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Stepanov-like pseudo almost automorphic solution to a parabolic evolution equation

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Abstract

In this paper, the parabolic evolution equation $u'(t) + A(t)u(t) = f(t)$ in a reflexive real Banach space is considered. Assuming strong monotonicity, pseudo almost automorphy and other appropriate conditions of the operators $A(t)$ and Stepanov-like pseudo almost automorphy of the forced term $f(t)$, we obtain the Stepanov-like pseudo almost automorphy of the solution to the evolution equation by using the almost automorphic component equation method. This paper extends a known result in the case where $A(\cdot)$ and f are almost automorphic in certain senses. Finally, a concrete example is given to illustrate our results.

MSC: Primary 34G20; secondary 43A60

Keywords: almost automorphic solution; pseudo almost automorphic solution; parabolic evolution equation; monotone operators

1 Introduction

The concept of almost automorphy is a generalization of almost periodicity. It has been introduced by Bochner in relation to some aspects of differential geometry [1–3]. Since then this notion has been generalized in different directions. For example, Veech [4] has studied almost automorphic functions on groups, Zaki [5] provided a clear presentation of the study of (weakly) almost automorphic functions in a Banach space. N'Guérékata introduce asymptotically almost automorphic functions and presented these topics before 2005 in his monographs [6, 7].

Recently, more general types of almost automorphy are developed (see Table 1 and the references cited therein). For clarity, the relationship between the various almost automorphy is depicted in Figure 1. It is worth mentioning that Zheng and Ding [8] have showed the completeness of *WPAA* function space, which is important progress in this area.

With the development of the theory of almost automorphy, its applications have attracted a great deal of attention of many mathematicians due to their significance and applications in physics, mathematical biology, control theory, and so on. The existence and uniqueness of (pseudo) almost automorphic type solution has been one of the most attracting topics in the context of various kinds of abstract differential equations [12, 13, 15], functional differential equations [16, 17], integro-differential equations [18, 19] and

Table 1 Recent development of almost automorphy

Function	Original reference
Pseudo almost automorphic (*PAA*)	Xiao *et al.* [9, 10]
Stepanov-like almost automorphic (S^pAA)	N'Guérékata and Pankov [11]
Weighted pseudo almost automorphic (*WPAA*)	Blot *et al.* [12]
Stepanov-like pseudo almost automorphic (S^pPAA)	Diagana [13]
Weighted Stepanov-like pseudo almost automorphic (S^pWPAA)	Xia and Fan [14]

Figure 1 Relationship between recently developed almost automorphy, where '→' denotes subset relation '⊂', *AA* is short for almost automorphic functions.

$$\begin{tikzcd} AA \arrow[r] \arrow[d] \arrow[dr] & PAA \arrow[r] \arrow[d] \arrow[dr] & WPAA \arrow[d] \\ S^pAA \arrow[r] & S^pPAA \arrow[r] & S^pWPAA \end{tikzcd}$$

fractional differential equations [20–22]. For more on these studies, we refer the reader to the references cited therein.

The abstract parabolic evolution equation can be applied to many self-organized models in the real world, such as semiconductor model, forest kinematic model, chemotaxis model, Lotka-Volterra competition model, and so on. The study of fixed point, almost periodic, and almost automorphic type solution of a parabolic evolution has practical significance. Let $\mathbf{X}$ be a separable reflexive embedded real Banach space. In this paper, we consider a Stepanov-like pseudo almost automorphic solution to a parabolic evolution equation of the form

$$u'(t) + A(t)u(t) = f(t), \quad t \in \mathbb{R}, \tag{1}$$

where the operator $A(t) : \mathbf{X} \to \mathbf{X}^*$ is strongly monotone, semicontinuous, $A(\cdot)$ is operator valued pseudo almost automorphic and the forced term $f \in L^p_{\mathrm{loc}}(\mathbb{R}; \mathbf{X}^*)$ $(1 \le p < \infty)$ is Stepanov-like pseudo almost automorphic.

Let $B(\cdot)$ and $g(\cdot)$ be the almost automorphic components of $A(\cdot)$ and $f(\cdot)$, respectively. We call

$$v'(t) + B(t)v(t) = g(t), \quad t \in \mathbb{R} \tag{2}$$

the almost automorphic component equation of (1). N'Guérékata and Pankov [11] have studied the almost automorphic solution to (2). Since Stepanov-like pseudo almost automorphy is more general than almost automorphy (see Figure 1), this paper is an extension of their results.

To show the Stepanov-like pseudo almost automorphy of a bounded solution u to (1), we link up (1) with (2). More explicitly, we first show that $B(t)$ also satisfies the conditions assumed on $A(t)$ when $A(\cdot)$ is uniformly continuous. Thus (2) admits an almost automorphic solution v by N'Guérékata and Pankov's results in [11]. Second, we subtract (2) from (1) and show that $u - v$ is a Stepanov-like ergodic perturbation. We call this method the almost automorphic component equation method. As auxiliary tools for this method, we explore some properties of uniformly continuous (Stepanov-like) pseudo almost automorphic functions. These properties also play a role when we reduce the solution of (1) from Stepanov-like pseudo almost automorphy to pseudo almost automorphy.

This paper is organized as follows. In Section 2, we present the preliminaries on almost automorphic type functions as well as some examples. We explore in Section 3 some properties of uniformly continuous (Stepanov-like) pseudo almost automorphic functions, which are auxiliary for our main results in Section 4. Finally, an example is given to illustrate our results.

2 Almost automorphic type functions

Let $\mathbf{X}$ be a real Banach space with the norm $\|\cdot\|$. $C(\mathbb{R},\mathbf{X})$ denotes the space of all continuous functions from $\mathbb{R}$ into $\mathbf{X}$, and $BC(\mathbb{R};\mathbf{X})$ consists of the bounded ones in $C(\mathbb{R},\mathbf{X})$. Equipped with the sup-norm $\|x\|_\infty = \sup_{t\in\mathbb{R}}\|x(t)\|$, $BC(\mathbb{R};\mathbf{X})$ is a Banach space. $C_u(\mathbb{R};\mathbf{X})$ denotes the set of all uniformly continuous functions in $C(\mathbb{R};\mathbf{X})$. $L^p_{\mathrm{loc}}(\mathbb{R};\mathbf{X}) = \{f:\mathbb{R}\to\mathbf{X} | f$ is measurable and $\int_K \|f(t)\|^p\,\mathrm{d}t < +\infty$ for each compact subset $K\subset\mathbb{R}\}$.

2.1 (Pseudo) almost automorphic functions

Definition 2.1 [2] A function $f\in C(\mathbb{R},\mathbf{X})$, is said to be almost automorphic in Bochner's sense if for every sequence of real numbers $\{s'_n\}$, there exists a subsequence $\{s_n\}$ such that

$$g(t) := \lim_{n\to\infty} f(t+s_n)$$

is well defined for each $t\in\mathbb{R}$, and

$$\lim_{n\to\infty} g(t-s_n) = f(t)$$

for each $t\in\mathbb{R}$. Denote by $AA(\mathbf{X})$ the set of all such (Bochner) almost automorphic functions.

If the convergences in this definition are uniform on $\mathbb{R}$, then f is almost periodic in the classical Bochner sense.

Theorem 2.2 ([6], Theorem 2.1.3)

(i) *If $f\in AA(\mathbf{X})$, then the range $R_f := \{f(t): t\in\mathbb{R}\}$ is relatively compact in $\mathbf{X}$, thus f is bounded*;

(ii) (*translation invariance*) *If $f\in AA(\mathbf{X})$, then $\tau_\alpha f\in AA(\mathbf{X})$ where $\tau_\alpha f(\cdot) = f(\cdot-\alpha)$*;

(iii) *Equipped with the sup-norm $\|\cdot\|_\infty$, $AA(\mathbf{X})$ is a Banach space.*

Remark 2.3 The function g in Definition 2.1 is measurable but not necessary continuous. However, if the convergences above are uniformly on compact intervals, *i.e.* in the Fréchet space $C(\mathbb{R},\mathbf{X})$, then g is continuous, which implies that f is uniformly continuous on $\mathbb{R}$ (*cf.* [23], Theorem 2.6). In the sequel, we will denote by $AA_u(\mathbf{X})$ the closed subspace of all functions $f\in AA(\mathbf{X})$ with $g\in C(\mathbb{R},\mathbf{X})$.

