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Existence results for fractional-order differential equations with nonlocal multi-point-strip conditions involving Caputo derivative

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Abstract

In this paper, we investigate the existence and uniqueness of solutions for a differential equation of fractional-order $q \in (1,2]$ subject to nonlocal boundary conditions involving Caputo derivative of the form

$$x(0) = \delta x(\sigma), \qquad a{}^cD^{\mu}x(\varrho_1) + b{}^cD^{\mu}x(\varrho_2) = c\int_{\beta_1}^{\beta_2} {}^cD^{\mu}x(s)\,ds,$$

$0 < \varrho_1 < \sigma < \beta_1 < \beta_2 < \varrho_2 < 1$, $0 < \mu < 1$, and δ, a, b, c are real constants. We make use of some standard tools of fixed point theory to obtain the desired results which are well illustrated with the aid of examples.

MSC: 34A08; 34B15

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1 Introduction

The study of fractional-order differential equations supplemented with a variety of initial and boundary conditions, such as classical, nonlocal, multi-point, periodic/anti-periodic, and integral boundary conditions, has attracted significant attention in recent years. In consequence, the literature on the topic is now much enriched and covers theoretical aspects as well as analytic/numerical methods for solving fractional-order initial and boundary value problems. The widespread applications of fractional calculus modeling techniques in several disciplines of applied and technical sciences have played a key role in the popularity of the subject. Examples include viscoelasticity, control theory, biological sciences, ecology, aerodynamics, electro-dynamics of complex medium, environmental issues, *etc.* An important and useful feature characterizing fractional-order differential and integral operators (in contrast to integer-order operators) is their nonlocal nature that accounts for the past and hereditary behavior of materials and processes involved in the real world problems. For examples and details, we refer the reader to the works [1–5].

Nonlocal conditions, introduced by Bitsadze and Samarskii [6], are regarded as more plausible than the classical initial/boundary conditions in view of their ability to describe certain peculiarities of chemical, physical or other processes happening inside the domain. Computational fluid dynamics (CFD) studies of blood flow indicate that it is not always possible to assume circular cross-section of blood arteries. Several approaches have been proposed to resolve this issue. However, the idea of introducing integral boundary conditions [7] is found to be quite a productive one. Also, integral boundary conditions are applied to regularize ill-posed parabolic backward problems in time partial differential equations, see, for example, mathematical models for bacterial self-regularization [8]. Some recent results on fractional-order boundary value problems involving nonlocal and integral boundary conditions can be found in [9–20] and the references cited therein.

In this paper, motivated by the utility of nonlocal integral boundary conditions in several diverse disciplines, we propose a new class of Caputo type nonlocal boundary value problems supplemented with integral boundary conditions. In precise terms, we consider the following problem:

$$ {}^cD^q x(t) = f(t,x), \quad 1 < q \leq 2, t \in [0,1], \tag{1.1}$$

$$ x(0) = \delta x(\sigma), \qquad a{}^cD^\mu x(\varrho_1) + b{}^cD^\mu x(\varrho_2) = c\int_{\beta_1}^{\beta_2} {}^cD^\mu x(s)\,ds, \tag{1.2}$$

where $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, $0 < \varrho_1 < \sigma < \beta_1 < \beta_2 < \varrho_2 < 1$, $0 < \mu < 1$, and δ, a, b, c are real constants. The integral boundary condition in (1.2) can be interpreted as the linear combination of the values of Caputo derivative of the unknown function of order $\mu \in (0,1)$ at nonlocal positions ϱ_1 and ϱ_2 (off the strip) is proportional to the strip contribution of the Caputo derivative of the unknown function, occupying the position (β_1, β_2).

The content of the paper is organized as follows. Section 2 is devoted to some basic concepts and a lemma concerning the unique solution of a linear variant of problem (1.1)-(1.2). In Section 3, we present our main results which are obtained via Krasnoselskii's fixed point theorem, Schauder type fixed point theorem, nonlinear alternative for single-valued maps and Banach's theorem. It is imperative to note that the exposition of indicated tools of fixed point theory is new in the context of problem (1.1)-(1.2). Finally, we discuss some examples for illustration of the main results.

2 Preliminaries

First of all, we recall some basic definitions.

Definition 2.1 [3] For at least n-times absolutely continuously differentiable function $h : [0,\infty) \to \mathbb{R}$, the Caputo derivative of fractional order q is defined as

$$ {}^cD^q h(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} h^{(n)}(s)\,ds, \quad n-1 < r < n, n = [q]+1, $$

where $[q]$ denotes the integer part of the real number q.

Definition 2.2 [3] The Riemann-Liouville fractional integral of order q for a continuous function h is defined as

$$I^q h(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{h(s)}{(t-s)^{1-q}}\, ds, \quad q > 0,$$

provided the integral exists.

Next we present an auxiliary lemma to define the solution for problem (1.1)-(1.2).

Lemma 2.3 *Let $y \in L[0,1]$ and $x \in AC^2[0,1]$. Then the solution of the linear fractional differential equation*

$$^cD^q x(t) = y(t), \quad 1 < q \leq 2, t \in [0,1], \tag{2.1}$$

supplemented with boundary conditions (1.2) *is equivalent to the integral equation*

$$\begin{aligned} x(t) = {} & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} y(s)\, ds + \frac{\delta}{1-\delta} \int_0^\sigma \frac{(\sigma-s)^{q-1}}{\Gamma(q)} y(s)\, ds \\ & + \left(\frac{\delta\sigma}{\mathcal{A}(1-\delta)} + \frac{t}{\mathcal{A}} \right) \left(-a \int_0^{\varrho_1} \frac{(\varrho_1 - s)^{q-\mu-1}}{\Gamma(q-\mu)} y(s)\, ds \right. \\ & \left. - b \int_0^{\varrho_2} \frac{(\varrho_2 - s)^{q-\mu-1}}{\Gamma(q-\mu)} y(s)\, ds + c \int_{\beta_1}^{\beta_2} \int_0^s \frac{(s-u)^{q-\mu-1}}{\Gamma(q-\mu)} y(u)\, du\, ds \right), \end{aligned} \tag{2.2}$$

where

$$\mathcal{A} = a \frac{\varrho_1^{1-\mu}}{\Gamma(2-\mu)} + b \frac{\varrho_2^{1-\mu}}{\Gamma(2-\mu)} - c \frac{({\beta_2}^{2-\mu} - \beta_1^{2-\mu})}{\Gamma(3-\mu)} \neq 0. \tag{2.3}$$

