_{t\in[0,T]_{\mathbb{T}}} |x(t)|$. It is easy to see that X is a Banach space with the norm $\|\cdot\|$.

For $x \in X$ which satisfies $x(t) \in \bar{D}$ for all $t \in [0,T]_{\mathbb{T}}$, we define an operator $\mathcal{T}(x_{\mathrm{Ker}}^{0}, x, \lambda)$ by

$$\mathcal{T}\bigl(x_{\mathrm{Ker}}^{0}, x, \lambda\bigr)(t) = \begin{pmatrix} x_{\mathrm{Ker}}^{0} + \frac{1}{T}\int_{0}^{T} Pf(\tau, x(\tau))\Delta\tau \\ x_{\mathrm{Ker}}^{0} - \lambda\mathcal{J}^{-1}(I-P)\int_{0}^{T} f(\tau, x(\tau))\Delta\tau + \lambda\int_{0}^{t} f(\tau, x(\tau))\Delta\tau \end{pmatrix}, \tag{14}$$

where $\lambda \in [0,1]$. We claim that each fixed point x of $\mathcal{T}$ in X is a solution of BVP (10)-(11).

In fact, if x is a fixed point of $\mathcal{T}$, we have

$$\begin{pmatrix} x_{\mathrm{Ker}}^{0} \\ x(t) \end{pmatrix} = \begin{pmatrix} x_{\mathrm{Ker}}^{0} + \frac{1}{T}\int_{0}^{T} Pf(\tau, x(\tau))\Delta\tau \\ x_{\mathrm{Ker}}^{0} - \lambda\mathcal{J}^{-1}(I-P)\int_{0}^{T} f(\tau, x(\tau))\Delta\tau + \lambda\int_{0}^{t} f(\tau, x(\tau))\Delta\tau \end{pmatrix}.$$

Thus

$$\frac{1}{T}\int_{0}^{T} Pf\bigl(\tau, x(\tau)\bigr)\Delta\tau = 0, \tag{15}$$

$$x(t) = x_{\mathrm{Ker}}^{0} - \lambda\mathcal{J}^{-1}(I-P)\int_{0}^{T} f\bigl(\tau, x(\tau)\bigr)\Delta\tau + \lambda\int_{0}^{t} f\bigl(\tau, x(\tau)\bigr)\Delta\tau. \tag{16}$$

By (16), we know that

$$x_{0} = x_{\mathrm{Ker}}^{0} - \lambda\mathcal{J}^{-1}(I-P)\int_{0}^{T} f\bigl(\tau, x(\tau)\bigr)\Delta\tau.$$

Thus

$$\begin{aligned} Qx_{0} &= Qx_{\mathrm{Ker}}^{0} - \lambda Q\mathcal{J}^{-1}(I-P)\int_{0}^{T} f\bigl(\tau, x(\tau)\bigr)\Delta\tau \\ &= x_{\mathrm{Ker}}^{0} - \lambda Q\mathcal{J}^{-1}(I-P)\int_{0}^{T} f\bigl(\tau, x(\tau)\bigr)\Delta\tau. \end{aligned}$$

Since (15) holds, we have

$$\begin{aligned} &\lambda(I-Q)\mathcal{J}^{-1}(I-P)\int_{0}^{T} f\bigl(\tau, x(\tau)\bigr)\Delta\tau \\ &\quad = \lambda(I-P)\int_{0}^{T} f\bigl(\tau, x(\tau)\bigr)\Delta\tau \end{aligned}$$

$$= \lambda(I-P)\int_0^T f(\tau,x(\tau))\Delta\tau + \lambda P\int_0^T f(\tau,x(\tau))\Delta\tau$$
$$= \lambda\int_0^T f(\tau,x(\tau))\Delta\tau.$$

Thus

$$\lambda Q\mathcal{J}^{-1}(I-P)\int_0^T f(\tau,x(\tau))\Delta\tau = \lambda\mathcal{J}^{-1}(I-P)\int_0^T f(\tau,x(\tau))\Delta\tau - \lambda\int_0^T f(\tau,x(\tau))\Delta\tau.$$