Example 2.4 The function

$$t\mapsto \cos\frac{1}{2-\sin t-\sin\pi t}$$

is almost automorphic but not in $AA_u(\mathbb{R})$, since it is not uniformly continuous.

Definition 2.5 [4] A sequence $x \in l^{\infty}(\mathbf{X})$ is said to be almost automorphic if for any sequence of integers $\{s'_n\}$, there exists a subsequence $\{s_n\}$ such that

$$\lim_{n\to\infty}\lim_{m\to\infty} x_{p+s_m-s_n} = x_p$$

for each $p \in \mathbb{Z}$.

Example 2.6 ([4], p.720) Let θ be an irrational real number. For $n \in \mathbb{Z}$, $\cos 2\pi n\theta$ is never zero, and one can define

$$f(n) = \operatorname{sgn}\cos 2\pi n\theta = \begin{cases} +1, & \cos 2\pi n\theta > 0; \\ -1, & \cos 2\pi n\theta < 0. \end{cases}$$

The sequence $\{f_n\}$ is almost automorphic but not almost periodic.

Define the function

$$f(t) = f(n) + (t-n)\bigl(f(n+1) - f(n)\bigr), \quad \forall t \in [n, n+1].$$

Then the function $f \in AA_u(\mathbb{R})$.

Definition 2.7 [24] A function $\phi \in BC(\mathbb{R};\mathbf{X})$ is named an ergodic perturbation if

$$\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\bigl\|\phi(t)\bigr\|\,\mathrm{d}t = 0.$$

We denote the set of all such functions by $PAP_0(\mathbf{X})$.

Definition 2.8 [10, 25] A function $f \in C(\mathbb{R};\mathbf{X})$ is called pseudo almost automorphic if it has the decomposition

$$f = g + \phi,$$

where $g \in AA(\mathbf{X})$ and $\phi \in PAP_0(\mathbf{X})$. Denote by $PAA(\mathbf{X})$ the set of all such pseudo almost automorphic functions. The functions g and ϕ are called the almost automorphic component and ergodic perturbation of f, respectively.

Obviously, both $PAP_0(\mathbf{X})$ and $PAA(\mathbf{X})$ are translation invariant. The following theorem was given by Basit, Zhang and Xiao *et al.* [25, 26] independently.

Theorem 2.9

(i) $PAA(\mathbf{X}) = AA(\mathbf{X}) \oplus PAP_0(\mathbf{X})$, *i.e. the decomposition in Definition* 2.8 *is unique*.

(ii) *Let* $f \in PAA(\mathbf{X})$ *and* g *be its almost automorphic component. Then* $\{g(t) : t \in \mathbb{R}\} \subset \overline{\{f(t) : t \in \mathbb{R}\}}$.

(iii) $PAA(\mathbf{X})$ *is a Banach space equipped with the sup-norm* $\|\cdot\|_{\infty}$.

Example 2.10 Let $\alpha \in [0,\infty)$ and $\gamma \in (0,\infty)$. It is not difficult to show that

$$f(t) = \cos\frac{1}{2-\sin t - \sin\pi t} + \frac{\mathrm{e}^{-\gamma|t|}}{(1+|t|)^{\alpha}}, \quad t \in \mathbb{R}$$

is a pseudo almost automorphic function.

2.2 Stepanov-like (pseudo) almost automorphic functions

This subsection is devoted to (pseudo) almost automorphic functions in the sense of Stepanov. We let $1 \leq p < \infty$ in this subsection.

Definition 2.11 [11] The Bochner transform $f^b(t,s)$, $t \in \mathbb{R}$, $s \in [0,1]$, of a function $f : \mathbb{R} \to \mathbf{X}$, is defined by $f^b(t,s) = f(t+s)$.

Definition 2.12 [11] The space $BS^p(\mathbf{X})$ of Stepanov bounded functions, with exponent p, consists of all measurable functions $f : \mathbb{R} \to \mathbf{X}$ such that $f^b \in L^\infty(\mathbb{R}; L^p(0,1;\mathbf{X}))$. This is a Banach space with the norm:

$$\|f\|_{S^p} = \|f^b\|_{L^\infty(\mathbb{R};L^p(0,1;\mathbf{X}))} = \sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \|f(\tau)\|^p \, \mathrm{d}\tau \right)^{1/p}.$$

Definition 2.13

(i) [11] The space $AS^p(\mathbf{X})$ of Stepanov-like almost automorphic functions consists of all $g \in BS^p(\mathbf{X})$ such that $g^b \in AA_u(L^p(0,1;\mathbf{X}))$.
(ii) [13] The space $PAP_0^p(\mathbf{X})$ of Stepanov-like ergodic perturbation functions consists of all $\phi \in BS^p(\mathbf{X})$ such that $\phi^b \in PAP_0(L^p(0,1;\mathbf{X}))$.
(iii) [13] The space $PAA^p(\mathbf{X})$ of Stepanov-like pseudo almost automorphic functions consists of all $f \in BS^p(\mathbf{X})$ such that $f = g + \phi$ where $g \in AS^p(\mathbf{X})$, $\phi \in PAP_0^p(\mathbf{X})$. We still call g and ϕ the almost automorphic component and ergodic perturbation of f, respectively.

Theorem 2.14 ([11], Theorem 2.3 and Remark 2.4, and [27], Theorems 3.2 and 3.3)

(i) $AS^p(\mathbf{X})$ *and* $PAA^p(\mathbf{X})$ *are Banach spaces with the norm* $\|\cdot\|_{S^p}$;
(ii) $AS^p(\mathbf{X}) \supset AA(\mathbf{X})$;
(iii) $PAA^p(\mathbf{X}) \supset PAA(\mathbf{X})$.

Example 2.15

(i) [11] Let $\{f_n\} \subset \mathbb{R}$ be an almost automorphic sequence, and $\epsilon \in (0, \frac{1}{2})$. Let $f(t) = f_n$ if $t \in (n-\epsilon, n+\epsilon)$ and $f(t) = 0$ otherwise. Then $f \in AS^p(\mathbb{R})$ for all $p \in [1,\infty)$ but $f \notin AA(\mathbb{R})$.
(ii) For a discontinuous Stepanov-like pseudo almost auromorphic function, we refer the readers to [27], Example 3.6.

3 Uniformly continuous (Stepanov-like) pseudo almost automorphic functions

As the examples in Section 2 indicated, pseudo almost automorphic functions may be not uniformly continuous and the ones in the sense of Stepanov may even be not continuous. It is interesting to investigate these functions under uniform continuity condition. Moreover, these results turn out to be useful in the next section when we deal with equation (1).

Proposition 3.1 *Let* $1 \leq p < \infty$. *Suppose* $f \in PAA^p(\mathbf{X}) \cap C_u(\mathbb{R};\mathbf{X})$ *and* $f = g + \phi$ *where* $g \in AS^p(\mathbf{X})$ *and* $\phi \in PAP_0^p(\mathbf{X})$. *Then* g *and* ϕ *are also in* $C_u(\mathbb{R};\mathbf{X})$.

