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Existence and localization of positive solutions for a fractional boundary value problem at resonance

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Abstract

We investigate the existence of positive solutions for a fractional boundary value problem at resonance. By means of a fixed point theorem of increasing operators, the minimal and maximal nonnegative solutions for the problem are obtained.

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Keywords: fractional boundary value problem; resonance; positive solutions; fixed point theorem

1 Introduction

We are concerned with the following fractional boundary value problem (P):

$$ {}^{c}D_{0^+}^{\alpha}u(t) = f\bigl(t,u(t)\bigr), \quad 0<t<1, \tag{1.1}$$

$$ u(0) = u'(0) = 0, \qquad u''(0) = 2u(1), \tag{1.2}$$

where ${}^{c}D_{0^+}^{\alpha}$ denotes the Caputo fractional derivative, $2 < \alpha < 3$. We assume that $f : [0,1] \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous. The boundary value problem (P) is said to be at resonance if the linear equation $Lu = {}^{c}D_{0^+}^{\alpha}u(t)$ with the boundary value conditions (1.2) has a nontrivial solution, *i.e.*, $\dim\ker L \geq 1$.

In recent years, there has been much work related to boundary value problems at resonance for ordinary or fractional differential equations. We refer the reader to [1–8] and the references therein. In most papers mentioned above, the coincidence degree theory was applied to establish existence theorems. In [9–11], the authors obtained the minimal and maximal positive solutions by using a fixed point theorem of increasing operators.

In this paper, we use this method to solve the boundary value problem (P). For the convenience of the reader, we recall some notations.

Let X and Y be real Banach spaces, $L : \operatorname{dom}(L) \subset X \to Y$ be a Fredholm operator of index zero. The map $N : X \to Y$ is called L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ and $K_p(I-Q)(\overline{\Omega})$ are both compact.

Let $P : X \to X$, $Q : Y \to Y$ be continuous projectors such that $\operatorname{Im} P = \operatorname{Ker} L$, $\operatorname{Ker} Q = \operatorname{Im} L$ and $X = \operatorname{Ker} L \oplus \operatorname{Ker} P$, $Y = \operatorname{Im} L \oplus \operatorname{Im} Q$. It follows that $L|_{\operatorname{Ker} P\cap \operatorname{dom}(L)} : \operatorname{Ker} P \cap \operatorname{dom}(L) \to \operatorname{Im}(L)$ is invertible.

We denote the inverse of L by $K_P : \operatorname{Im} L \to \operatorname{Ker} P \cap \operatorname{dom}(L)$. Moreover, since $\dim \operatorname{Im} Q = \dim \operatorname{Ker} L < \infty$, there exists an isomorphism $J : \operatorname{Im} Q \to \operatorname{Ker} L = \operatorname{Im} P$. Set $H = L + J^{-1}P$, then $H : \operatorname{dom}(L) \subset X \to Y$ is a linear bijection with bounded inverse and $(JQ + K_p(I - Q))(L + J^{-1}P) = (L + J^{-1}P)(JQ + K_p(I - Q)) = I$. From [12] we see that $K_1 = H(K \cap \operatorname{dom}(L))$ is a cone in Y and we have the following theorem.

Theorem 1.1 [12] *$N(u) + J^{-1}P(u) = H(\overline{u})$, where $\overline{u} = P(u) + JQN(u) + K_p(I - Q)N(u)$ and $\overline{u}$ is uniquely determined.*

As a consequence of the above theorem, the author obtained the equivalence of the following two assertions:

(i) $P + JQN + K_p(I - Q)N : K \cap \operatorname{dom}(L) \to K \cap \operatorname{dom}(L)$,

(ii) $N + J^{-1}P : K \cap \operatorname{dom}(L) \to K_1$.

Now we introduce the notion of lower and upper solutions.

Definition 1.2 [6] Let K be a normal cone in a Banach space X, $u_0 \leq v_0$, and $u_0, v_0 \in K \cap \operatorname{dom}(L)$ are said to be coupled lower and upper solutions of the equation $Lu = Nu$ if

$$\begin{cases} Lu_0 \leq Nu_0, \\ Lv_0 \geq Nv_0. \end{cases}$$

Theorem 1.3 [6] *Let $L : \operatorname{dom}(L) \subset X \to Y$ be a Fredholm operator of index zero, K be a normal cone in Banach space X, $u_0, v_0 \in K \cap \operatorname{dom}(L)$, $u_0 \leq v_0$, and $N : [u_0, v_0] \to Y$ be L-compact and continuous. Suppose that the following conditions are satisfied*:

(C_1) *u_0 and v_0 are coupled lower and upper solutions of the equation $Lu = Nu$.*

(C_2) *$N + J^{-1}P : K \cap \operatorname{dom}(L) \to K_1$ is an increasing operator.*

Then the equation $Lu = Nu$ has a minimal solution u^* and a maximal solution v^* in $[u_0, v_0]$.