Then

$$\begin{aligned} Qx_0 &= x^0_{\mathrm{Ker}} - \lambda Q\mathcal{J}^{-1}(I-P)\int_0^T f(\tau,x(\tau))\Delta\tau \\ &= x^0_{\mathrm{Ker}} - \lambda\mathcal{J}^{-1}(I-P)\int_0^T f(\tau,x(\tau))\Delta\tau + \lambda\int_0^T f(\tau,x(\tau))\Delta\tau \\ &= x(T). \end{aligned} \tag{17}$$

By (16) and (17), we know that (12) holds. Thus

$$x^0_{\perp} = -\lambda\mathcal{J}^{-1}(I-P)\int_0^T f(\tau,x(\tau))\Delta\tau.$$

Then

$$\begin{aligned} x(t) &= x^0_{\mathrm{Ker}} - \lambda\mathcal{J}^{-1}(I-P)\int_0^T f(\tau,x(\tau))\Delta\tau + \lambda\int_0^t f(\tau,x(\tau))\Delta\tau \\ &= x^0_{\mathrm{Ker}} + x^0_{\perp} + \lambda\int_0^t f(\tau,x(\tau))\Delta\tau \\ &= x_0 + \lambda\int_0^t f(\tau,x(\tau))\Delta\tau. \end{aligned}$$

It means that the fixed point x is a solution of BVP (10)-(11).

Now, we need to prove the existence of fixed points of $\mathcal{T}$.

Take a constant M which satisfies $M > \sup_{t\in[0,T]_{\mathbb{T}}, x\in\bar{D}} |f(t,x)|$, and let

$$X_\lambda = \left\{x\in X : \left|\frac{x(t)-x(s)}{t-s}\right| \le \lambda M \ \forall t\neq s\right\}.$$

It is easy to make a retraction $\alpha_\lambda : X \to X_\lambda$.

Define an operator $\widehat{\mathcal{T}}(x^0_{\mathrm{Ker}}, x, \lambda)$ by

$$\widehat{\mathcal{T}}(x^0_{\mathrm{Ker}}, x, \lambda)(t) = \begin{pmatrix} x^0_{\mathrm{Ker}} + \frac{1}{T}\int_0^T Pf(\tau,\alpha_\lambda\circ x(\tau))\Delta\tau \\ \alpha_\lambda\circ x^0_{\mathrm{Ker}} - \lambda\mathcal{J}^{-1}(I-P)\int_0^T f(\tau,\alpha_\lambda\circ x(\tau))\Delta\tau \\ +\lambda\int_0^t f(\tau,\alpha_\lambda\circ x(\tau))\Delta\tau \end{pmatrix}. \tag{18}$$

Obviously,

$$H(x^0_{\mathrm{Ker}}, x, \lambda) = \widehat{\mathcal{T}}(x^0_{\mathrm{Ker}}, x, \lambda) \tag{19}$$

is a homotopy, where

$$\begin{aligned}&\bigl(x^0_{\mathrm{Ker}}, x, \lambda\bigr) \in \bigl(D \cap \operatorname{Ker}(I-Q)\bigr) \times \tilde{D} \times [0,1],\\&\tilde{D} = \bigl\{x \in X : x(t) \in D\ \forall t \in [0,T]_{\mathbb{T}}\bigr\}.\end{aligned}$$

We claim that

$$0 \notin (\mathrm{id} - H)\bigl(\partial\bigl(\bigl(D \cap \operatorname{Ker}(I-Q)\bigr) \times \tilde{D}\bigr) \times [0,1]\bigr), \tag{20}$$

where id is the identity operator.