Proof It is well known that the general solution of the fractional differential equation (2.1) can be written as

$$x(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} y(s)\, ds + c_0 + c_1 t, \tag{2.4}$$

where $c_0, c_1 \in \mathbb{R}$ are arbitrary constants. Applying the boundary conditions (1.2) and using (2.3), we find that

$$\begin{aligned} c_0 = {} & \frac{\delta}{1-\delta} \int_0^\sigma \frac{(\sigma-s)^{q-1}}{\Gamma(q)} y(s)\, ds + \frac{\delta\sigma}{\mathcal{A}(1-\delta)} \left(-a \int_0^{\varrho_1} \frac{(\varrho_1 - s)^{q-\mu-1}}{\Gamma(q-\mu)} y(s)\, ds \right. \\ & \left. - b \int_0^{\varrho_2} \frac{(\varrho_2 - s)^{q-\mu-1}}{\Gamma(q-\mu)} y(s)\, ds + c \int_{\beta_1}^{\beta_2} \int_0^s \frac{(s-u)^{q-\mu-1}}{\Gamma(q-\mu)} y(u)\, du\, ds \right) \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} c_1 = {} & \frac{1}{\mathcal{A}} \left(-a \int_0^{\varrho_1} \frac{(\varrho_1 - s)^{q-\mu-1}}{\Gamma(q-\mu)} y(s)\, ds \right. \\ & \left. - b \int_0^{\varrho_2} \frac{(\varrho_2 - s)^{q-\mu-1}}{\Gamma(q-\mu)} y(s)\, ds + c \int_{\beta_1}^{\beta_2} \int_0^s \frac{(s-u)^{q-\mu-1}}{\Gamma(q-\mu)} y(u)\, du\, ds \right). \end{aligned} \tag{2.6}$$

Substituting the values of c_0, c_1 in (2.4), we get (2.2). This completes the proof. □

3 Main results

In relation to problem (1.1)-(1.2), we define an operator $\mathcal{S}:\mathcal{C}\longrightarrow\mathcal{C}$ via Lemma 2.3 as follows:

$$\begin{aligned}(\mathcal{S}x)(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f\big(s,x(s)\big)\,ds + \frac{\delta}{1-\delta}\int_0^\sigma \frac{(\sigma-s)^{q-1}}{\Gamma(q)} f\big(s,x(s)\big)\,ds \\ &\quad + \left(\frac{\delta\sigma}{\mathcal{A}(1-\delta)} + \frac{t}{\mathcal{A}}\right)\left(-a\int_0^{\varrho_1} \frac{(\varrho_1-s)^{q-\mu-1}}{\Gamma(q-\mu)} f\big(s,x(s)\big)\,ds \right.\\ &\quad - b\int_0^{\varrho_2} \frac{(\varrho_2-s)^{q-\mu-1}}{\Gamma(q-\mu)} f\big(s,x(s)\big)\,ds \\ &\quad \left. + c\int_{\beta_1}^{\beta_2}\int_0^s \frac{(s-u)^{q-\mu-1}}{\Gamma(q-\mu)} f\big(u,x(u)\big)\,du\,ds\right), \end{aligned} \tag{3.1}$$

where $\mathcal{C}=C([0,1],\mathbb{R})$ denote the Banach space of all continuous functions from [0,1] to $\mathbb{R}$ endowed with the norm: $\|x\| = \sup\{|x(t)|, t\in[0,1]\}$ and $\mathcal{A}$ is given by (2.3). Observe that problem (1.1)-(1.2) has solutions if and only if the operator $\mathcal{S}$ has fixed points. For computational convenience, we set

$$\begin{aligned}\vartheta_1 &= \frac{1}{\Gamma(q+1)} + \frac{|\delta|\sigma^q}{|1-\delta|\Gamma(q+1)} + \left(\frac{|\delta|\sigma}{|\mathcal{A}(1-\delta)|} + \frac{1}{|\mathcal{A}|}\right) \\ &\quad \times\left(|a|\frac{\varrho_1^{q-\mu}}{\Gamma(q-\mu+1)} + |b|\frac{\varrho_2^{q-\mu}}{\Gamma(q-\mu+1)} + |c|\frac{({\beta_2}^{q-\mu+1}-{\beta_1}^{q-\mu+1})}{\Gamma(q-\mu+2)}\right), \end{aligned} \tag{3.2}$$

$$\vartheta_2 = \vartheta_1 - \frac{1}{\Gamma(q+1)}. \tag{3.3}$$

Now we state the known fixed point results which we need in the forthcoming analysis.