Moreover, $u^* = \lim_{n\to\infty} u_n$, and $v^* = \lim_{n\to\infty} v_n$, where

$$\begin{aligned} &u_n = \left(L + J^{-1}P\right)^{-1}\left(N + J^{-1}P\right)u_{n-1}, \\ &v_n = \left(L + J^{-1}P\right)^{-1}\left(N + J^{-1}P\right)v_{n-1}, \quad \textit{for } n = 1, 2, 3, \ldots, \\ &u_0 \leq u_1 \leq u_2 \leq \cdots \leq u_n \leq \cdots \leq v_n \leq \cdots \leq v_2 \leq v_1 \leq v_0. \end{aligned}$$

2 Preliminaries

Now, we introduce some notations, definitions and preliminary facts which will be used throughout this paper.

Definition 2.1 The Riemann-Liouville fractional integral operator of order $\alpha > 0$ of a function g is defined by

$$I^{\alpha}_{a^+} g(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t - s)^{\alpha-1} g(s)\, ds,$$

provided that the right side integral is pointwise defined on $(0, +\infty)$.

Definition 2.2 The Caputo fractional derivative of order $\alpha > 0$ of a continuous function g is given by

$$^{c}D^{\alpha}_{a^+}g(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-s)^{n-\alpha-1}g^{(n)}(s)\,ds,$$

when n is the smallest integer greater than or equal to α, provided that the right side integral is pointwise defined on $(0,+\infty)$.

Lemma 2.3 *For $\alpha > 0$, $g \in C([0,1],\mathbb{R})$, the homogeneous fractional differential equation $^{c}D^{\alpha}_{a^+}g(t) = 0$ has a solution $g(t) = c_0 + c_1t + c_2t^2 + \cdots + c_{n-1}t^{n-1}$, where $c_i \in \mathbb{R}$, $i = 0,\dots,n-1$, here n is the smallest integer greater than or equal to α.*

Let $X = Y = C[0,1]$ equipped with the norm $\|u\| = \sup_{t\in[0,1]}|u(t)|$ and $K = \{u \in X : u(t) \geq 0, t \in [0,1]\}$.

Define the operators L and N, respectively, by $L : \operatorname{dom}(L) \subset X \to Y$

$$Lu(t) = {}^{c}D^{\alpha}_{0^+}u(t),$$

$\operatorname{dom}(L) = \{u \in AC^2[0,1] : {}^{c}D^{\alpha}_{0^+}u(t) \in C[0,1], u(0) = u'(0) = 0, u''(0) = 2u(1)\}$ and $N : X \to Y$

$$Nu(t) = f\big(t,u(t)\big), \quad \forall t \in [0,1],$$

then the boundary value problem (P) can be written as $Lu = Nu$, $u \in K \cap \operatorname{dom}(L)$.

Lemma 2.4 *We have*

$$\operatorname{Ker} L = \big\{u \in \operatorname{dom}(L) : u(t) = ct^2, c \in \mathbb{R}, \forall t \in [0,1]\big\}$$

and

$$\operatorname{Im} L = \left\{y \in Y : \int_0^1 (1-s)^{\alpha-1}y(s)\,ds = 0\right\}.$$

Proof By Lemma 2.3, the function $u(t) = c_0 + c_1t + c_2t^2$, $c_0, c_1, c_2 \in \mathbb{R}$ is the solution of $Lu(t) = D^{\alpha}_{0^+}u(t) = 0$. Taking into account the boundary conditions (1.2), we get

$$\operatorname{Ker} L = \big\{u \in \operatorname{dom}(L) : u(t) = ct^2, c \in \mathbb{R}, \forall t \in [0,1]\big\}.$$

Let us show that

$$\operatorname{Im} L = \left\{y \in Y : \int_0^1 (1-s)^{\alpha-1}y(s)\,ds = 0\right\}.$$