Suppose, on the contrary, that there exists $(\hat{x}^0_{\mathrm{Ker}}, \hat{x}, \hat{\lambda}) \in \partial((D \cap \operatorname{Ker}(I-Q)) \times \tilde{D}) \times [0,1]$, such that $(\mathrm{id} - H)(\hat{x}^0_{\mathrm{Ker}}, \hat{x}, \hat{\lambda}) = 0$. As $\hat{x}^0_{\mathrm{Ker}} \in \partial(D \cap \operatorname{Ker}(I-Q)) \subset \partial D$ is contradictory to (H_1); we know that $\hat{x}^0_{\mathrm{Ker}} \notin \partial(D \cap \operatorname{Ker}(I-Q))$. In other words, $\hat{x} \in \partial\tilde{D}$. Then (20) can be proved as follows:

(i) When $\hat{\lambda} = 0$, by the definition of set X_λ, we have

$$X_0 = \biggl\{x \in X : \biggl|\frac{x(t) - x(s)}{t-s}\biggr| \le 0\ \forall t \neq s\biggr\}.$$

Hence $\alpha_0 \circ x(t) \equiv \alpha_0 \circ x(0)$ for all $t \in [0,T]_{\mathbb{T}}$. Since $(\mathrm{id} - H)(\hat{x}^0_{\mathrm{Ker}}, \hat{x}, 0) = 0$, we have

$$\begin{pmatrix}\hat{x}^0_{\mathrm{Ker}}\\ \hat{x}(t)\end{pmatrix} = \begin{pmatrix}\hat{x}^0_{\mathrm{Ker}} + \frac{1}{T}\int_0^T Pf(\tau, \alpha_\lambda \circ \hat{x}(\tau))\Delta\tau\\ \alpha_0 \circ \hat{x}^0_{\mathrm{Ker}}\end{pmatrix}.$$

It means that $\hat{x}(t) \equiv \hat{x}(0)$ for all $t \in [0,T]_{\mathbb{T}}$. Take $\hat{x}(t) = p$, we have $\alpha_0 \circ \hat{x}^0_{\mathrm{Ker}} = \hat{x}(t) = p$. Consequently $\frac{1}{T}\int_0^T Pf(\tau, p)\Delta\tau = 0$, and this is equivalent to $g(p) = 0$ by the definition of $g(a)$. Notice that $\hat{x} \in \partial\tilde{D}$ and $\tilde{D} = \{x \in X : x(t) \in D\ \forall t \in [0,T]_{\mathbb{T}}\}$. Hence there exists $t_0 \in [0,T]_{\mathbb{T}}$ such that $\hat{x}(t_0) \in \partial D$. As $\hat{x}(t) \equiv p$ for all $t \in [0,T]_{\mathbb{T}}$, we obtain $p \in \partial D$. Thus we have $p \in \partial D$ and $g(p) = 0$. It is contradictory to (H_2) because the Brouwer degree $\deg(g, D, 0) \neq 0$.

(ii) When $\hat{\lambda} \in (0,1]$, as $0 = (\mathrm{id} - H)(\hat{x}^0_{\mathrm{Ker}}, \hat{x}, \hat{\lambda})$, we have

$$\begin{pmatrix}\hat{x}^0_{\mathrm{Ker}}\\ \hat{x}(t)\end{pmatrix} = \begin{pmatrix}\hat{x}^0_{\mathrm{Ker}} + \frac{1}{T}\int_0^T Pf(\tau, \alpha_{\hat{\lambda}} \circ \hat{x}(\tau))\Delta\tau\\ \alpha_{\hat{\lambda}} \circ \hat{x}^0_{\mathrm{Ker}} - \hat{\lambda}\mathcal{J}^{-1}(I-P)\int_0^T f(\tau, \alpha_{\hat{\lambda}} \circ \hat{x}(\tau))\Delta\tau\\ +\hat{\lambda}\int_0^t f(\tau, \alpha_{\hat{\lambda}} \circ \hat{x}(\tau))\Delta\tau\end{pmatrix}.$$

