Advances in Difference Equations
a SpringerOpen Journal

RESEARCH　　　　　　　　　　　　　　　　　　　　　　　　　**Open Access**

CrossMark

Existence of solutions converging to zero for nonlinear delayed differential systems

Josef Rebenda[1] and Zdeněk Šmarda[1,2*]

*Correspondence:
smarda@feec.vutbr.cz
[1] CEITEC BUT, Brno University of
Technology, Technicka 3058/10,
Brno, 61600, Czech Republic
[2] Department of Mathematics, Brno
University of Technology, Brno,
Czech Republic

Abstract

We present a result about an interesting asymptotic property of real two-dimensional delayed differential systems satisfying certain sufficient conditions. We employ two previous results, which were obtained using a Razumikhin-type modification of the Ważewski topological method for retarded differential equations and the method of a Lyapunov-Krasovskii functional. The result is illustrated by a nontrivial explanatory example.

MSC: 34K12; 34K20

Keywords: differential system with delays; stability of solutions

1 Introduction

Various properties of solutions of differential equations with delay were extensively studied recently. Among others we mention [1–9] and the references therein. The results contained in this paper are a generalization of previous research published in [10–12] and [13].

Our aim here is to study the asymptotic behavior of solutions of the following system of differential equations:

$$x'(t) = A(t)x(t) + \sum_{k=1}^{m} B_k(t)x(\theta_k(t)) + h(t,x(t),x(\theta_1(t)),\ldots,x(\theta_m(t))), \qquad (1)$$

where $t - \theta_k(t) \geq 0$ are bounded nonconstant delays satisfying $\lim_{t\to\infty} \theta_k(t) = \infty$, $\theta_k(t)$ are real functions,

$$h(t,x,y) = (h_1(t,x,y_1,\ldots,y_m), h_2(t,x,y_1,\ldots,y_m))$$

is a real vector function, where $x = (x_1, x_2)$, $y_k = (y_{1k}, y_{2k})$, and

$$A(t) = (a_{ij}(t)), \qquad B_k(t) = (b_{ijk}(t)), \quad i,j = 1,2; k = 1,\ldots,m,$$

are real square matrices.

In this paper, we introduce an interesting result, which is a combination of two theorems presented in [14], one regarding the instability of solutions, the other one dealing with the existence of bounded solutions.

It is supposed that the function h satisfies the Carathéodory conditions on $[t_0, \infty) \times \mathbb{R}^{2(m+1)}$, the functions b_{ijk} are locally Lebesgue integrable on $[t_0, \infty)$, and the functions θ_k, a_{ij} are locally absolutely continuous on $[t_0, \infty)$.

Since we study two-dimensional systems, we use a transformation into complex variables to simplify the system (1) into one equation with complex coefficients.

The complex variables are defined as $z = x_1 + ix_2$, $w_1 = y_{11} + iy_{12}, \ldots, w_m = y_{m1} + iy_{m2}$. Using this transformation we get

$$
z'(t) = a(t)z(t) + b(t)\bar{z}(t) + \sum_{k=1}^{m} \left[A_k(t)z\big(\theta_k(t)\big) + B_k(t)\bar{z}\big(\theta_k(t)\big) \right]
$$
$$
+ g\big(t, z(t), z(\theta_1(t)), \ldots, z(\theta_m(t))\big), \tag{2}
$$

where we assume ($J = [t_0, \infty)$):

- $A_k, B_k \in L_{\mathrm{loc}}(J, \mathbb{C})$ (*i.e.* locally Lebesgue integrable complex-valued functions on J) for $k = 1, \ldots, m$,
- $\theta_k \in AC_{\mathrm{loc}}(J, \mathbb{R})$ (*i.e.* locally absolutely continuous real-valued functions on J) for $k = 1, \ldots, m$,
- $a, b \in AC_{\mathrm{loc}}(J, \mathbb{C})$ (*i.e.* locally absolutely continuous complex-valued functions on J),
- $g \in K(J \times \mathbb{C}^{m+1}, \mathbb{C})$ (*i.e.* a complex-valued function which satisfies the Carathéodory conditions on $J \times \mathbb{C}^{m+1}$).

Obviously, the function g is in general dependent on $\bar{z}$ as well as on every $\bar{z}(\theta_k)$. However, the fact that the function g satisfies the Carathéodory conditions enables us to significantly simplify the notation by using only z, since the validity of the Carathéodory conditions is not violated by composing with continuous functions $\bar{z}$, $\bar{w}_k$, and θ_k.

The relations between the functions are the following:

$$
a(t) = \frac{1}{2}\big(a_{11}(t) + a_{22}(t)\big) + \frac{i}{2}\big(a_{21}(t) - a_{12}(t)\big),
$$

$$
b(t) = \frac{1}{2}\big(a_{11}(t) - a_{22}(t)\big) + \frac{i}{2}\big(a_{21}(t) + a_{12}(t)\big),
$$

$$
A_k(t) = \frac{1}{2}\big(b_{11k}(t) + b_{22k}(t)\big) + \frac{i}{2}\big(b_{21k}(t) - b_{12k}(t)\big),
$$

$$
B_k(t) = \frac{1}{2}\big(b_{11k}(t) - b_{22k}(t)\big) + \frac{i}{2}\big(b_{21k}(t) + b_{12k}(t)\big),
$$

$$
g(t, z, w_1, \ldots, w_m)
$$
$$
= h_1\left(t, \frac{1}{2}(z + \bar{z}), \frac{1}{2i}(z - \bar{z}), \frac{1}{2}(w_1 + \bar{w}_1), \ldots, \frac{1}{2i}(w_m - \bar{w}_m)\right)
$$
$$
+ ih_2\left(t, \frac{1}{2}(z + \bar{z}), \frac{1}{2i}(z - \bar{z}), \frac{1}{2}(w_1 + \bar{w}_1), \frac{1}{2i}(w_1 - \bar{w}_1), \ldots, \frac{1}{2i}(w_m - \bar{w}_m)\right).
$$