Lemma 3.1 (Krasnoselskii [21]) *Let $\mathcal{Q}$ be a closed, convex, bounded and nonempty subset of a Banach space Y. Let ϕ_1, ϕ_2 be operators such that* (a) $\phi_1\nu_1+\phi_2\nu_2\in\mathcal{Q}$ *whenever* $\nu_1,\nu_2\in\mathcal{Q}$; (b) ϕ_1 *is compact and continuous*; *and* (c) ϕ_2 *is a contraction mapping. Then there exists $\nu\in\mathcal{Q}$ such that $\nu=\phi_1\nu+\phi_2\nu$.*

Lemma 3.2 [21] *Let X be a Banach space. Assume that $T: X\longrightarrow X$ is a completely continuous operator and the set $V=\{u\in X \mid u=\epsilon Tu, 0<\epsilon<1\}$ is bounded. Then T has a fixed point in X.*

Lemma 3.3 (Nonlinear alternative for single-valued maps [22]) *Let E be a Banach space, E_1 be a closed, convex subset of E, V be an open subset of E_1, and $0\in V$. Suppose that $\mathcal{U}:\overline{V}\longrightarrow E_1$ is a continuous, compact* (*that is, $\mathcal{U}(\overline{V})$ is a relatively compact subset of E_1*) *map. Then either*

(i) *$\mathcal{U}$ has a fixed point in $\overline{V}$, or*

(ii) *there is $x\in\partial V$* (*the boundary of V in E_1*) *and $\kappa\in(0,1)$ with $x=\kappa\,\mathcal{U}(x)$.*

Our first existence result is based on Lemma 3.1.

Theorem 3.4 *Let $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ be a continuous function satisfying the following conditions*:

($\mathcal{H}_1$) $|f(t,x)-f(t,y)| \leq \ell|x-y|$, $\ell > 0$, $\forall t \in [0,1]$, $x,y \in \mathbb{R}$;
($\mathcal{H}_2$) $|f(t,x)| \leq \omega(t)$, $\forall(t,x) \in [0,1] \times \mathbb{R}$, $\omega \in C([0,1],\mathbb{R}^+)$.

Then problem (1.1)-(1.2) *has at least one solution on* $[0,1]$ *if* $\ell\vartheta_2 < 1$, *where* ϑ_2 *is given by* (3.3).

Proof Let us introduce a set $B_\rho = \{x \in \mathcal{C} : \|x\| \leq \rho\}$ with $\rho \geq \|\omega\|\vartheta_1$, where $\|\omega\| = \sup\{|\omega(t)|, t \in [0,1]\}$ and ϑ_1 is given by (3.2). Define the operators $\mathcal{S}_1$ and $\mathcal{S}_2$ on B_ρ as

$$\begin{aligned}
(\mathcal{S}_1x)(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f\bigl(s,x(s)\bigr)\,ds,\\
(\mathcal{S}_2x)(t) &= \frac{\delta}{1-\delta}\int_0^\sigma \frac{(\sigma-s)^{q-1}}{\Gamma(q)} f\bigl(s,x(s)\bigr)\,ds\\
&\quad + \left(\frac{\delta\sigma}{\mathcal{A}(1-\delta)} + \frac{t}{\mathcal{A}}\right)\left(-a\int_0^{\varrho_1} \frac{(\varrho_1-s)^{q-\mu-1}}{\Gamma(q-\mu)} f\bigl(s,x(s)\bigr)\,ds\right.\\
&\quad \left. - b\int_0^{\varrho_2} \frac{(\varrho_2-s)^{q-\mu-1}}{\Gamma(q-\mu)} f\bigl(s,x(s)\bigr)\,ds + c\int_{\beta_1}^{\beta_2}\int_0^s \frac{(s-u)^{q-\mu-1}}{\Gamma(q-\mu)} f\bigl(u,x(u)\bigr)\,du\,ds\right).
\end{aligned}$$

For $x,y \in B_\rho$, it is easy to show that $\|(\mathcal{S}_1x)+(\mathcal{S}_2y)\| \leq \|\omega\|\vartheta_1 \leq \rho$, which implies that $\mathcal{S}_1x + \mathcal{S}_2y \in B_\rho$. Applying the condition ($\mathcal{H}_1$), we find that

$$\begin{aligned}
&\bigl\|(\mathcal{S}_2x)-(\mathcal{S}_2y)\bigr\|\\
&\quad \leq \sup_{t\in[0,1]}\left\{\frac{|\delta|}{|1-\delta|}\int_0^\sigma \frac{(\sigma-s)^{q-1}}{\Gamma(q)}\bigl|f\bigl(s,x(s)\bigr)-f\bigl(s,y(s)\bigr)\bigr|\,ds\right.\\
&\qquad + \left(\frac{|\delta|\sigma}{|\mathcal{A}(1-\delta)|} + \frac{t}{|\mathcal{A}|}\right)\left(|a|\int_0^{\varrho_1} \frac{(\varrho_1-s)^{q-\mu-1}}{\Gamma(q-\mu)}\bigl|f\bigl(s,x(s)\bigr)-f\bigl(s,y(s)\bigr)\bigr|\,ds\right.\\
&\qquad + |b|\int_0^{\varrho_2} \frac{(\varrho_2-s)^{q-\mu-1}}{\Gamma(q-\mu)}\bigl|f\bigl(s,x(s)\bigr)-f\bigl(s,y(s)\bigr)\bigr|\,ds\\
&\qquad \left.\left. + |c|\int_{\beta_1}^{\beta_2}\int_0^s \frac{(s-u)^{q-\mu-1}}{\Gamma(q-\mu)}\bigl|f\bigl(u,x(u)\bigr)-f\bigl(u,y(u)\bigr)\bigr|\,du\,ds\right)\right\}\\
&\quad \leq \ell\sup_{t\in[0,1]}\left\{\frac{|\delta|\sigma^q}{|1-\delta|\Gamma(q+1)} + \left(\frac{|\delta|\sigma}{|\mathcal{A}(1-\delta)|} + \frac{t}{|\mathcal{A}|}\right)\left(|a|\frac{\varrho_1^{q-\mu}}{\Gamma(q-\mu+1)}\right.\right.\\
&\qquad \left.\left. + |b|\frac{\varrho_2^{q-\mu}}{\Gamma(q-\mu+1)} + |c|\frac{(\beta_2^{q-\mu+1}-\beta_1^{q-\mu+1})}{\Gamma(q-\mu+2)}\right)\right\}\|x-y\|\\
&\quad \leq \ell\vartheta_2\|x-y\|,
\end{aligned}$$