Proof Since $f \in C_u(\mathbb{R};\mathbf{X})$, for any $\epsilon > 0$ there exists a $\delta = \delta(\epsilon) > 0$ such that

$$\|f(x_1) - f(x_2)\| < \epsilon, \quad \forall x_1, x_2 \in \mathbb{R}, |x_1 - x_2| \leq \delta.$$

For all $t \in \mathbb{R}$, by Theorem 2.9(ii), there exists a sequence $\{t_n\} \subset \mathbb{R}$ such that $f^b(t_n,\cdot) \to g^b(t,\cdot)$ in $L^p(0,1;\mathbf{X})$ as $n \to \infty$. Then there exists a subsequence $\{t_{n_k}\} \subset \{t_n\}$ such that $f^b(t_{n_k},\cdot) \to g^b(t,\cdot)$ almost everywhere in $[0,1]$ as $k \to \infty$. So we can choose a finite δ-net $\{s_1,s_2,\ldots,s_m\}$ of the interval $[0,1]$ such that $f^b(t_{n_k},s_j)$ converges to $g^b(t,s_j)$ as $n \to \infty$ for each $j \in \{1,2,\ldots,m\}$. Let the integer $K = K(\epsilon) > 0$ satisfy

$$\left\|f^b(t_{n_{k+p}},s_j) - f^b(t_{n_k},s_j)\right\| < \epsilon, \quad \forall k > K, \forall j = 1,2,\ldots,m, \forall p \in \mathbb{N}.$$

For each $s \in [0,1]$, there exists a $j(s) \in \{1,2,\ldots,m\}$ such that $|s - s_{j(s)}| < \delta$. Then for any $k > K$ and $p \in \mathbb{N}$, we have

$$\begin{aligned}\left\|f^b(t_{n_{k+p}},s) - f^b(t_{n_k},s)\right\| &\leq \left\|f^b(t_{n_{k+p}},s) - f^b(t_{n_{k+p}},s_{j(s)})\right\| \\ &\quad + \left\|f^b(t_{n_{k+p}},s_{j(s)}) - f^b(t_{n_k},s_{j(s)})\right\| \\ &\quad + \left\|f^b(t_{n_k},s_{j(s)}) - f^b(t_{n_k},s)\right\| \\ &< 3\epsilon.\end{aligned}$$

This implies that $\{f^b(t_{n_k},\cdot)\}$ is a Cauchy sequence in $C([0,1];\mathbf{X})$ under the sup-norm. Obviously, it converges to $g^b(t,\cdot)$.

For arbitrary $s_1,s_2 \in [0,1]$ satisfying $|s_1 - s_2| < \delta$, we have

$$\left\|g(t+s_1) - g(t+s_2)\right\| = \lim_{k\to\infty}\left\|f(t_{n_k}+s_1) - f(t_{n_k}+s_2)\right\| \leq \epsilon.$$

Since the δ is independent of t, it follows that g is uniformly continuous on $\mathbb{R}$. Therefore, so is $\phi = f - g$. □

In view of Theorem 2.14(iii) that $PAA^p(\mathbf{X}) \supset PAA(\mathbf{X})$, we get the following result.

Corollary 3.2 *Let $f \in PAA(\mathbf{X}) \cap C_u(\mathbb{R};\mathbf{X})$ and $f = g + \phi$ where $g \in AA(\mathbf{X})$ and $\phi \in PAP_0(\mathbf{X})$. Then g and ϕ are also in $C_u(\mathbb{R};\mathbf{X})$*

Under the uniform continuity condition, we find that a Stepanov-like pseudo almost automorphic function space reduces to a pseudo almost automorphic function space.

Proposition 3.3 *Let $f \in C_u(\mathbb{R};\mathbf{X})$ and $1 \leq p < \infty$. Then the following statements hold*:

(i) *$f \in AS^p(\mathbf{X})$ implies that $f \in AA(\mathbf{X})$*;

(ii) *$f \in PAP_0^p(\mathbf{X})$ implies that $f \in PAP_0(\mathbf{X})$*;

(iii) *$f \in PAA^p(\mathbf{X})$ implies that $f \in PAA(\mathbf{X})$*.

Proof Statement (i) is given by Ding *et al.* in [28].

Now suppose $f \in PAP_0^p(\mathbf{X})$ is uniformly continuous on $\mathbb{R}$. Let $f_n(t) = n\int_0^{\frac{1}{n}} f(t+s)\,\mathrm{d}s$ for each $t \in \mathbb{R}$, $n \in \mathbb{N}$. It is easy to see that $f_n \in BC(\mathbb{R};\mathbf{X})$ converges uniformly to f on $\mathbb{R}$ as $n \to \infty$. Since $PAP_0(\mathbf{X})$ is a closed subspace of $BC(\mathbb{R};\mathbf{X})$, we only need to show that $f_n \in PAP_0(\mathbf{X})$ for each $n \in \mathbb{N}$.

When $1 < p < \infty$, let $1/p + 1/p' = 1$. For any $T > 0$, using the Hölder inequality, we have

$$\begin{aligned}\frac{1}{2T}\int_{-T}^{T}\|f_n(t)\|\,\mathrm{d}t &\le n\frac{1}{2T}\int_{-T}^{T}\int_0^{\frac{1}{n}}\|f(t+s)\|\,\mathrm{d}s\,\mathrm{d}t \\ &\le n\frac{1}{2T}\int_{-T}^{T}\left(\int_0^{\frac{1}{n}}\|f(t+s)\|^p\,\mathrm{d}s\right)^{\frac{1}{p}}\left(\frac{1}{n}\right)^{\frac{1}{p'}}\mathrm{d}t \\ &\le n^{\frac{1}{p}}\frac{1}{2T}\int_{-T}^{T}\left(\int_0^{1}\|f(t+s)\|^p\,\mathrm{d}s\right)^{\frac{1}{p}}\mathrm{d}t.\end{aligned}$$

It follows that

$$\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\|f_n(t)\|\,\mathrm{d}t \le n^{\frac{1}{p}}\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\left(\int_0^{1}\|f(t+s)\|^p\,\mathrm{d}s\right)^{\frac{1}{p}}\mathrm{d}t = 0,$$

i.e., $f_n \in PAP_0(\mathbf{X})$.

The argument for the case $p = 1$ is simpler in the same way. Thus, statement (ii) has been proved.

Suppose $f \in PAA^p(\mathbf{X}) \cap C_u(\mathbb{R};\mathbf{X})$ and $f = g + \phi$ where $g \in AS^p(\mathbf{X})$ and $\phi \in PAP_0^p(\mathbf{X})$. It follows from Proposition 3.1 that g and ϕ are both in $C_u(\mathbb{R};\mathbf{X})$. Then statement (iii) is implied by statements (i) and (ii). □

Definition 3.4 A sequence $\{x_n\} \subset \mathbf{X}$ is named an ergodic perturbation sequence if

$$\lim_{N\to\infty}\frac{1}{2N+1}\sum_{k=-N}^{N}\|x_k\| = 0.$$

Denote the set of all such sequences by $PAP_0(\mathbb{Z};\mathbf{X})$.

Proposition 3.5 *Let $f \in PAP_0(\mathbf{X}) \cap C_u(\mathbb{R};\mathbf{X})$. Then for arbitrary $l > 0$ and $a \in \mathbb{R}$, the positive sequence $\{f_i : f_i = \max_{s\in[a+il,a+(i+1)l]}\|f(s)\|\}$ is in $PAP_0(\mathbb{Z};\mathbb{R})$.*

Proof Since $PAP_0(\mathbf{X})$ is translation invariant, we may assume $a = 0$.