For $y \in \operatorname{Im} L$, there exists $u \in \operatorname{dom}(L)$ such that $y = Lu \in Y$. By Lemma 2.3, it follows that

$$u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)\,ds + c_0 + c_1t + c_2t^2.$$

It is easy to get

$$u'(t) = \frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2}y(s)\,ds + c_1 + 2c_2t,$$
$$u''(t) = \frac{1}{\Gamma(\alpha-2)}\int_0^t (t-s)^{\alpha-3}y(s)\,ds + 2c_2,$$

then the boundary conditions (1.2) imply

$$\int_0^1 (1-s)^{\alpha-1}y(s)\,ds = 0.$$

On the other hand, suppose that $y \in Y$ and satisfies $\int_0^1 (1-s)^{\alpha-1}y(s)\,ds = 0$. Let $u(t) = I_{0^+}^{\alpha}y(t) + ct^2$, then $u \in \operatorname{dom}(L)$ and $D_{0^+}^{\alpha}u(t) = y(t)$. Thus, $y \in \operatorname{Im} L$. □

Now, define the operators $P : X \to X$ by

$$Pu(t) = \frac{1}{2}\alpha(\alpha+1)(\alpha+2)t^2\int_0^1 (1-s)^{\alpha-1}u(s)\,ds$$

and $Q : Y \to Y$ by

$$Qy(t) = \alpha\int_0^1 (1-s)^{\alpha-1}y(s)\,ds, \quad \forall t \in [0,1].$$

It is easy to see that the operators P and Q are both projectors. In fact, for $t \in [0,1]$,

$$\begin{aligned} P^2u(t) &= P(Pu)(t) \\ &= \frac{1}{2}\alpha(\alpha+1)(\alpha+2)t^2\int_0^1 (1-s)^{\alpha-1}(Pu)(s)\,ds \\ &= \frac{1}{4}\alpha^2(\alpha+1)^2(\alpha+2)^2t^2\int_0^1 (1-s)^{\alpha-1}u(s)\,ds\int_0^1 (1-s)^{\alpha-1}s^2\,ds \\ &= \frac{1}{2}\alpha(\alpha+1)(\alpha+2)t^2\int_0^1 (1-s)^{\alpha-1}u(s)\,ds = Pu(t). \end{aligned}$$

Similarly we show that Q is a projector. Obviously, $\operatorname{Im} P = \operatorname{Ker} L$ and $\operatorname{Ker} Q = \operatorname{Im} L$.

Lemma 2.5 *The operator $L : \operatorname{dom}(L) \subset X \to Y$ is a Fredholm operator of index zero, and its inverse $K_p : \operatorname{Im} L \to \operatorname{dom}(L) \cap \operatorname{Ker} P$ is given by*

$$K_py(t) = \int_0^1 k(t,s)y(s)\,ds, \quad \forall t \in [0,1],$$

where

$$k(t,s) = \begin{cases} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \alpha(\alpha+1)(\alpha+2)t^2\frac{\Gamma(\alpha)}{2\Gamma(2\alpha)}(1-s)^{2\alpha-1}, & 0 \le s \le t \le 1, \\ -\alpha(\alpha+1)(\alpha+2)t^2\frac{\Gamma(\alpha)}{2\Gamma(2\alpha)}(1-s)^{2\alpha-1}, & 0 \le t \le s \le 1. \end{cases} \tag{2.1}$$

Proof From $u = (u - Pu) + Pu$ it follows that $X = \operatorname{Ker} P + \operatorname{Ker} L$. By simple calculation, we obtain $\operatorname{Ker} L \cap \operatorname{Ker} P = \{0\}$, then $X = \operatorname{Ker} L \oplus \operatorname{Ker} P$. By the same idea we prove that $Y = \operatorname{Im} L \oplus \operatorname{Im} Q$. Thus

$$\dim \operatorname{Ker} L = \dim \operatorname{Im} Q = \operatorname{co}\dim \operatorname{Im} L = 1.$$

This means that L is a Fredholm operator of index zero.