which, in view of the condition $\ell\vartheta_2 < 1$, implies that the operator $\mathcal{S}_2$ is a contraction. Further $\mathcal{S}_1$ is continuous in view of the continuity of f. Also, $\mathcal{S}_1$ is uniformly bounded on B_ρ as

$$\begin{aligned}
\|\mathcal{S}_1x\| &\leq \sup_{t\in[0,1]}\left\{\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\bigl|f\bigl(s,x(s)\bigr)\bigr|\,ds\right\} \leq \sup_{t\in[0,1]}\left\{\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\omega(s)\,ds\right\}\\
&\leq \|\omega\|\sup_{t\in[0,1]}\left\{\frac{t^q}{\Gamma(q+1)}\right\} \leq \frac{\|\omega\|}{\Gamma(q+1)}.
\end{aligned}$$

With $\sup_{(t,x)\in[0,1]\times B_\rho} |f(t,x)| = f_m < \infty$ and $0 < t_1 < t_2 < T$, we have

$$\begin{aligned}
&\big|(\mathcal{S}_1x)(t_2) - (\mathcal{S}_1x)(t_1)\big| \\
&= \left|\int_0^{t_2} \frac{(t_2 - s)^{q-1}}{\Gamma(q)} f\big(s,x(s)\big)\,ds - \int_0^{t_1} \frac{(t_1 - s)^{q-1}}{\Gamma(q)} f\big(s,x(s)\big)\,ds\right| \\
&\le \int_0^{t_1} \frac{[(t_2 - s)^{q-1} - (t_1 - s)^{q-1}]}{\Gamma(q)} \big|f\big(s,x(s)\big)\big|\,ds \\
&\quad + \int_{t_1}^{t_2} \frac{(t_2 - s)^{q-1}}{\Gamma(q)} \big|f\big(s,x(s)\big)\big|\,ds \le \frac{f_m}{\Gamma(q+1)}\big[2|t_2 - t_1|^q + \big|t_2^q - t_1^q\big|\big],
\end{aligned}$$

which tends to zero independent of x as $t_2 - t_1 \longrightarrow 0$. This shows that $\mathcal{S}_1$ is relatively compact on B_ρ. Hence, by the Arzelá-Ascoli theorem, $\mathcal{S}_1$ is compact on B_ρ. Thus all the assumptions of Lemma 3.1 are satisfied. So problem (1.1)-(1.2) has at least one solution on $[0,1]$. This completes the proof. □

In the next result, we make use of Lemma 3.2.

Theorem 3.5 *Assume that there exists a positive constant L_1 such that $|f(t,x)| \le L_1$ for all $t \in [0,1]$, $x \in \mathcal{C}$. Then there exists at least one solution for problem* (1.1)-(1.2).

Proof In the first step, it will be shown that the operator $\mathcal{S}$ is completely continuous. Clearly the continuity of $\mathcal{S}$ follows from the continuity of f. Let $\mathcal{V} \subset C([0,1],\mathbb{R}^+)$ be bounded. Then $\forall x \in \mathcal{V}$, it is easy to establish that $|(\mathcal{S}x)(t)| \le L_1\vartheta_1 = L_2$. Furthermore, we find that

$$\begin{aligned}
\big|(\mathcal{S}x)'(t)\big| &= \bigg|\int_0^t \frac{(t-s)^{q-2}}{\Gamma(q-1)} f\big(s,x(s)\big)\,ds \\
&\quad + \frac{1}{\mathcal{A}}\bigg(-a\int_0^{\varrho_1} \frac{(\varrho_1 - s)^{q-\mu-1}}{\Gamma(q-\mu)} f\big(s,x(s)\big)\,ds \\
&\quad - b\int_0^{\varrho_2} \frac{(\varrho_2 - s)^{q-\mu-1}}{\Gamma(q-\mu)} f\big(s,x(s)\big)\,ds \\
&\quad + c\int_{\beta_1}^{\beta_2}\int_0^s \frac{(s-u)^{q-\mu-1}}{\Gamma(q-\mu)} f\big(u,x(u)\big)\,du\,ds\bigg)\bigg| \\
&\le L_1\bigg(\frac{1}{\Gamma(q)} + \frac{1}{|\mathcal{A}|}\bigg[|a|\frac{\varrho_1^{q-\mu}}{\Gamma(q-\mu+1)} \\
&\quad + |b|\frac{\varrho_2^{q-\mu}}{\Gamma(q-\mu+1)} + |c|\frac{(\beta_2^{q-\mu+1} - \beta_1^{q-\mu+1})}{\Gamma(q-\mu+2)}\bigg]\bigg) = L_3.
\end{aligned}$$

Hence, for $t_1, t_2 \in [0,1]$, it follows that

$$\big|(\mathcal{S}x)(t_1) - (\mathcal{S}x)(t_2)\big| \le \int_{t_1}^{t_2} \big|(\mathcal{S}x)'(s)\big|\,ds \le L_3(t_2 - t_1).$$

Therefore, $\mathcal{S}$ is equicontinuous on $[0,1]$. Thus, by the Arzelá-Ascoli theorem, the operator $\mathcal{S}$ is completely continuous.