Thus

$$\frac{1}{T}\int_0^T Pf\bigl(\tau, \alpha_{\hat{\lambda}} \circ \hat{x}(\tau)\bigr)\Delta\tau = 0$$

and

$$\begin{aligned}\hat{x}(t) = {}&\alpha_{\hat{\lambda}} \circ \hat{x}^0_{\mathrm{Ker}} - \hat{\lambda}\mathcal{J}^{-1}(I-P)\int_0^T f\bigl(\tau, \alpha_{\hat{\lambda}} \circ \hat{x}(\tau)\bigr)\Delta\tau\\&+ \hat{\lambda}\int_0^t f\bigl(\tau, \alpha_{\hat{\lambda}} \circ \hat{x}(\tau)\bigr)\Delta\tau.\end{aligned} \tag{21}$$

Note that

$$\begin{aligned}\left|\frac{\hat{x}(t)-\hat{x}(s)}{t-s}\right| &= \frac{1}{|t-s|}\left|\hat{\lambda}\int_s^t f\big(\tau,\alpha_{\hat{\lambda}}\circ\hat{x}(\tau)\big)\Delta\tau\right| \\ &\le \frac{\hat{\lambda}}{|t-s|}\int_s^t \big|f\big(\tau,\alpha_{\hat{\lambda}}\circ\hat{x}(\tau)\big)\big|\Delta\tau \\ &\le \hat{\lambda}M.\end{aligned}$$

By the definition of X_λ, we obtain $\hat{x}\in X_{\hat{\lambda}}$, which means that $\alpha_{\hat{\lambda}}\circ\hat{x}=\hat{x}$. Thus (21) is equivalent to

$$\hat{x}(t)=\hat{x}^0_{\mathrm{Ker}}-\hat{\lambda}\mathcal{J}^{-1}(I-P)\int_0^T f\big(\tau,\hat{x}(\tau)\big)\Delta\tau+\hat{\lambda}\int_0^t f\big(\tau,\hat{x}(\tau)\big)\Delta\tau.$$

By a similar discussion to (16), we can prove that $\hat{x}(t)$ is a solution of BVP (10)-(11). By hypothesis (H_1), we know that $\hat{x}(t)\notin\partial D$ for any $t\in[0,T]_{\mathbb{T}}$. It is contradictory to $\hat{x}\in\partial\tilde{D}$.

By (i) and (ii), we obtain

$$0\notin(\mathrm{id}-H)\big(\partial\big(\big(D\cap\mathrm{Ker}(I-Q)\big)\times\tilde{D}\big)\times[0,1]\big).$$

Therefore, by the homotopy invariance and the theory of the Brouwer degree, we have

$$\begin{aligned}&\deg\big(\mathrm{id}-H\big(x^0_{\mathrm{Ker}},\cdot,1\big),\big(D\cap\mathrm{Ker}(I-Q)\big)\times\tilde{D},0\big)\\ &\quad=\deg\big(\mathrm{id}-H\big(x^0_{\mathrm{Ker}},\cdot,0\big),\big(D\cap\mathrm{Ker}(I-Q)\big)\times\tilde{D},0\big)\\ &\quad=\deg\big(g,D\cap\mathrm{Ker}(I-Q),0\big)\\ &\quad\neq 0.\end{aligned}$$

It means that there exists $\hat{x}_*\in\tilde{D}$, such that

$$\begin{pmatrix}\hat{x}^0_{*\,\mathrm{Ker}}\\ \hat{x}_*(t)\end{pmatrix}=\widehat{\mathcal{T}}\big(\hat{x}^0_{*\,\mathrm{Ker}},\hat{x}_*(t),1\big). \tag{22}$$

Similar to the proof in (ii), we get $\hat{x}_*\in X_1$. Then

$$\widehat{\mathcal{T}}\big(\hat{x}^0_{*\,\mathrm{Ker}},\hat{x}_*(t),1\big)=\mathcal{T}\big(\hat{x}^0_{*\,\mathrm{Ker}},\hat{x}_*(t),1\big). \tag{23}$$

By (22) and (23), we see that $\hat{x}_*$ is a fixed point of $\mathcal{T}$ in X. Thus $\hat{x}_*(t)$ is a solution of BVP (10)-(11).