Suppose the sequence $\{f_i\}$ is not in $PAP_0(\mathbb{Z};\mathbb{R})$. Then there exist an $\epsilon > 0$ and a strictly increasing sequence $\{n_k\} \subset \mathbb{N}$ such that

$$\lim_{k\to\infty}\frac{1}{2n_k+1}\sum_{i=-n_k}^{n_k} f_i > \epsilon.$$

Because of the uniform continuity of f, there exists a positive δ ($< \frac{l}{2}$) such that

$$\|f(x_1) - f(x_2)\| < \frac{\epsilon}{2}, \quad \forall x_1, x_2 \in \mathbb{R}, |x_1 - x_2| < \delta.$$

For all $i \in \mathbb{Z}$, let $f_i = \|f(s_i)\|$, $s_i \in [il, (i+1)l]$. If $s_i \ge (i + \frac{1}{2})l$, then

$$\int_{il}^{(i+1)l}\|f(t)\|\,\mathrm{d}t \ge \int_{s_i-\delta}^{s_i}\|f(t)\|\,\mathrm{d}t \ge \int_{s_i-\delta}^{s_i}\left(f_i - \frac{\epsilon}{2}\right)\mathrm{d}t = \left(f_i - \frac{\epsilon}{2}\right)\delta.$$

The case $s_i \le (i + \frac{1}{2})l$ can be treated analogously.

For the mean of $\|f(\cdot)\|$ on $[-n_k l, n_k l]$, we have

$$\begin{aligned}\frac{1}{2n_k l}\int_{-n_k l}^{n_k l}\|f(t)\|\,\mathrm{d}t &= \frac{1}{2n_k l}\sum_{i=-n_k}^{n_k-1}\int_{il}^{(i+1)l}\|f(t)\|\,\mathrm{d}t \geq \frac{1}{2n_k l}\sum_{i=-n_k}^{n_k-1}\left(f_i-\frac{\epsilon}{2}\right)\delta \\ &= \frac{\delta}{l}\frac{2n_k+1}{2n_k}\frac{1}{2n_k+1}\sum_{i=-n_k}^{n_k}\left(f_i-\frac{\epsilon}{2}\right)-\frac{1}{2n_k l}\left(f_{n_k}-\frac{\epsilon}{2}\right)\delta.\end{aligned}$$

Letting $k\to\infty$ in the formula above, we get

$$\lim_{k\to\infty}\frac{1}{2n_k l}\int_{-n_k l}^{n_k l}\|f(t)\|\,\mathrm{d}t \geq \frac{\delta}{l}\lim_{k\to\infty}\frac{1}{2n_k+1}\sum_{i=-n_k}^{n_k}\left(f_i-\frac{\epsilon}{2}\right) > \frac{\delta\epsilon}{2l} > 0,$$

which contradicts the fact that $f\in PAP_0(\mathbf{X})$. □

4 Pseudo almost automorphic monotone evolution equation

Let $\mathbf{X}$ be a separable reflexive embedded real Banach space; this means that there is a Hilbert space $\mathbf{H}$ such that the embedding $\mathbf{X}\subset\mathbf{H}\subset\mathbf{X}^*$ is dense and continuous, and that the bilinear form (y,x) ($y\in\mathbf{X}^*$, $x\in\mathbf{X}$) coincides with the scalar product on $\mathbf{H}$ whenever $x,y\in\mathbf{H}$. We shall use the following notation for the norms: $\|x\|$ is an $\mathbf{X}$-norm, $\|y\|_*$ is an $\mathbf{X}^*$-norm, and $|x|_{\mathbf{H}}$ is an $\mathbf{H}$-norm.

In this section, we consider the equation

$$u'(t)+A(t)u(t)=f(t), \tag{3}$$

where $A(t):\mathbf{X}\to\mathbf{X}^*$ and $f:\mathbb{R}\to\mathbf{X}^*$.

For the existence and uniqueness of solution to (3) on the whole line $\mathbb{R}$, we refer to the following conditions:

(H1) (compactness condition) the embedding $\mathbf{X}\subset\mathbf{H}$ is compact;

(H2) for each $t\in\mathbb{R}$ the operator $A(t):\mathbf{X}\to\mathbf{X}^*$ is semicontinuous, *i.e.*, the function $\lambda\mapsto(A(t)(\upsilon+\lambda\omega),u)$ is continuous for each υ, ω, and u in $\mathbf{X}$;

(H3) (coercivity) there exist $\alpha>0$ and $\beta\in\mathbb{R}$ such that

$$\bigl(A(t)v,v\bigr)\geq\alpha\|v\|^p+\beta$$

for all $t\in\mathbb{R}$ and $v\in\mathbf{X}$, where $p\geq 2$ is a constant;

(H4) (strong monotonicity) there exist two constants γ and q such that $\gamma>0$, $2\leq q\leq p$, and

$$\bigl(A(t)v-A(t)w,v-w\bigr)\geq\gamma|v-w|_{\mathbf{H}}^q$$

for all $t\in\mathbb{R}$, v and w in $\mathbf{X}$ with p from (H3);

(H5) $A(t)0=0$;

(H6) there exist $c_1>0$, $c_2\in\mathbb{R}$ such that

$$\bigl\|A(t)v\bigr\|_*\leq c_1\|v\|^{p-1}+c_2$$

for all $t\in\mathbb{R}$ and $v\in\mathbf{X}$ with p from (H3);

(H7) for any $v \in \mathbf{X}$ and any bounded set $U \subset \mathbf{X}$ the family of functions $\{(A(\cdot)u, v), u \in U\}$ is equicontinuous on any compact subinterval of $\mathbb{R}$.

Condition (H5) is imposed for simplicity and estimates of solutions [29], p.60, and we will see it is not a necessary restriction of (3); in the case we will consider (see Corollary 4.6 below).

In what follows p' stands for the conjugate exponent $\frac{1}{p} + \frac{1}{p'} = 1$.

Suppose that $f \in L^{p'}_{\mathrm{loc}}(a, b; \mathbf{X}^*)$. Then a function $u : (a, b) \to \mathbf{X}$ is a (weak) solution to (3) if $u \in L^p_{\mathrm{loc}}(a, b; \mathbf{X})$ and the derivative u' is to be understood in the weak sense [30]. From (3) and (H6), it follows that $u' \in L^{p'}_{\mathrm{loc}}(a, b; \mathbf{X}^*)$. Hence, $u \in C((a, b); \mathbf{H})$ (see [30]). Moreover, for any such function u:

$$\int_{t_1}^{t_2} \bigl(u'(t), u(t)\bigr)\,\mathrm{d}t = \frac{1}{2}\bigl(\bigl|u(t_2)\bigr|^2_{\mathbf{H}} - \bigl|u(t_1)\bigr|^2_{\mathbf{H}}\bigr), \quad [t_1, t_2] \subset [a, b].$$

The following lemma comes from [29], Theorem 3.2.5 and Remark 3.2.7.

Lemma 4.1 *Under the conditions* (H1)-(H7), *suppose that* $f \in BS^{p'}(\mathbf{X}^*)$. *Then* (3) *has a unique solution u such that* $u \in BS^p(\mathbf{X}) \cap BC(\mathbb{R}; \mathbf{H})$. *Moreover, there is a bound of the form*

$$\|u\|_\infty \le C_1\bigl(\|f\|_{S^{p'}}\bigr), \qquad \|u\|_{S^p} \le C_2\bigl(\|f\|_{S^{p'}}\bigr),$$

where C_1 *and* C_2 *are increasing functions on* $[0, \infty)$ *that depend only on constants involved in* (H6).