Next, we consider the set $\mathcal{U} = \{x \in \mathcal{C} : \xi \mathcal{S}x, 0 < \xi < 1\}$ and show that $\mathcal{U}$ is bounded. Then, for $t \in [0,1]$, we have

$$\begin{aligned} x(t) = {} & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f\big(s,x(s)\big)\,ds + \frac{\delta}{1-\delta}\int_0^\sigma \frac{(\sigma-s)^{q-1}}{\Gamma(q)} f\big(s,x(s)\big)\,ds \\ & + \left(\frac{\delta\sigma}{\mathcal{A}(1-\delta)} + \frac{t}{\mathcal{A}}\right)\left(-a\int_0^{\varrho_1} \frac{(\varrho_1-s)^{q-\mu-1}}{\Gamma(q-\mu)} f\big(s,x(s)\big)\,ds \right. \\ & \left. - b\int_0^{\varrho_2} \frac{(\varrho_2-s)^{q-\mu-1}}{\Gamma(q-\mu)} f\big(s,x(s)\big)\,ds + c\int_{\beta_1}^{\beta_2}\int_0^s \frac{(s-u)^{q-\mu-1}}{\Gamma(q-\mu)} f\big(u,x(u)\big)\,du\,ds\right) \end{aligned}$$

and $|x(t)| = \xi|(\mathcal{S}x)(t)| \le L_1\vartheta_1 = L_2$. In consequence, we get $\|x\| \le L_2$, $\forall x \in \mathcal{U}$, $t \in [0,1]$. So $\mathcal{U}$ is bounded. Thus, by the application of Lemma 3.2, problem (1.1)-(1.2) has at least one solution. This completes the proof. □

The following theorem deals with the uniqueness of solutions for problem (1.1)-(1.2).

Theorem 3.6 *Let $f : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function satisfying the condition* $(\mathcal{H}_1)$ *and that $\ell\vartheta_1 < 1$, where ϑ_1 is given by* (3.2). *Then there exists a unique solution for problem* (1.1)-(1.2) *on* $[0,1]$.

Proof In the first step, we consider the operator $\mathcal{S} : \mathcal{C} \longrightarrow \mathcal{C}$ defined by (3.1) and show that $\mathcal{S}\mathcal{E}_r \subset \mathcal{E}_r$, where $\mathcal{E}_r = \{x \in \mathcal{C} : \|x\| \le r\}$ with $\sup_{t\in[0,1]} |f(t,0)| = \alpha$ and $r > \frac{\vartheta_1\alpha}{1-\vartheta_1\ell}$ for $x \in \mathcal{E}_r$, $t \in [0,1]$. Using the fact that $|f(s,x(s))| = |f(s,x(s)) - f(s,0) + f(s,0)| \le \ell r + \alpha$, we get

$$\begin{aligned} \|(\mathcal{S}x)\| \le {} & \sup_{t\in[0,1]}\left\{\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\big|f\big(s,x(s)\big)\big|\,ds + \frac{|\delta|}{|1-\delta|}\int_0^\sigma \frac{(\sigma-s)^{q-1}}{\Gamma(q)}\big|f\big(s,x(s)\big)\big|\,ds \right. \\ & + \left(\frac{|\delta|\sigma}{|\mathcal{A}(1-\delta)|} + \frac{t}{|\mathcal{A}|}\right)\left(|a|\int_0^{\varrho_1} \frac{(\varrho_1-s)^{q-\mu-1}}{\Gamma(q-\mu)}\big|f\big(s,x(s)\big)\big|\,ds \right. \\ & + |b|\int_0^{\varrho_2} \frac{(\varrho_2-s)^{q-\mu-1}}{\Gamma(q-\mu)}\big|f\big(s,x(s)\big)\big|\,ds \\ & \left.\left. + |c|\int_{\beta_1}^{\beta_2}\int_0^s \frac{(s-u)^{q-\mu-1}}{\Gamma(q-\mu)}\big|f\big(u,x(u)\big)\big|\,du\,ds\right)\right\} \\ \le {} & (\ell r + \alpha)\vartheta_1 \le r. \end{aligned}$$

This shows that $\mathcal{S}\mathcal{E}_r \subset \mathcal{E}_r$, where we have used (3.2).

Now, for $x, y \in \mathcal{C}$ and for each $t \in [0,1]$, we obtain

$$\begin{aligned} \|\mathcal{S}x - \mathcal{S}y\| \le {} & \sup_{t\in[0,1]}\left\{\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\big|f\big(s,x(s)\big) - f\big(s,y(s)\big)\big|\,ds \right. \\ & + \frac{|\delta|}{|1-\delta|}\int_0^\sigma \frac{(\sigma-s)^{q-1}}{\Gamma(q)}\big|f\big(s,x(s)\big) - f\big(s,y(s)\big)\big|\,ds \\ & + \left(\frac{|\delta|\sigma}{|\mathcal{A}(1-\delta)|} + \frac{t}{|\mathcal{A}|}\right)\left(|a|\int_0^{\varrho_1} \frac{(\varrho_1-s)^{q-\mu-1}}{\Gamma(q-\mu)}\big|f\big(s,x(s)\big) - f\big(s,y(s)\big)\big|\,ds \right. \\ & + |b|\int_0^{\varrho_2} \frac{(\varrho_2-s)^{q-\mu-1}}{\Gamma(q-\mu)}\big|f\big(s,x(s)\big) - f\big(s,y(s)\big)\big|\,ds \end{aligned}$$

$$
\begin{aligned}
&+ |c| \int_{\beta_1}^{\beta_2} \int_0^s \frac{(s-u)^{q-\mu-1}}{\Gamma(q-\mu)} \bigl|f\bigl(u,x(u)\bigr) - f\bigl(u,y(u)\bigr)\bigr| \, du \, ds \biggr) \biggr\} \\
&\leq \ell \vartheta_1 \|x-y\|,
\end{aligned}
$$

which, by the assumption $\ell\vartheta_1 < 1$, implies that the operator $\mathcal{S}$ is a contraction. Thus, Banach's contraction mapping principle applies and there exists a unique solution for problem (1.1)-(1.2). This completes the proof. □

Our final result relies on Lemma 3.3 (nonlinear alternative for single-valued maps).