Let us find the expression of $K_p : \operatorname{Im} L \to \operatorname{dom}(L) \cap \operatorname{Ker} P$. Let $u \in \operatorname{dom}(L) \cap \operatorname{Ker} P$, then $y(t) = {}^cD^{\alpha}_{0^+} u(t) \in \operatorname{Im} L$ and

$$K_P y(t) = u(t) = I^{\alpha}_{0^+} y(t) + Ct^2 = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\,ds + Ct^2. \tag{2.2}$$

Since $u \in \operatorname{dom}(L) \cap \operatorname{Ker} P$,

$$\begin{aligned} 0 &= \int_0^1 (1-t)^{\alpha-1} u(t)\,dt \\ &= \frac{1}{\Gamma(\alpha)} \int_0^1 (1-t)^{\alpha-1} \int_0^t (t-s)^{\alpha-1} y(s)\,ds\,dt + C \int_0^1 t^2 (1-t)^{\alpha-1}\,dt \\ &= \frac{1}{\Gamma(\alpha)} \int_0^1 y(s) \int_s^1 (1-t)^{\alpha-1}(t-s)^{\alpha-1}\,dt\,ds + \frac{2C}{\alpha(\alpha+1)(\alpha+2)}, \end{aligned}$$

thus

$$\begin{aligned} C &= -\frac{\alpha(\alpha+1)(\alpha+2)}{2\Gamma(\alpha)} \int_0^1 y(s) \int_s^1 (1-t)^{\alpha-1}(t-s)^{\alpha-1}\,dt\,ds, \\ &= -\alpha(\alpha+1)(\alpha+2) \frac{\Gamma^2(\alpha)}{2\Gamma(2\alpha)} \int_0^1 (1-s)^{2\alpha-1} y(s)\,ds. \end{aligned}$$

Substituting C by its value in (2.2) we get

$$\begin{aligned} (K_P y)(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\,ds \\ &\quad - \alpha(\alpha+1)(\alpha+2) t^2 \frac{\Gamma(\alpha)}{2\Gamma(2\alpha)} \int_0^1 (1-s)^{2\alpha-1} y(s)\,ds \\ &= \int_0^1 k(t,s) y(s)\,ds, \end{aligned}$$

where $k(t,s)$ is given by (2.1). □

3 Main result

Define the isomorphism $J : \operatorname{Im} Q \to \operatorname{Ker} L$ by $J(c) = \frac{1}{2}(\alpha+1)(\alpha+2)ct^2$. We have the following result.

Lemma 3.1 *We have*

$$\bigl(JQN + K_p(I-Q)N\bigr) u(t) = \int_0^1 G(t,s) f\bigl(s, u(s)\bigr)\,ds,$$

where

$$G(t,s)=\begin{cases}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}-\alpha(\alpha+1)(\alpha+2)t^2\frac{\Gamma(\alpha)}{2\Gamma(2\alpha)}(1-s)^{2\alpha-1}\\ \quad+\alpha(\alpha+1)(\alpha+2)t^2(\frac{1}{2}+\frac{\Gamma(\alpha)}{4\Gamma(2\alpha)})(1-s)^{\alpha-1}-\frac{t^\alpha}{\Gamma(\alpha)}(1-s)^{\alpha-1}, & 0\le s\le t\le 1,\\ -\alpha(\alpha+1)(\alpha+2)t^2\frac{\Gamma(\alpha)}{2\Gamma(2\alpha)}(1-s)^{2\alpha-1}\\ \quad+\alpha(\alpha+1)(\alpha+2)t^2(\frac{1}{2}+\frac{\Gamma(\alpha)}{4\Gamma(2\alpha)})(1-s)^{\alpha-1}-\frac{t^\alpha}{\Gamma(\alpha)}(1-s)^{\alpha-1}, & 0\le t\le s\le 1.\end{cases}$$

G is continuous and nonnegative on $[0,1]\times[0,1]$.

Proof We have

$$\begin{aligned} QNu(t)&=\alpha\int_0^1(1-s)^{\alpha-1}f\bigl(s,u(s)\bigr)\,ds,\\ K_p(I-Q)Nu(t)&=\int_0^1 k(t,s)f\bigl(s,u(s)\bigr)\,ds\\ &\quad-\alpha\biggl(\int_0^1(1-s)^{\alpha-1}f\bigl(s,u(s)\bigr)\,ds\biggr)\biggl(\int_0^1 k(t,s)\,ds\biggr).\end{aligned}$$