In [29], p.53, Pankov introduced the space $Y_{p,\mathbf{X}}$ ($1 \le p < \infty$) of all operators A from $\mathbf{X}$ into $\mathbf{X}^*$ satisfying inequality

$$\|Ax\|_* \le a_1\|x\|^{p-1} + a_2,$$

where $a_1 > 0$ and $a_2 \in \mathbb{R}$ are constants depending on the operator. For any two operators A_1 and A_2 in $Y_{p,\mathbf{X}}$, we define their sum by $(A_1 + A_2)x = A_1x + A_2x$, $\forall x \in \mathbf{X}$ and the product of a scalar $\lambda \in \mathbb{R}$ and $A \in Y_{p,\mathbf{X}}$ by $(\lambda A)x = \lambda Ax$, $\forall x \in \mathbf{X}$. Obviously, $Y_{p,\mathbf{X}}$ is a vector space. For all $A \in Y_{p,\mathbf{X}}$, we define the norm

$$\|A\|_{Y_{p,\mathbf{X}}} = \sup_{x \in \mathbf{X}} \frac{\|Ax\|_*}{1 + \|x\|^{p-1}}.$$

It is easy to show that equipped with this norm, $Y_{p,\mathbf{X}}$ is a Banach space.

Remark 4.2 If the function $A(\cdot) \in BC(\mathbb{R}; Y_{p,\mathbf{X}})$, then the conditions (H6) and (H7) will be satisfied. In fact,

$$\bigl\|A(t)v\bigr\|_* \le \bigl\|A(t)\bigr\|_{Y_{p,\mathbf{X}}}\bigl(\|v\|^{p-1} + 1\bigr).$$

So we can choose $c_1 = c_2 = \|A(\cdot)\|_\infty$ in the condition (H6). Given $v \in \mathbf{X}$, a bounded set $U \subset \mathbf{X}$ and a compact subinterval J of $\mathbb{R}$, for any $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\bigl\|A(t_1) - A(t_2)\bigr\|_{Y_{p,\mathbf{X}}} < \epsilon \cdot \frac{1}{(1 + \sup_{u \in U}\|u\|^{p-1})\|v\| + 1}, \quad \forall t_1, t_2 \in J, |t_1 - t_2| < \delta.$$

It follows that for all $u \in U$ and $t_1, t_2 \in J$, $|t_1 - t_2| < \delta$,

$$\bigl|\bigl(A(t_1)u, v\bigr) - \bigl(A(t_2)u, v\bigr)\bigr| \leq \bigl\|A(t_1) - A(t_2)\bigr\|_{Y_{p,\mathbf{X}}}\bigl(1 + \|u\|^{p-1}\bigr)\|v\| < \epsilon.$$

Thus, the condition (H7) is satisfied.

Theorem 4.3 (Main result) *Under the conditions* (H1)-(H5), *suppose that* $A(\cdot) \in PAA(Y_{p,\mathbf{X}}) \cap C_u(\mathbb{R}; Y_{p,\mathbf{X}})$ *and* $f \in PAA^{p'}(\mathbf{X}^*)$. *Then* (3) *has a unique solution* u *such that* $u \in PAA^q(\mathbf{H}) \cap BC(\mathbb{R}; \mathbf{H}) \cap BS^p(\mathbf{X})$ *and its almost automorphic component is in* $AA_u(\mathbf{H}) \cap BS^p(\mathbf{X})$.

Let $A(\cdot) = B(\cdot) + \Phi(\cdot)$, $f = g + \phi$, where $B(\cdot)$ and g are the almost automorphic components of $A(\cdot)$ and f respectively. We call the equation

$$v'(t) + B(t)v(t) = g(t) \tag{4}$$

the almost automorphic component of (3).

We claim that under the conditions of Theorem 4.3, conditions (H2)-(H7) still hold when $A(t)$ is replaced with $B(t)$.

Firstly, we show that $B(t)$ is semicontinuous, *i.e.*, for each $t \in \mathbb{R}$ and any v, w, and u in $\mathbf{X}$ the function $\lambda \mapsto (B(t)(v + \lambda w), u)$ is continuous. By Theorem 2.9(ii), there exists a sequence $\{t_n\} \subset \mathbb{R}$ such that $\lim_{n\to\infty} A(t_n) = B(t)$ in $Y_{p,\mathbf{X}}$. Since $A(t_n)$ is semicontinuous, the function $\lambda \mapsto (A(t_n)(v + \lambda w), u)$ is continuous for each $n \in \mathbb{N}$. We have

$$\begin{aligned}
&\bigl|\bigl(A(t_n)(v + \lambda w), u\bigr) - \bigl(B(t)(v + \lambda w), u\bigr)\bigr| \\
&\quad\leq \bigl\|A(t_n)(v + \lambda w) - B(t)(v + \lambda w)\bigr\|_* \|u\| \\
&\quad\leq \bigl\|A(t_n) - B(t)\bigr\|_{Y_{p,\mathbf{X}}}\bigl(1 + \bigl(\|v\| + |\lambda|\|w\|\bigr)^{p-1}\bigr)\|u\|,
\end{aligned}$$

which implies that $\lambda \mapsto (A(t_n)(v + \lambda w), u)$ converges to $\lambda \mapsto (B(t)(v + \lambda w), u)$ uniformly on any compact interval of $\mathbb{R}$. So $\lambda \mapsto (B(t)(v + \lambda w), u)$ is continuous on $\mathbb{R}$.

Using the t and $\{t_n\}$ in the paragraph above, we have

$$\bigl\|A(t_n)v - B(t)v\bigr\|_* \leq \bigl\|A(t_n) - B(t)\bigr\|_{Y_{p,\mathbf{X}}}\bigl(1 + \|v\|^{p-1}\bigr) \to 0, \quad \text{as } n \to \infty, \forall v \in \mathbf{X}.$$

Thus, for any v and w in $\mathbf{X}$,

$$\begin{aligned}
&\bigl(B(t)v - B(t)w, v - w\bigr) \\
&\quad= \bigl(B(t)v - A(t_n)v - \bigl(B(t)w - A(t_n)w\bigr), v - w\bigr) + \bigl(A(t_n)v - A(t_n)w, v - w\bigr) \\
&\quad\geq \bigl(B(t)v - A(t_n)v, v - w\bigr) - \bigl(B(t)w - A(t_n)w, v - w\bigr) + \gamma|v - w|_{\mathbf{H}}^q.
\end{aligned}$$

Noting that $(B(t)v - B(t)w, v - w)$ is independent of n above, we get

$$\bigl(B(t)v - B(t)w, v - w\bigr) \geq \gamma|v - w|_{\mathbf{H}}^p$$

by letting $n \to \infty$. So condition (H4) holds for $B(t)$. The cases for conditions (H3) and (H5) can be shown in the same way.

It follows from Corollary 3.2 that $B(\cdot) \in AA_u(Y_{p,\mathbf{X}})$. By Remark 4.2, conditions (H6) and (H7) hold for $B(t)$.

Now, for the almost automorphic component (4), we have the following result, which is due to N'Guérékata and Pankov [11], Theorem 4.1.

Lemma 4.4 *Under the conditions of Theorem* 4.3, (4) *has a unique solution v such that $v \in AA_u(\mathbf{H}) \cap BS^p(\mathbf{X})$.*

Remark 4.5 For each $t \in \mathbb{R}$, the operator $B(t)$ is assumed to be continuous from $\mathbf{X}$ into $\mathbf{X}^*$ in [11], Section 4. The semicontinuity is weaker than continuity but Lemma 4.4 still hold under this weaker condition. For this, we advise the readers to refer to [29], p.57, and the proof of [11], Theorem 4.1.

To prove Theorem 4.3, a natural idea is that if (3) has a solution u in $PAA^P(\mathbf{H})$, then its almost automorphic component should be the solution $v \in AA_u(\mathbf{H})$ of (4), *i.e.*, $u - v$ is an ergodic perturbation in the sense of Stepanov.

Proof of Theorem 4.3 It follows from Lemma 4.1 and Remark 4.2 that (3) has a unique solution u such that $u \in BS^p(\mathbf{X}) \cap BC(\mathbb{R}; \mathbf{H})$.