(II) If $\mathrm{Ker}(I-Q)=\{0\}$.

In this case, $(I-Q)^{-1}$ exists. Then

$$x_0=-\lambda\mathcal{J}^{-1}(I-P)\int_0^T f\big(\tau,x(\tau)\big)\Delta\tau.$$

Hypothesis (H_2) will not be needed anymore. Consider the homotopy

$$H(x,\lambda)=-\lambda\mathcal{J}^{-1}(I-P)\int_0^T f\big(\tau,\alpha_\lambda\circ x(\tau)\big)\Delta\tau+\lambda\int_0^t f\big(\tau,\alpha_\lambda\circ x(\tau)\big)\Delta\tau.$$

Similar to the proof when $\operatorname{Ker}(I-Q)\neq\{0\}$, we have $0\notin(\mathrm{id}-H)(\partial\tilde{D}\times[0,1])$. Hence

$$\begin{aligned}\deg\bigl(\mathrm{id}-H(\cdot,1),\tilde{D},0\bigr)&=\deg\bigl(\mathrm{id}-H(\cdot,0),\tilde{D},0\bigr)\\&=\deg(\mathrm{id},\tilde{D},0)\\&=1.\end{aligned}$$

It means that there exists $\hat{x}_*$, which satisfies $\hat{x}_*(t)\in D$ for all $t\in[0,T]_{\mathbb{T}}$, such that

$$\hat{x}_*(t)=\hat{x}_*(0)+\int_0^t f\bigl(\tau,\hat{x}_*(\tau)\bigr)\Delta\tau.$$

Therefore $\hat{x}_*(t)$ is a solution of BVP (10)-(11).

By Lemma 2.1 and the proofs in (I) and (II), it is easy to see that APS (6) has a (Q,T)-affine-periodic solution $x_*(t)$, which is an extension of $\hat{x}_*(t)$ on $\mathbb{T}$. By hypothesis (H_1) and $\hat{x}_*(t)\in D$ for all $t\in[0,T]_{\mathbb{T}}$, we know that $x_*(t)\in D$ for all $t\in[0,T]_{\mathbb{T}}$.

4 Proof of Corollary 1.1

Consider the auxiliary equation (10) of the system (6),

$$x^{\Delta}(t)=\lambda f\bigl(t,x(t)\bigr).$$

Let

$$\begin{aligned}&V\bigl(x(t)\bigr)=\frac{1}{2}\bigl|x(t)\bigr|^2,\\&D=\bigl\{p\in R^n:V(p)<M+1\bigr\}.\end{aligned}$$

Clearly, D is bounded. We claim that for $\lambda\in(0,1]$, every possible (Q,T)-affine-periodic solution $x(t)$ of (10) satisfies (H_1).

In fact, assume that $x(t)$ is a (Q,T)-affine-periodic solution of (10), $Q\in O(n)$. Set $u(t)=V(x(t))$. Then

$$u(t+T)=V\bigl(x(t+T)\bigr)=V\bigl(Qx(t)\bigr)=\frac{1}{2}\bigl|Qx(t)\bigr|^2=\frac{1}{2}\bigl|x(t)\bigr|^2=V\bigl(x(t)\bigr)=u(t),$$

hence $u:\mathbb{T}\to R^1_+$ is T-periodic. Thereby there exists $t_j\in[0,T)\cap\mathbb{T}$ and $t_0\in[0,T]\cap\mathbb{T}$ with $t_j\nearrow t_0$ or $t_j\searrow t_0$ such that

$$u(t_j)\to\sup_{[0,T]_{\mathbb{T}}}u(t),\quad t_j\to t_0. \tag{24}$$

By Definition 1.9, we know that $x\in C(\mathbb{T},R^n)$. Hence $u(t_0)=\sup_{[0,T]_{\mathbb{T}}}u(t)$. By Theorem 2.2, we have