Then

$$\begin{aligned} &\bigl(JQN+K_p(I-Q)N\bigr)u(t)\\ &\quad=\frac{\alpha(\alpha+1)(\alpha+2)t^2}{2}\int_0^1(1-s)^{\alpha-1}f\bigl(s,u(s)\bigr)\,ds\\ &\qquad+\int_0^1 k(t,s)f\bigl(s,u(s)\bigr)\,ds-\alpha\biggl(\int_0^1(1-s)^{\alpha-1}f\bigl(s,u(s)\bigr)\,ds\biggr)\biggl(\int_0^1 k(t,s)\,ds\biggr)\\ &\quad=\frac{\alpha(\alpha+1)(\alpha+2)t^2}{2}\int_0^1(1-s)^{\alpha-1}f\bigl(s,u(s)\bigr)\,ds\\ &\qquad+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f\bigl(s,u(s)\bigr)\,ds\\ &\qquad-\alpha(\alpha+1)(\alpha+2)t^2\frac{\Gamma(\alpha)}{2\Gamma(2\alpha)}\int_0^1(1-s)^{2\alpha-1}f\bigl(s,u(s)\bigr)\,ds\\ &\qquad+\biggl(-\frac{t^\alpha}{\Gamma(\alpha)}+\alpha(\alpha+1)(\alpha+2)t^2\frac{\Gamma(\alpha)}{4\Gamma(2\alpha)}\biggr)\biggl(\int_0^1(1-s)^{\alpha-1}f\bigl(s,u(s)\bigr)\,ds\biggr).\end{aligned}$$

It is easy to see that G is continuous according to both variables $s,t\in[0,1]$. Let $t\le s\le 1$, then

$$\begin{aligned} G(t,s)&=-\alpha(\alpha+1)(\alpha+2)t^2\frac{\Gamma(\alpha)}{2\Gamma(2\alpha)}(1-s)^{2\alpha-1}\\ &\qquad+\alpha(\alpha+1)(\alpha+2)t^2\biggl(\frac{1}{2}+\frac{\Gamma(\alpha)}{4\Gamma(2\alpha)}\biggr)(1-s)^{\alpha-1}-\frac{t^\alpha}{\Gamma(\alpha)}(1-s)^{\alpha-1}\\ &\ge\biggl(-\alpha(\alpha+1)(\alpha+2)t^2\frac{\Gamma(\alpha)}{4\Gamma(2\alpha)}-\frac{t^2}{\Gamma(\alpha)}+\frac{1}{2}\alpha(\alpha+1)(\alpha+2)t^2\biggr)(1-s)^{\alpha-1}\\ &\ge\biggl(-\alpha(\alpha+1)(\alpha+2)\frac{\Gamma(\alpha)}{4\Gamma(2\alpha)}-\frac{1}{\Gamma(\alpha)}+\frac{1}{2}\alpha(\alpha+1)(\alpha+2)\biggr)t^2(1-s)^{\alpha-1}\\ &\ge 6t^2(1-s)^{\alpha-1}\ge 0.\end{aligned}$$

Similarly we get, for $0 \leq s \leq t \leq 1$,

$$G(t,s) \geq \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + 6t^2(1-s)^{\alpha-1} \geq 0.$$

The proof is complete. □

Lemma 3.2 *The operator N is L-compact and continuous on $\overline{\Omega}$, where Ω is any open bounded subset of $K \cap \operatorname{dom}(L)$.*

Proof We have to prove that $QN(\overline{\Omega})$ and $K_p(I-Q)(\overline{\Omega})$ are both compact. Let $u \in \overline{\Omega}$ and $M = \max(f(s,u(s)), 0 \leq s \leq 1, u \in \overline{\Omega})$, remarking that $|k(t,s)| \leq 21$, we easily get

$$\begin{aligned} \bigl|K_p(I-Q)Nu(t)\bigr| \leq{} & \int_0^1 f\bigl(s,u(s)\bigr)\bigl|k(t,s)\bigr|\,ds \\ & + \alpha\biggl(\int_0^1 (1-s)^{\alpha-1} f\bigl(s,u(s)\bigr)\,ds\biggr)\biggl(\int_0^1 \bigl|k(t,s)\bigr|\,ds\biggr) \leq 42M, \end{aligned}$$

thus $\|K_p(I-Q)Nu\| \leq 42M$, so $K_p(I-Q)N$ is uniformly bounded on $\overline{\Omega}$.