Let $z = u - v$ where v is the one in Lemma 4.4. Then z satisfies

$$z'(t) + A(t)u(t) - A(t)v(t) = \phi(t) - \Phi(t)v(t).$$

Multiplying it by $z(t)$, we get

$$\bigl(z'(t), z(t)\bigr) + \bigl(A(t)u(t) - A(t)v(t), z(t)\bigr) = \bigl(\phi(t), z(t)\bigr) - \bigl(\Phi(t)v(t), z(t)\bigr).$$

It follows that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\bigl|z(t)\bigr|_{\mathbf{H}}^2 + \gamma\bigl|z(t)\bigr|_{\mathbf{H}}^q \le \bigl\|\phi(t)\bigr\|_*\bigl\|z(t)\bigr\| + \bigl\|\Phi(t)v(t)\bigr\|_*\bigl\|z(t)\bigr\|.$$

Integrating the inequality above from t to $t+1$ yields

$$\begin{aligned}
&\frac{1}{2}\bigl(\bigl|z(t+1)\bigr|_{\mathbf{H}}^2 - \bigl|z(t)\bigr|_{\mathbf{H}}^2\bigr) + \gamma\int_t^{t+1}\bigl|z(s)\bigr|_{\mathbf{H}}^q\,\mathrm{d}s \\
&\quad\le \int_t^{t+1}\bigl\|\phi(s)\bigr\|_*\bigl\|z(s)\bigr\|\,\mathrm{d}s + \int_t^{t+1}\bigl\|\Phi(s)v(s)\bigr\|_*\bigl\|z(s)\bigr\|\,\mathrm{d}s \\
&\quad\le \|z\|_{S^p}\biggl(\int_t^{t+1}\bigl\|\phi(s)\bigr\|_*^{p'}\,\mathrm{d}s\biggr)^{\frac{1}{p'}} + \|z\|_{S^p}\biggl(\int_t^{t+1}\bigl\|\Phi(s)v(s)\bigr\|_*^{p'}\,\mathrm{d}s\biggr)^{\frac{1}{p'}}.
\end{aligned}$$

For any $T > 0$, integrating the formula above from $-T$ to T and dividing it by $2T$, we obtain

$$\begin{aligned}
&\frac{1}{4T}\int_{-T}^T\bigl(\bigl|z(t+1)\bigr|_{\mathbf{H}}^2 - \bigl|z(t)\bigr|_{\mathbf{H}}^2\bigr)\,\mathrm{d}t + \gamma\frac{1}{2T}\int_{-T}^T\int_t^{t+1}\bigl|z(s)\bigr|_{\mathbf{H}}^q\,\mathrm{d}s\,\mathrm{d}t \\
&\quad\le \|z\|_{S^p}\frac{1}{2T}\int_{-T}^T\biggl(\int_t^{t+1}\bigl\|\phi(s)\bigr\|_*^{p'}\,\mathrm{d}s\biggr)^{\frac{1}{p'}}\,\mathrm{d}t \\
&\qquad + \|z\|_{S^p}\frac{1}{2T}\int_{-T}^T\biggl(\int_t^{t+1}\bigl\|\Phi(s)v(s)\bigr\|_*^{p'}\,\mathrm{d}s\biggr)^{\frac{1}{p'}}\,\mathrm{d}t. \qquad (5)
\end{aligned}$$

From Lemma 4.1 we know both u and v are bounded with values in $\mathbf{H}$ and also bounded in $BS^p(\mathbf{X})$, then so is $z = u - v$.

For the first term on the left hand side of inequality (5), we have

$$\begin{aligned}
&\frac{1}{4T}\int_{-T}^{T}\left(\left|z(t+1)\right|_{\mathbf{H}}^2 - \left|z(t)\right|_{\mathbf{H}}^2\right)\mathrm{d}t \\
&\quad = \frac{1}{4T}\int_{T}^{T+1}\left|z(t)\right|_{\mathbf{H}}^2\,\mathrm{d}t - \frac{1}{4T}\int_{-T}^{-T+1}\left|z(t)\right|_{\mathbf{H}}^2\,\mathrm{d}t \\
&\quad \le \frac{1}{2T}\sup_{t\in\mathbb{R}}\left|z(t)\right|_{\mathbf{H}}^2 \to 0, \quad \text{as } T\to+\infty.
\end{aligned} \tag{6}$$

Since $\phi \in PAP_0^{p'}(\mathbf{X}^*)$, the first term on the right hand side of inequality (5) converges to zero as $T\to+\infty$, *i.e.*,

$$\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\left(\int_t^{t+1}\left\|\phi(s)\right\|_*^{p'}\,\mathrm{d}s\right)^{\frac{1}{p'}}\mathrm{d}t = 0. \tag{7}$$

For the second term on the right hand side of inequality (5), we have

$$\begin{aligned}
&\frac{1}{2T}\int_{-T}^{T}\left(\int_t^{t+1}\left\|\Phi(s)v(s)\right\|_*^{p'}\,\mathrm{d}s\right)^{\frac{1}{p'}}\mathrm{d}t \\
&\quad \le \frac{1}{2T}\int_{-T}^{T}\left(\int_t^{t+1}\left[\left\|\Phi(s)\right\|_{Y_{p,\mathbf{X}}}\left(1+\left\|v(s)\right\|^{p-1}\right)\right]^{p'}\mathrm{d}s\right)^{\frac{1}{p'}}\mathrm{d}t \\
&\quad \le \frac{1}{2T}\int_{-T}^{T}\max_{\lambda\in[t,t+1]}\left\|\Phi(\lambda)\right\|_{Y_{p,\mathbf{X}}}\left(\int_t^{t+1}\left(1+\left\|v(s)\right\|^{p-1}\right)^{p'}\mathrm{d}s\right)^{\frac{1}{p'}}\mathrm{d}t \\
&\quad \le \left(1+\|v\|_{S^p}^{\frac{p}{p'}}\right)\frac{1}{2T}\int_{-T}^{T}\max_{\lambda\in[t,t+1]}\left\|\Phi(\lambda)\right\|_{Y_{p,\mathbf{X}}}\mathrm{d}t,
\end{aligned} \tag{8}$$

where the Minkowski inequality is used in the last inequality.

Let $[t]$ denote the largest integer which is not larger than t for each $t\in\mathbb{R}$. Then, for $T>0$ sufficient large,

$$\begin{aligned}
&\frac{1}{2T}\int_{-T}^{T}\max_{\lambda\in[t,t+1]}\left\|\Phi(\lambda)\right\|_{Y_{p,\mathbf{X}}}\mathrm{d}t \\
&\quad \le \frac{1}{2[T]}\int_{-[T]-1}^{[T]+1}\max_{\lambda\in[t,t+1]}\left\|\Phi(\lambda)\right\|_{Y_{p,\mathbf{X}}}\mathrm{d}t \\
&\quad \le \frac{1}{2[T]}\sum_{k=-[T]-1}^{[T]}\int_k^{k+1}\max_{\lambda\in[t,t+1]}\left\|\Phi(\lambda)\right\|_{Y_{p,\mathbf{X}}}\mathrm{d}t \\
&\quad \le \frac{1}{2[T]}\sum_{k=-[T]-1}^{[T]}\max_{\lambda\in[k,k+2]}\left\|\Phi(\lambda)\right\|_{Y_{p,\mathbf{X}}}.
\end{aligned} \tag{9}$$

It is easy to see that for each $k\in\mathbb{Z}$,

$$\max_{\lambda\in[k,k+2]}\left\|\Phi(\lambda)\right\|_{Y_{p,\mathbf{X}}} \le \max_{\lambda\in[k,k+1]}\left\|\Phi(\lambda)\right\|_{Y_{p,\mathbf{X}}} + \max_{\lambda\in[k+1,k+2]}\left\|\Phi(\lambda)\right\|_{Y_{p,\mathbf{X}}}. \tag{10}$$