Theorem 3.7 *Let* $f : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ *be a continuous function, and assume that*

($\mathcal{H}_3$) *there exist a function* $p \in C([0,1], \mathbb{R}^+)$ *and a nondecreasing function* $\varphi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ *such that* $|f(t,x)| \leq p(t)\varphi(\|x\|)$, $\forall (t,x) \in [0,1] \times \mathbb{R}$;

($\mathcal{H}_4$) *there exists a constant* $M > 0$ *such that*

$$
\begin{aligned}
M\biggl[\varphi(M)\|p\| \biggl\{ &\frac{1}{\Gamma(q+1)} + \frac{|\delta|\sigma^q}{|1-\delta|\Gamma(q+1)} + \biggl(\frac{|\delta|\sigma}{|\mathcal{A}(1-\delta)|} + \frac{1}{|\mathcal{A}|} \biggr) \\
&\times \biggl(|a| \frac{\varrho_1^{q-\mu}}{\Gamma(q-\mu+1)} + |b| \frac{\varrho_2^{q-\mu}}{\Gamma(q-\mu+1)} + |c| \frac{({\beta_2}^{q-\mu+1} - {\beta_1}^{q-\mu+1})}{\Gamma(q-\mu+2)} \biggr) \biggr\} \biggr]^{-1} > 1.
\end{aligned}
$$

Then problem (1.1)-(1.2) *has at least one solution on* $[0,1]$.

Proof We complete the proof in several steps. As a first step, we show that the operator $\mathcal{S} : \mathcal{C} \longrightarrow \mathcal{C}$ defined by (3.1) maps bounded sets into bounded sets in $\mathcal{C}$. For a positive number ν, let $\mathbf{B}_\nu = \{x \in \mathcal{C} : \|x\| \leq \nu\}$ be a bounded set in $\mathcal{C}$. Then, for $x \in \mathbf{B}_\nu$, using ($\mathcal{H}_3$) we obtain

$$
\begin{aligned}
\bigl|(\mathcal{S}x)(t)\bigr| \leq{}& \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} p(s)\varphi\bigl(\|x\|\bigr) \, ds + \frac{|\delta|}{|1-\delta|} \int_0^\sigma \frac{(\sigma-s)^{q-1}}{\Gamma(q)} p(s)\varphi\bigl(\|x\|\bigr) \, ds \\
&+ \biggl(\frac{|\delta|\sigma}{|\mathcal{A}(1-\delta)|} + \frac{t}{|\mathcal{A}|} \biggr) \biggl(|a| \int_0^{\varrho_1} \frac{(\varrho_1-s)^{q-\mu-1}}{\Gamma(q-\mu)} p(s)\varphi\bigl(\|x\|\bigr) \, ds \\
&+ |b| \int_0^{\varrho_2} \frac{(\varrho_2-s)^{q-\mu-1}}{\Gamma(q-\mu)} p(s)\varphi\bigl(\|x\|\bigr) \, ds \\
&+ |c| \int_{\beta_1}^{\beta_2} \int_0^s \frac{(s-u)^{q-\mu-1}}{\Gamma(q-\mu)} p(u)\varphi\bigl(\|x\|\bigr) \, du \, ds \biggr) \\
\leq{}& \varphi(\nu)\|p\| \biggl\{ \frac{1}{\Gamma(q+1)} + \frac{|\delta|\sigma^q}{|1-\delta|\Gamma(q+1)} + \biggl(\frac{|\delta|\sigma}{|\mathcal{A}(1-\delta)|} + \frac{1}{|\mathcal{A}|} \biggr) \\
&\times \biggl(|a| \frac{\varrho_1^{q-\mu}}{\Gamma(q-\mu+1)} + |b| \frac{\varrho_2^{q-\mu}}{\Gamma(q-\mu+1)} + |c| \frac{({\beta_2}^{q-\mu+1} - \beta_1^{q-\mu+1})}{\Gamma(q-\mu+2)} \biggr) \biggr\}.
\end{aligned}
$$

Next, it will be shown that the operator $\mathcal{S}$ maps bounded sets into equicontinuous sets of $\mathcal{C}$. Let $t_1, t_2 \in [0,1]$ with $t_1 < t_2$. Then, for $x \in \mathbf{B}_\nu$, we have

$$
\begin{aligned}
&\bigl|(\mathcal{S}x)(t_2) - (\mathcal{S}x)(t_1)\bigr| \\
&\quad\leq \biggl| \int_0^{t_1} \frac{[(t_2-s)^{q-1} - (t_1-s)^{q-1}]}{\Gamma(q)} f\bigl(s,x(s)\bigr) \, ds + \int_{t_1}^{t_2} \frac{(t_2-s)^{q-1}}{\Gamma(q)} f\bigl(s,x(s)\bigr) \biggr|
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{|t_2-t_1|}{|A|}\biggl(|a|\int_0^{\varrho_1}\frac{(\varrho_1-s)^{q-\mu-1}}{\Gamma(q-\mu)}\bigl|f\bigl(s,x(s)\bigr)\bigr|\,ds \\
&+|b|\int_0^{\varrho_2}\frac{(\varrho_2-s)^{q-\mu-1}}{\Gamma(q-\mu)}\bigl|f\bigl(s,x(s)\bigr)\bigr|\,ds+|c|\int_{\beta_1}^{\beta_2}\int_0^s\frac{(s-u)^{q-\mu-1}}{\Gamma(q-\mu)}\bigl|f\bigl(u,x(u)\bigr)\bigr|\,du\,ds\biggr) \\
&\le\varphi(\nu)\|p\|\biggl[\frac{[2|t_2-t_1|^q+|t_2^q-t_1^q|]}{\Gamma(q+1)}+\frac{|t_2-t_1|}{|\mathcal{A}|}\biggl(|a|\frac{\varrho_1^{q-\mu}}{\Gamma(q-\mu+1)} \\
&+|b|\frac{\varrho_2^{q-\mu}}{\Gamma(q-\mu+1)}+|c|\frac{({\beta_2}^{q-\mu+1}-{\beta_1}^{q-\mu+1})}{\Gamma(q-\mu+2)}\biggr)\biggr].
\end{aligned}
$$