Let $0 \leq t_1 < t_2 \leq 1$, then

$$\begin{aligned} & \bigl|K_p(I-Q)Nu(t_2) - K_p(I-Q)Nu(t_1)\bigr| \\ & \quad \leq \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \bigl((t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\bigr) f\bigl(s,u(s)\bigr)\,ds \\ & \qquad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} f\bigl(s,u(s)\bigr)\,ds \\ & \qquad + \alpha(\alpha+1)(\alpha+2)\bigl(t_2^2 - t_1^2\bigr)\frac{\Gamma(\alpha)}{2\Gamma(2\alpha)} \int_0^1 (1-s)^{2\alpha-1} f\bigl(s,u(s)\bigr)\,ds \\ & \qquad + \biggl(\int_0^1 (1-s)^{\alpha-1} f\bigl(s,u(s)\bigr)\,ds\biggr)\biggl(\bigl(t_2^\alpha - t_1^\alpha\bigr) + (\alpha+1)(\alpha+2)\bigl(t_2^2 - t_1^2\bigr)\frac{\Gamma(\alpha)}{2\Gamma(2\alpha)}\biggr) \\ & \quad \leq \frac{M}{\Gamma(\alpha)}\biggl[\alpha(t_2 - t_1) + \frac{\Gamma^2(\alpha)}{2\Gamma(2\alpha)}(\alpha+1)^2(\alpha+2)\bigl(t_2^2 - t_1^2\bigr) + \bigl(t_2^\alpha - t_1^\alpha\bigr)\biggr]. \end{aligned}$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to 0, consequently $K_p(I-Q)(\overline{\Omega})$ is equicontinuous. By means of the Arzela-Ascoli theorem we conclude that $K_p(I-Q)(\overline{\Omega})$ is compact. Similarly we prove that $QN(\overline{\Omega})$ is compact. □

Theorem 3.3 *Assume that*:

(H_1) *There exist $u_0, v_0 \in K \cap \operatorname{dom}(L)$ such that $u_0 \leq v_0$ and*

$$\begin{cases} {}^cD_{0^+}^\alpha u_0(t) \leq f(t,u_0(t)), & \forall t \in [0,1], \\ {}^cD_{0^+}^\alpha v_0(t) \geq f(t,v_0(t)), & \forall t \in [0,1]. \end{cases}$$

(H_2) *For any $x, y \in K \cap \operatorname{dom}(L)$, $u_0(t) \leq y(t) \leq x(t) \leq v_0(t)$, $\forall t \in [0,1]$, the function f satisfies*

$$f\bigl(t,x(t)\bigr) - f\bigl(t,y(t)\bigr) \geq -\alpha\biggl(\int_0^1 (1-t)^{\alpha-1} x(t)\,dt - \int_0^1 (1-t)^{\alpha-1} y(t)\,dt\biggr).$$

Then the boundary value problem (P) *has a minimal solution* u^* *and a maximal solution* v^* *in* $[u_0, v_0]$.

Proof We will prove that all conditions of Theorem 1.3 are satisfied. From the proof of Lemma 2.5, we know that L is a Fredholm operator of index zero. In view of condition (H_1), we get $Lu_0 \leq Nu_0$ and $Lv_0 \geq Nv_0$, so condition (C_1) of Theorem 1.3 holds. For $u \in K$, we have

$$\begin{aligned}
&\bigl(P + JQN + K_p(I - Q)N\bigr)u(t) \\
&\quad= \frac{1}{2}\alpha(\alpha + 1)(\alpha + 2)t^2 \int_0^1 (1 - s)^{\alpha-1}u(s) + \int_0^1 G(t,s)f\bigl(s,u(s)\bigr)\,ds.
\end{aligned}$$

Since $G(t,s)$ is continuous and nonnegative for $t, s \in [0,1]$, $(P + JQN + K_p(I - Q)N)(K) \subset K$. By virtue of the equivalence assertions, we conclude that $N + J^{-1}P : K \cap \mathrm{dom}(L) \to K_1$. Condition (H_2) implies that $N + J^{-1}P : K \cap \mathrm{dom}(L) \to K_1$ is a monotone increasing operator, in fact for $x, y \in K \cap \mathrm{dom}(L)$, $y(t) \leq x(t)$, $\forall t \in [0,1]$, we have

$$\begin{aligned}
&\bigl(N + J^{-1}P\bigr)x(t) - \bigl(N + J^{-1}P\bigr)y(t) \\
&\quad= f\bigl(t,x(t)\bigr) - f\bigl(t,y(t)\bigr) + \alpha\biggl(\int_0^1 (1 - t)^{\alpha-1}x(t)\,dt - \int_0^1 (1 - t)^{\alpha-1}y(t)\,dt\biggr) \geq 0,
\end{aligned}$$