Combining inequalities (9) and (10), we obtain

$$\begin{aligned}
&\frac{1}{2T}\int_{-T}^{T}\max_{\lambda\in[t,t+1]}\|\Phi(\lambda)\|_{Y_{p,\mathbf{X}}}\,\mathrm{d}t \\
&\quad\le \frac{1}{2[T]}\sum_{k=-[T]-1}^{[T]}\max_{\lambda\in[k,k+1]}\|\Phi(\lambda)\|_{Y_{p,\mathbf{X}}}+\frac{1}{2[T]}\sum_{k=-[T]}^{[T]+1}\max_{\lambda\in[k,k+1]}\|\Phi(\lambda)\|_{Y_{p,\mathbf{X}}} \\
&\quad\le \frac{2([T]+1)+1}{[T]}\frac{1}{2([T]+1)+1}\sum_{k=-[T]-1}^{[T]+1}\max_{\lambda\in[k,k+1]}\|\Phi(\lambda)\|_{Y_{p,\mathbf{X}}}.
\end{aligned}\tag{11}$$

By Corollary 3.2, we know that $\Phi:\mathbb{R}\to Y_{p,\mathbf{X}}$ is uniformly continuous. Then it follows from Proposition 3.5 that the sequence

$$\Bigl\{\max_{\lambda\in[k,k+1]}\|\Phi(\lambda)\|_{Y_{p,\mathbf{X}}}:k\in\mathbb{Z}\Bigr\}\in PAP_0(\mathbb{Z};\mathbb{R}).$$

Thus, inequality (11) implies that

$$\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\max_{\lambda\in[t,t+1]}\|\Phi(\lambda)\|_{Y_{p,\mathbf{X}}}\,\mathrm{d}t=0.\tag{12}$$

It follows from (8) and (12) that

$$\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\biggl(\int_{t}^{t+1}\|\Phi(s)v(s)\|_{*}^{p'}\,\mathrm{d}s\biggr)^{\frac{1}{p'}}\,\mathrm{d}t=0.\tag{13}$$

Letting $T\to+\infty$ and substituting (6), (7), and (13) into inequality (5), we obtain

$$\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\int_{t}^{t+1}|z(s)|_{\mathbf{H}}^{q}\,\mathrm{d}s\,\mathrm{d}t=0.\tag{14}$$

By the Hölder inequality, it is easy to see that

$$\frac{1}{2T}\int_{-T}^{T}\biggl(\int_{t}^{t+1}|z(s)|_{\mathbf{H}}^{q}\,\mathrm{d}s\biggr)^{\frac{1}{q}}\,\mathrm{d}t\le\biggl(\frac{1}{2T}\int_{-T}^{T}\int_{t}^{t+1}|z(s)|_{\mathbf{H}}^{q}\,\mathrm{d}s\,\mathrm{d}t\biggr)^{\frac{1}{q}}.$$

Thus we obtain from (14)

$$\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\biggl(\int_{t}^{t+1}|z(s)|_{\mathbf{H}}^{q}\,\mathrm{d}s\biggr)^{\frac{1}{q}}\,\mathrm{d}t=0,$$

i.e., $z\in PAP_0^q(\mathbf{H})$. The proof is complete. □

For the general case $A(t)0\neq 0$, we have the following result.

Corollary 4.6 *Theorem* 4.3 *still holds without condition* (H5) *that* $A(t)0=0$ *for each* $t\in\mathbb{R}$.

Proof We may consider $u\mapsto A(t)u-A(t)0$ and the function $f(t)-A(t)0$ instead of $A(t)u$ and $f(t)$, respectively, in (3). It is easy to show that under the above transformation conditions (H2)-(H4) still hold. Now, we show that $t\mapsto[u\mapsto A(t)u-A(t)0]\in PAA(Y_{p,\mathbf{X}})\cap C_u(\mathbb{R};Y_{p,\mathbf{X}})$ and $t\mapsto A(t)0\in PAA(\mathbf{X}^*)$.

Since $PAA(Y_{p,\mathbf{X}})$ and $C_u(\mathbb{R};Y_{p,\mathbf{X}})$ are linear spaces and $A(\cdot)\in PAA(Y_{p,\mathbf{X}})\cap C_u(\mathbb{R};Y_{p,\mathbf{X}})$, to show $t\mapsto[u\mapsto A(t)u-A(t)0]\in PAA(Y_{p,\mathbf{X}})\cap C_u(\mathbb{R};Y_{p,\mathbf{X}})$, we only need to show $t\mapsto[u\mapsto A(t)0]\in PAA(Y_{p,\mathbf{X}})\cap C_u(\mathbb{R};Y_{p,\mathbf{X}})$.

For any $A\in Y_{p,\mathbf{X}}$, $\|u\mapsto A0\|_{Y_{p,\mathbf{X}}}=\sup_{x\in\mathbf{X}}\frac{\|A0\|_*}{1+\|u\|^{p-1}}=\frac{\|A0\|_*}{1+\|0\|^{p-1}}\le\|A\|_{Y_{p,\mathbf{X}}}$. For any $t_1,t_2\in\mathbb{R}$, we have

$$\begin{aligned}\big\|\big[u\mapsto A(t_1)0\big]-\big[u\mapsto A(t_2)0\big]\big\|_{Y_{p,\mathbf{X}}}&=\big\|u\mapsto\big(A(t_1)-A(t_2)\big)0\big\|_{Y_{p,\mathbf{X}}}\\&\le\big\|A(t_1)-A(t_2)\big\|_{Y_{p,\mathbf{X}}}.\end{aligned}$$

Thus, $A(\cdot)\in C_u(\mathbb{R};Y_{p,\mathbf{X}})$ implies that $t\mapsto[u\mapsto A(t)0]\in C_u(\mathbb{R};Y_{p,\mathbf{X}})$ also.

We write $u\mapsto A(t)0=[u\mapsto B(t)0]+[u\mapsto\Phi(t)0]$ for each $t\in\mathbb{R}$ and claim that $t\mapsto[u\mapsto B(t)0]\in AA(Y_{p,\mathbf{X}})$ and $t\mapsto[u\mapsto\Phi(t)0]\in PAP_0(Y_{p,\mathbf{X}})$.

Suppose there exist a sequence $\{t_n\}\subset\mathbb{R}$ and a function $\tilde{B}:\mathbb{R}\to Y_{p,\mathbf{X}}$ such that

$$B(t+t_n)\to\tilde{B}(t)\quad\text{and}\quad\tilde{B}(t-t_n)\to B(t)$$

in $Y_{p,\mathbf{X}}$ for each $t\in\mathbb{R}$ as $n\to\infty$. Thus, for each $t\in\mathbb{R}$, we have

$$\begin{aligned}\big\|\big[u\mapsto B(t+t_n)0\big]-\big[u\mapsto\tilde{B}(t)0\big]\big\|_{Y_{p,\mathbf{X}}}&=\big\|u\mapsto\big(B(t+t_n)-\tilde{B}(t)\big)0\big\|_{Y_{p,\mathbf{X}}}\\&\le\big\|B(t+t_n)-\tilde{B}(t)\big\|_{Y_{p,\mathbf{X}}}\to0,\end{aligned}$$

as $n\to\infty$.

Similarly, we can show $u\mapsto\tilde{B}(t-t_n)0$ converges to $u\mapsto B(t)0$ in $Y_{p,\mathbf{X}}$ for each $t\in\mathbb{R}$ as $n\to\infty$. Thus, $t\mapsto[u\mapsto B(t)0]\in AA(Y_{p,\mathbf{X}})$.