$$\begin{aligned}u^{\Delta}(t_0)&=\biggl\langle\int_0^1\bigl\{x(t_0)+h\mu(t_0)x^{\Delta}(t_0)\bigr\}\,\mathrm{d}h,x^{\Delta}(t_0)\biggr\rangle\\&=\biggl\langle\int_0^1\bigl\{x(t_0)+h\mu(t_0)f\bigl(t_0,x(t_0)\bigr)\bigr\}\,\mathrm{d}h,f\bigl(t_0,x(t_0)\bigr)\biggr\rangle\end{aligned}$$

$$= \langle x(t_0), f(t_0, x(t_0)) \rangle + \frac{1}{2}\mu(t_0) |f(t_0, x(t_0))|^2$$
$$\leq 0.$$

By (7), this result yields

$$|x(t_0)| < M.$$

Consequently, by the definition of D and (24), we have

$$x(t) \in D \quad \forall t.$$

Thus (H_1) holds.

If $\operatorname{Ker}(I - Q) = \{0\}$, by the proof of Theorem 1.1, we know that the system (6) admits a (Q, T)-affine-periodic solution.

Now we prove that if $\operatorname{Ker}(I - Q) \neq \{0\}$,

$$\deg(g, D \cap \operatorname{Ker}(I - Q), 0) \neq 0.$$

Indeed, let $B_M = \{p \in R^n : |p| < M\}$. Consider the homotopy

$$H(p, \lambda) = \lambda \operatorname{sgn}(\langle \nabla V(\cdot), Pf(t, \cdot)\rangle|_{\partial(B_M \cap \operatorname{Ker}(I-Q))}) \nabla V(p) + (1 - \lambda) g(p),$$

where $(p, \lambda) \in (B_{M_0} \cap \operatorname{Ker}(I - Q)) \times [0, 1]$. It follows that

$$\begin{aligned} \langle \nabla V(p), H(p, \lambda)\rangle = {}& \lambda \operatorname{sgn}(\langle \nabla V(\cdot), Pf(t, \cdot)\rangle|_{\partial(B_M \cap \operatorname{Ker}(I-Q))}) |\nabla V(p)|^2 \\ & + (1 - \lambda)\langle \nabla V(p), g(p)\rangle. \end{aligned} \tag{25}$$

For any $(p, t) \in \partial(B_M \cap \operatorname{Ker}(I - Q)) \times R^1$, by (8), we know that the sign of $\langle \nabla V(p), Pf(t, p)\rangle$ does not change. By the definition of $g(a)$, we have

$$\langle \nabla V(p), g(p)\rangle = \left\langle \nabla V(p), \frac{1}{T}\int_0^T Pf(s, p)\Delta s \right\rangle = \frac{1}{T}\int_0^T \langle \nabla V(p), Pf(s, p)\rangle \Delta s.$$

It means that $\langle \nabla V(p), g(p)\rangle$ always has the same sign with $\langle \nabla V(p), Pf(t, p)\rangle$. Also, by (8), we know that $|\nabla V(p)| \neq 0$ when $p \in \partial(B_M \cap \operatorname{Ker}(I - Q))$. Consequently, the right hand side of (25) is nonzero.

Thus

$$\langle \nabla V(p), H(p, \lambda)\rangle \neq 0 \quad \forall (p, \lambda) \in \partial(B_M \cap \operatorname{Ker}(I - Q)) \times [0, 1],$$

which implies that $0 \notin H(\partial(B_M \cap \operatorname{Ker}(I - Q)) \times [0, 1])$.

The homotopy invariance of the Brouwer degree implies

$$\begin{aligned} & \deg(g, B_M \cap \operatorname{Ker}(I - Q), 0) \\ & \quad = \deg(\operatorname{sgn}(\langle \nabla V(\cdot), Pf(t, \cdot)\rangle|_{\partial(B_M \cap \operatorname{Ker}(I-Q))}) \nabla V, B_M \cap \operatorname{Ker}(I - Q), 0) \end{aligned}$$

$$= \deg\big(\operatorname{sgn}\big(\langle \nabla V(\cdot), Pf(t,\cdot)\rangle|_{\partial(B_M\cap \operatorname{Ker}(I-Q))}\big)\operatorname{id}, B_M\cap \operatorname{Ker}(I-Q), 0\big)$$
$$\neq 0.$$

Hence hypothesis (H_2) holds. Thus Corollary 1.1 follows from Theorem 1.1.