so condition (C_2) is satisfied. Finally, we conclude by Theorem 1.3 that the equation $Lu = Nu$ has a minimal solution u^* and a maximal solution v^* in $[u_0, v_0]$, where $u^* = \lim_{n\to\infty} u_n$ and $v^* = \lim_{n\to\infty} v_n$, uniformly according to t, the sequences u_n and v_n are defined by

$$\begin{aligned}
u_n &= \bigl(L + J^{-1}P\bigr)^{-1}\bigl(N + J^{-1}P\bigr)u_{n-1} \\
&= \bigl(JQ + K_p(I - Q)\bigr)\bigl(N + J^{-1}P\bigr)u_{n-1} \\
&= \bigl(JQ + K_p(I - Q)\bigr)\biggl(f\bigl(s, u_{n-1}(s)\bigr) + \alpha \int_0^1 (1 - s)^{\alpha-1}u_{n-1}(s)\,ds\biggr) \\
&= \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1}f\bigl(s, u_{n-1}(s)\bigr)\,ds \\
&\quad+ \alpha(\alpha + 1)(\alpha + 2)t^2 \frac{\Gamma(\alpha)}{4\Gamma(2\alpha)} \int_0^1 (1 - s)^{\alpha-1}f\bigl(s, u_{n-1}(s)\bigr)\,ds \\
&\quad- \frac{t^\alpha}{\Gamma(\alpha)} \int_0^1 (1 - s)^{\alpha-1}f\bigl(s, u_{n-1}(s)\bigr)\,ds \\
&\quad- \alpha(\alpha + 1)(\alpha + 2)\frac{t^2\Gamma(\alpha)}{2\Gamma(2\alpha)} \int_0^1 (1 - s)^{2\alpha-1}f\bigl(s, u_{n-1}(s)\bigr)\,ds \\
&\quad+ \frac{1}{2}\alpha(\alpha + 1)(\alpha + 2)t^2 \int_0^1 (1 - s)^{\alpha-1}u_{n-1}(s)\,ds,
\end{aligned}$$

similarly we get the expression of v_n, moreover, we have

$$u_0 \leq u_1 \leq u_2 \leq \cdots \leq u_n \leq \cdots \leq v_n \leq \cdots \leq v_2 \leq v_1 \leq v_0. \qquad \square$$

Example 3.4 Let us consider the following fractional boundary value problem:

$$\begin{cases} {}^cD_{0^+}^{\frac{5}{2}}u(t) = t^2 + \frac{u}{u+1}, & 0 < t < 1, \\ u(0) = u'(0) = 0, & u''(0) = 2u(1). \end{cases} \tag{3.1}$$

We can choose

$$u_0(t) = \frac{1}{\Gamma(\frac{5}{2})}\int_0^t (t-s)^{\frac{3}{2}} s^2\,ds \le \frac{1}{\Gamma(\frac{5}{2})}\int_0^t (t-s)^{\frac{3}{2}}(s+1)^2\,ds = v_0(t),$$

then

$$\begin{aligned} &{}^cD_{0^+}^{\frac{5}{2}}u_0(t) = t^2 \le (t+1)^2 = {}^cD_{0^+}^{\frac{5}{2}}v_0(t), \\ &{}^cD_{0^+}^{\frac{5}{2}}u_0(t) \le f\bigl(t, u_0(t)\bigr), \qquad {}^cD_{0^+}^{\frac{5}{2}}v_0(t) \ge f\bigl(t, v_0(t)\bigr), \quad \forall t \in [0,1]. \end{aligned}$$

For any $x, y \in K \cap \operatorname{dom}(L)$, we have

$$\left(t^2 + \frac{x}{x+1}\right) - \left(t^2 + \frac{y}{y+1}\right) \ge -\frac{5}{2}\left(\int_0^1 (1-t)^{\alpha-1}x(t)\,dt - \int_0^1 (1-t)^{\alpha-1}y(t)\,dt\right),$$

where $u_0(t) \le y(t) \le x(t) \le v_0(t)$, $\forall t \in [0,1]$. Then, by Theorem 3.3, the boundary value problem (3.1) has a minimal solution u^* and a maximal solution v^* in $[u_0, v_0]$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed in writing this paper. All authors read and approved the final manuscript.

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