Clearly, the right-hand side tends to zero independent of $x\in\mathbf{B}_\nu$ as $t_2\longrightarrow t_1$. Thus, by the Arzelá theorem, the operator $\mathcal{S}$ is completely continuous.

Let x be a solution of problem (1.1)-(1.2). Then, for $\lambda\in(0,1)$, using the method of computation employed to show the boundedness of the operator $\mathcal{S}$, it can be found that

$$
\begin{aligned}
\bigl|x(t)\bigr|=\bigl|\lambda(\mathcal{S}x)(t)\bigr|\le\varphi\bigl(\|x\|\bigr)\|p\|\biggl[&\frac{1}{\Gamma(q+1)}+\frac{|\delta|\sigma^q}{|1-\delta|\Gamma(q+1)} \\
&+\biggl(\frac{1}{|\mathcal{A}|}+\frac{|\delta|\sigma}{|\mathcal{A}(1-\delta)|}\biggr)\biggl(|c|\frac{({\beta_2}^{q-\mu+1}-{\beta_1}^{q-\mu+1})}{\Gamma(q+1)}+\frac{(|a|\varrho_1^{q-\mu}+|b|\varrho_2^{q-\mu})}{\Gamma(q-\mu+1)}\biggr)\biggr].
\end{aligned}
$$

In consequence, we get

$$
\begin{aligned}
&\|x\|\biggl[\varphi\bigl(\|x\|\bigr)\|p\|\biggl\{\frac{1}{\Gamma(q+1)}+\frac{|\delta|\sigma^q}{|1-\delta|\Gamma(q+1)}+\biggl(\frac{|\delta|\sigma}{|\mathcal{A}(1-\delta)|}+\frac{1}{|\mathcal{A}|}\biggr) \\
&\times\biggl(|a|\frac{\varrho_1^{q-\mu}}{\Gamma(q-\mu+1)}+|b|\frac{\varrho_2^{q-\mu}}{\Gamma(q-\mu+1)}+|c|\frac{({\beta_2}^{q-\mu+1}-{\beta_1}^{q-\mu+1})}{\Gamma(q-\mu+2)}\biggr)\biggr\}\biggr]^{-1}\le 1.
\end{aligned}
$$

In view of $(\mathcal{H}_4)$, there exists M such that $\|x\|\neq M$. Let us select a set $\mathcal{N}=\{x\in\mathcal{C}:\|x\|<M+1\}$. Observe that the operator $\mathcal{S}:\overline{\mathcal{N}}\longrightarrow\mathcal{C}$ is continuous and completely continuous. From the choice of $\mathcal{N}$, there is no $x\in\partial\mathcal{N}$ such that $x=\lambda\mathcal{S}(x)$ for some $\lambda\in(0,1)$. Thus, by Lemma 3.3, we deduce that the operator $\mathcal{S}$ has a fixed point $x\in\overline{\mathcal{N}}$ which is a solution of problem (1.1)-(1.2). This completes the proof. □

Example 3.8 Consider a fractional boundary value problem given by

$$
\begin{cases}
{}^cD^{3/2}x(t)=\frac{t}{9}\sin x+\frac{e^{-t}}{9}\tan^{-1}x+\frac{1}{2}, & t\in[0,1], \\
x(0)=\frac{1}{2}x(1/3), & 2{}^cD^{1/2}x(1/6)+3{}^cD^{1/2}x(3/4)=\int_{3/7}^{1/2}{}^cD^{1/2}x(s)\,ds.
\end{cases}
\tag{3.4}
$$

Here, $\delta=1/2$, $q=3/2$, $a=2$, $b=3$, $c=1$, $\sigma=1/3$, $\varrho_1=1/6$, $\varrho_2=3/4$, $\beta_1=3/7$, $\beta_2=1/2$, $\mu=1/2$ and $f(t,x)=\frac{t}{9}\sin x+\frac{e^{-t}}{9}\tan^{-1}x$. With the given data, $\ell=\frac{2}{9}$,

$$
\begin{aligned}
|\mathcal{A}|&=\biggl|a\frac{\varrho_1^{1-\mu}}{\Gamma(2-\mu)}+b\frac{\varrho_2^{1-\mu}}{\Gamma(2-\mu)}-c\frac{(\beta_2^{2-\mu}-\beta_1^{2-\mu})}{\Gamma(3-\mu)}\biggr|=3.797291, \\
\vartheta_1&=\frac{1}{\Gamma(q+1)}+\frac{|\delta|\sigma^q}{|1-\delta|\Gamma(q+1)}+\biggl(\frac{|\delta|\sigma}{|\mathcal{A}(1-\delta)|}+\frac{1}{|\mathcal{A}|}\biggr) \\
&\quad\times\biggl(|a|\frac{\varrho_1^{q-\mu}}{\Gamma(q-\mu+1)}+|b|\frac{\varrho_2^{q-\mu}}{\Gamma(q-\mu+1)}+|c|\frac{(\xi^{q-\mu+1}-\beta_1^{q-\mu+1})}{\Gamma(q-\mu+2)}\biggr)=1.815607.
\end{aligned}
$$

Clearly $\ell < \frac{1}{\vartheta_1}$. Thus all the conditions of Theorem 3.6 are satisfied and, consequently, there exists a unique solution for problem (3.4) on $[0,1]$.