5 Examples

As an application of Theorem 1.1, Corollary 1.1 is very useful and more directly. In this section, we will show two examples and prove the existence of affine-periodic solutions of them by using Corollary 1.1.

Example 5.1 Let $\mathbb{T}$ be a 2π-periodic time scale. Consider the system

$$x^{\Delta}(t) = x^3(t) + \sin t. \tag{26}$$

Let M be a constant large enough. Set $V(x) = \frac{1}{2}x^2, f(t,x(t)) = x^3(t) + \sin t$. We have

$$\begin{aligned} f\big(t+2\pi, x(t)\big) &= x^3(t) + \sin(t+2\pi) \\ &= x^3(t) + \sin t \\ &= f\big(t, x(t)\big). \end{aligned}$$

Hence (26) is a 2π-affine-periodic system. Then when $|x| > M$, it is easy to see that

$$\begin{aligned} (V\circ x)^{\Delta}(t) &= x^{\Delta}(t)\int_0^1 \big\{V'\big(x(t) + h\mu(t)x^{\Delta}(t)\big)\big\}\,\mathrm{d}h \\ &= f(t,x)\int_0^1 \big\{x(t) + h\mu(t)f(t,x)\big\}\,\mathrm{d}h \\ &= xf(t,x) + \frac{1}{2}\mu(t)f(t,x)^2 \\ &= x^4 + x\sin t + \frac{1}{2}\mu(t)\big(x^3 + \sin t\big)^2 > \delta > 0 \quad \forall t\in\mathbb{T}. \end{aligned}$$

Also, since $P = \operatorname{id}$, we have

$$\big|x^4 + x\sin t\big| > 0 \quad \forall |x| > M, t\in\mathbb{T}.$$

By Corollary 1.1, (26) has a 2π-periodic solution.

Example 5.2 Let $\mathbb{T}$ be a 1-periodic time scale. Consider the system

$$x^{\Delta} = |x|^2 x + (\sin t, \cos t, \sin 2\pi t, \cos 2\pi t)^{\mathrm{T}}. \tag{27}$$

Set

$$Q = \begin{pmatrix} \cos(2\pi-1) & -\sin(2\pi-1) & 0 & 0 \\ \sin(2\pi-1) & \cos(2\pi-1) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then (27) is a $(Q,1)$-affine-periodic system. Let

$$f(t,x) = |x|^2 x + (\sin t, \cos t, \sin 2\pi t, \cos 2\pi t)^{\mathrm{T}},$$

and M a constant large enough. When $|x| > M$, similar to Example 5.1, it is easy to see that

$$\langle x(t), f(t, x(t))\rangle + \frac{1}{2}\mu(t)\left|f(t, x(t))\right|^2 \geq \delta > 0 \quad \forall t \in \mathbb{T}.$$

Also, since $P : R^4 \to \mathrm{Ker}(I - Q)$ is an orthogonal projection,

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

we have

$$\left|\langle x, Pf(t,x)\rangle\right| > 0 \quad \forall |x| > M, t \in \mathbb{T}.$$

By Corollary 1.1, (27) has a $(Q,1)$-affine-periodic solution.

6 Conclusion

In this paper, we considered the affine-periodic systems on time scales and provided Theorem 1.1. This theorem asserts the existence of affine-periodic solutions in a certain topological formalism. As an application of Theorem 1.1, we proved Corollary 1.1 and gave two examples. The affine-periodicity is a new and attractive topic in this area, and many significant problems remain to be further studied. In our forthcoming papers, we will discuss the affine-periodic solutions for impulsive equations. Besides, affine-periodic solutions of higher-order equations and delay equations *etc.* are also very interesting problems.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors jointly worked on deriving the results and approved the final manuscript.

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