The function $t\mapsto[u\mapsto\Phi(t)0]$ is bounded, because $\|u\mapsto\Phi(t)0\|_{Y_{p,\mathbf{X}}}\le\|\Phi(t)\|_{Y_{p,\mathbf{X}}}\le\|\Phi\|_\infty<\infty$ for any $t\in\mathbb{R}$. Moreover, since $\Phi(\cdot)\in PAP_0(Y_{p,\mathbf{X}})$,

$$\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\big\|u\mapsto\Phi(t)0\big\|_{Y_{p,\mathbf{X}}}\,\mathrm{d}t\le\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\big\|\Phi(t)\big\|_{Y_{p,\mathbf{X}}}\,\mathrm{d}t=0.$$

Thus, $t\mapsto[u\mapsto\Phi(t)0]\in PAP_0(Y_{p,\mathbf{X}})$.

So far we have shown $t\mapsto[u\mapsto A(t)0]\in PAA(Y_{p,\mathbf{X}})\cap C_u(\mathbb{R};Y_{p,\mathbf{X}})$.

Noting that $\|u\mapsto A0\|_{Y_{p,\mathbf{X}}}=\|A0\|_*$ for each $A\in Y_{p,\mathbf{X}}$, one can show $t\mapsto A(t)0\in PAA(\mathbf{X}^*)\cap C_u(\mathbb{R};\mathbf{X}^*)$ as in the case of the function $t\mapsto[u\mapsto A(t)0]$ above with only some adaption of notations. □

From the proof of [11], Lemma 4.3, we get the following lemma.

Lemma 4.7 *Under conditions* (H1)-(H5), *suppose further that* $A(\cdot):\mathbb{R}\to Y_{p,\mathbf{X}}$ *is bounded and uniformly continuous and* $f\in BS^{p'}(\mathbf{X}^*)$ *satisfies*

$$\lim_{\delta\to0}\int_t^{t+\delta}\big\|f(s)\big\|_*^{p'}\,\mathrm{d}s=0\tag{15}$$

uniformly with respect to $t\in\mathbb{R}$. *Then the solution* u *of* (3) *in Lemma* 4.1 *is uniformly continuous with values in* $\mathbf{H}$.

Corollary 4.8 *Under the conditions of Theorem* 4.3, *suppose further that* f *satisfies* (15) *uniformly with respect to* $t \in \mathbb{R}$. *Then the solution* u *of* (3) *in Theorem* 4.3 *is in* $PAA(\mathbf{H}) \cap C_u(\mathbb{R}; \mathbf{H}) \cap BS^p(\mathbf{X})$.

Proof From Theorem 4.3, we know that $u \in PAA^q(\mathbf{H})$. Lemma 4.7 implies that $u \in C_u(\mathbb{R}; \mathbf{H})$. Then it follows from Proposition 3.3(iii) that $u \in PAA(\mathbf{H})$. □

Remark 4.9 A sufficient condition for (15) is that the ergodic perturbation ϕ of f is essentially bounded. By the Minkowski inequality, we have

$$\left(\int_t^{t+\delta} \|f(s)\|_*^{p'} \, ds\right)^{\frac{1}{p'}} \le \left(\int_t^{t+\delta} \|g(s)\|_*^{p'} \, ds\right)^{\frac{1}{p'}} + \left(\int_t^{t+\delta} \|\phi(s)\|_*^{p'} \, ds\right)^{\frac{1}{p'}}. \tag{16}$$

Due to Stepanov-like almost automorphy of g, the range of its Bochner transform is precompact in the space $L^{p'}(0,1; \mathbf{X}^*)$. Hence, the first term on the right hand side of inequality (16) converges to 0 uniformly on $\mathbb{R}$ as $\delta \to 0$. Since ϕ is essentially bounded, the second term on the right hand side of inequality (16) also converges to 0 uniformly on $\mathbb{R}$ as $\delta \to 0$. Thus, (15) holds uniformly on $\mathbb{R}$.

Because $t \mapsto A(t)0$ is in $PAA(\mathbf{X}^*)$ (and thus bounded, so is its ergodic perturbation), Corollary 4.8 still hold without condition (H5) that $A(t)0 = 0$.

5 An example

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, and denote $\mathbf{X} = W_0^{1,p}(\Omega)$, $\mathbf{H} = L^2(\Omega)$, $\mathbf{X}^* = W^{-1,p'}(\Omega)$. We consider the operator

$$A(t)u = -\sum_{i=1}^{n} a_i(t) \frac{\partial}{\partial x_i} a\left(\frac{\partial u}{\partial x_i}\right),$$

where $a_i \in PAA(\mathbb{R}) \cap C_u(\mathbb{R}; \mathbb{R})$ has a strictly positive infimum, and $a : \mathbb{R} \to \mathbb{R}$ is a continuous monotone increasing function such that

$$c_1 \cdot \left(|\xi|^p - 1\right) \le a(\xi) \cdot \xi \le c_2 \cdot \left(|\xi|^p + 1\right), \tag{17}$$

where c_1 and c_2 are positive constants. It is not difficult to see that $A(t) : \mathbf{X} \to \mathbf{X}^*$ is a monotone continuous operator. Moreover, (17) implies (H3) and $u \mapsto \frac{\partial}{\partial x_i} a(\frac{\partial u}{\partial x_i}) \in Y_{p,\mathbf{X}}$. Noting that $Y_{p,\mathbf{X}}$ is a Banach space, one can obtain $A(\cdot) \in PAA(Y_{p,\mathbf{X}})$ and is uniformly continuous.

If $\frac{1}{p} - \frac{1}{n} < \frac{1}{2}$, then $\mathbf{X} \subset \mathbf{H}$, and this embedding is compact [31]. We assume additionally that the function a satisfies the inequality

$$\left[a(\xi) - a(\eta)\right] \cdot (\xi - \eta) \ge \alpha |\xi - \eta|^q, \quad \alpha > 0, p \ge q \ge 2. \tag{18}$$

Then there exists a constant $\gamma > 0$ such that

$$\left(A(t)v - A(t)w, v - w\right) \ge \gamma \|v - w\|_{W_0^{1,q}}^q \ge \gamma \|v - w\|_{L^q(\Omega)}^q \ge \gamma \|v - w\|_{\mathbf{H}}^q$$

for all $t \in \mathbb{R}$, v and w in $\mathbf{X}$. Thus (H4) holds for $A(\cdot)$.

For such a function $a(\cdot)$ described above, we have $a(\xi) = |\xi|^{p-2}\xi$ as an example with $p = q$ in (18). Applying Theorem 4.3 and Corollary 4.8, we obtain

Corollary 5.1 *Suppose $f \in PAA^p(\mathbf{X}^*)$ and all the conditions above for $A(\cdot)$ hold. Then the equation*

$$u_t + A(t)u = f(t,x) \tag{19}$$

has a unique solution u such that $u \in PAA^q(\mathbf{H}) \cap BC(\mathbb{R};\mathbf{H}) \cap BS^p(\mathbf{X})$ and its almost automorphic component in $AA_u(\mathbf{H}) \cap BS^p(\mathbf{X})$ is the solution of the equation

$$v_t - \sum_{i=1}^n \tilde{a}_i(t) \frac{\partial}{\partial x_i} a\left(\frac{\partial v}{\partial x_i}\right) = g(t,x),$$

where $\tilde{a}_i$ and g are the almost automorphic components of a_i and f, respectively. Moreover, if f satisfies

$$\lim_{\delta \to 0} \int_t^{t+\delta} \|f(s,\cdot)\|_{\mathbf{X}^*}^{p'} \, \mathrm{d}s = 0$$

uniformly with respect to $t \in \mathbb{R}$, then $u \in PAA(\mathbf{H}) \cap C_u(\mathbb{R};\mathbf{H}) \cap BS^p(\mathbf{X})$.

Remark 5.2 In the case $a_i(t) \equiv 1$ and $a(\xi) = |\xi|^{p-2}\xi$,

$$-A(t)u = \sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u}{\partial x_i}\right)$$

is typical in nonlinear elliptic operators.

Moreover, if $p = 2$, then (19) is a standard heat equation.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors have read and approved the final manuscript.

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