Example 3.9 Consider problem (3.4) with

$$f(t,x) = (t/3 + 1)(\cos x + \sin x/5). \tag{3.5}$$

Clearly, $|f(t,x)| \leq p(t)\varphi(|x|)$, where $p(t) = \frac{t}{3} + 1$, $\varphi(|x|) = 1 + \frac{|x|}{5}$. By the assumption $(\mathcal{H}_4)$:

$$M\left\{\varphi(M)\|p\|\left\{\frac{1}{\Gamma(q+1)} + \frac{|\delta|\sigma^q}{|1-\delta|\Gamma(q+1)} + \left(\frac{|\delta|\sigma}{|\mathcal{A}(1-\delta)|} + \frac{1}{|\mathcal{A}|}\right)\right.\right.$$
$$\left.\left.\times\left(|a|\frac{\varrho_1^{q-\mu}}{\Gamma(q-\mu+1)} + |b|\frac{\varrho_2^{q-\mu}}{\Gamma(q-\mu+1)} + |c|\frac{(\xi^{q-\mu+1} - \beta_1^{q-\mu+1})}{\Gamma(q-\mu+2)}\right)\right\}\right\}^{-1} > 1,$$

we find that $M > 4.692964$. As the hypothesis of Theorem 3.7 is satisfied, therefore, its conclusion implies that there exists at least one solution for problem (3.4) with $f(t,x)$ given by (3.5).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Each of the authors, BA, AA and AA, contributed to each part of this work equally and read and approved the final version of the manuscript.

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References

1. Podlubny, I: Fractional Differential Equations. Academic Press, San Diego (1999)
2. Zaslavsky, GM: Hamiltonian Chaos and Fractional Dynamics. Oxford University Press, Oxford (2005)
3. Kilbas, AA, Srivastava, HM, Trujillo, JJ: Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Studies, vol. 204. Elsevier, Amsterdam (2006)
4. Sabatier, J, Agrawal, OP, Machado, JAT (eds.): Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering. Springer, Dordrecht (2007)
5. Konjik, S, Oparnica, L, Zorica, D: Waves in viscoelastic media described by a linear fractional model. Integral Transforms Spec. Funct. **22**, 283-291 (2011)
6. Bitsadze, A, Samarskii, A: On some simple generalizations of linear elliptic boundary problems. Dokl. Math. **10**, 398-400 (1969)
7. Ahmad, B, Alsaedi, A, Alghamdi, BS: Analytic approximation of solutions of the forced Duffing equation with integral boundary conditions. Nonlinear Anal., Real World Appl. **9**, 1727-1740 (2008)
8. Čiegis, R, Bugajev, A: Numerical approximation of one model of the bacterial self-organization. Nonlinear Anal., Model. Control **17**, 253-270 (2012)
9. Ahmad, B, Eloe, P: A nonlocal boundary value problem for a nonlinear fractional differential equation with two indices. Commun. Appl. Nonlinear Anal. **17**, 69-80 (2010)
10. Bai, ZB, Sun, W: Existence and multiplicity of positive solutions for singular fractional boundary value problems. Comput. Math. Appl. **63**, 1369-1381 (2012)
11. Ahmad, B, Malar, K, Karthikeyan, K: A study of nonlocal problems of impulsive integrodifferential equations with measure of noncompactness. Adv. Differ. Equ. **2013**, 205 (2013)
12. Zhai, C, Xu, L: Properties of positive solutions to a class of four-point boundary value problem of Caputo fractional differential equations with a parameter. Commun. Nonlinear Sci. Numer. Simul. **19**, 2820-2827 (2014)
13. Punzo, F, Terrone, G: On the Cauchy problem for a general fractional porous medium equation with variable density. Nonlinear Anal. **98**, 27-47 (2014)
14. Zhang, L, Ahmad, B, Wang, G: The existence of an extremal solution to a nonlinear system with the right-handed Riemann-Liouville fractional derivative. Appl. Math. Lett. **31**, 1-6 (2014)
15. Henderson, J, Kosmatov, N: Eigenvalue comparison for fractional boundary value problems with the Caputo derivative. Fract. Calc. Appl. Anal. **17**, 872-880 (2014)
16. Ahmad, B, Agarwal, RP: Some new versions of fractional boundary value problems with slit-strips conditions. Bound. Value Probl. **2014**, 175 (2014)

17. Henderson, J, Luca, R, Tudorache, A: On a system of fractional differential equations with coupled integral boundary conditions. Fract. Calc. Appl. Anal. **18**, 361-386 (2015)
18. Ahmad, B, Ntouyas, SK, Alsaedi, A, Alzahrani, F: New fractional-order multivalued problems with nonlocal nonlinear flux type integral boundary conditions. Bound. Value Probl. **2015**, 83 (2015)
19. Ding, Y, Wei, Z, Xu, J, O'Regan, D: Extremal solutions for nonlinear fractional boundary value problems with *p*-Laplacian. J. Comput. Appl. Math. **288**, 151-158 (2015)
20. Zhang, L, Ahmad, B, Wang, G: Successive iterations for positive extremal solutions of nonlinear fractional differential equations on a half line. Bull. Aust. Math. Soc. **91**, 116-128 (2015)
21. Smart, DR: Fixed Point Theorems. Cambridge University Press, Cambridge (1980)
22. Granas, A, Dugundji, J: Fixed Point Theory. Springer, New York (2003)