Conversely, putting

$$
a_{11}(t) = \mathrm{Re}\big[a(t) + b(t)\big], \qquad a_{12}(t) = \mathrm{Im}\big[b(t) - a(t)\big],
$$
$$
a_{21}(t) = \mathrm{Im}\big[a(t) + b(t)\big], \qquad a_{22}(t) = \mathrm{Re}\big[a(t) - b(t)\big],
$$
$$
b_{11k}(t) = \mathrm{Re}\big[A_k(t) + B_k(t)\big], \qquad b_{12k}(t) = \mathrm{Im}\big[B_k(t) - A_k(t)\big],
$$

$$b_{21k}(t) = \text{Im}\big[A_k(t) + B_k(t)\big], \qquad b_{22k}(t) = \text{Re}\big[A_k(t) - B_k(t)\big],$$

$$h_1(t, x, y_1, \ldots, y_m) = \text{Re}\, g(t, x_1 + ix_2, y_{11} + iy_{12}, \ldots, y_{m1} + iy_{m2}),$$

$$h_2(t, x, y_1, \ldots, y_m) = \text{Im}\, g(t, x_1 + ix_2, y_{11} + iy_{12}, \ldots, y_{m1} + iy_{m2}),$$

equation (2) can be written in the real form (1) as well.

The equivalence of the dynamical invariants and asymptotic properties of the solutions of the real system (1) and the complex equation (2) is shown in [15] for the simple case covering ordinary differential equations.

In this paper we consider (2) in the case when

$$\liminf_{t \to \infty}\big(\big|a(t)\big| - \big|b(t)\big|\big) > 0 \tag{3}$$

and study the behavior of the solutions of (2) under this assumption, which generally means that $\det \mathsf{A}(t) > 0$ for t sufficiently large. This situation corresponds to the case when the equilibrium point 0 of the autonomous homogeneous system

$$x' = \mathsf{A}x, \tag{4}$$

where A is supposed to be a regular constant matrix, is a center, a focus or a node. Such a situation has some geometrical aspects, which are used in an analysis of the transformed equation (2). See [15] for more details.

Further, we suppose that (2) satisfies the uniqueness property of solutions.

2 Preliminaries
Throughout this paper we will assume that

$$\liminf_{t \to \infty}\big(\big|a(t)\big| - \big|b(t)\big|\big) > 0, \quad t \geq \theta_k(t) \geq t - r \text{ for } t \geq t_0 + r, \tag{5}$$

where $r > 0$ is a constant, which means that the delays θ_k are bounded. This is the same case as considered in [14]. Similar results for a case different from (5) were obtained in [7].

Then there are numbers $T \geq t_0 + r$ and $\mu > 0$ such that

$$\big|a(t)\big| > \big|b(t)\big| + \mu \quad \text{for } t \geq T. \tag{6}$$

Denote

$$c(t) = \frac{\bar{a}(t)b(t)}{|a(t)|}, \qquad \gamma(t) = \big|a(t)\big| + \sqrt{\big|a(t)\big|^2 - \big|b(t)\big|^2} \tag{7}$$

and

$$\vartheta(t) = \frac{\text{Re}(\gamma(t)\gamma'(t) - \bar{c}(t)c'(t)) - |\gamma(t)c'(t) - \gamma'(t)c(t)|}{\gamma^2(t) - |c(t)|^2},$$

$$\alpha(t) = 1 - \left|\frac{b(t)}{a(t)}\right| \text{sgn}\, \text{Re}\, a(t). \tag{8}$$

Moreover, we assume the following conditions to be valid:

(i) The numbers $T \geq t_0 + r$ and $\mu > 0$ satisfy condition (6).

(ii) There are functions $\varrho, \varkappa, \kappa_k : [T, \infty) \to \mathbb{R}$, where ϱ is continuous on $[T, \infty)$, such that

$$\left| \gamma(t) g(t, z, w_1, \dots, w_m) + c(t) \bar{g}(t, z, w_1, \dots, w_m) \right|$$

$$\leq \varkappa(t) \left| \gamma(t) z + c(t) \bar{z} \right| + \sum_{k=1}^{m} \kappa_k(t) \left| \gamma(\theta_k(t)) w_k + c(\theta_k(t)) \bar{w}_k \right| + \varrho(t)$$

for $t \geq T$, $z, w_k \in \mathbb{C}$ ($k = 1, \dots, m$).

(ii$_n$) There are numbers $R_n \geq 0$ and functions $\varkappa_n, \kappa_{nk} : [T, \infty) \to \mathbb{R}$ satisfying the inequality

$$\left| \gamma(t) g(t, z, w_1, \dots, w_m) + c(t) \bar{g}(t, z, w_1, \dots, w_m) \right|$$

$$\leq \varkappa_n(t) \left| \gamma(t) z + c(t) \bar{z} \right| + \sum_{k=1}^{m} \kappa_{nk}(t) \left| \gamma(\theta_k(t)) w_k + c(\theta_k(t)) \bar{w}_k \right|$$

for $t \geq \tau_n \geq T$, $|z| + \sum_{k=1}^{m} |w_k| > R_n$.

(iii) The function $\beta \in AC_{\text{loc}}([T, \infty), \mathbb{R}_-^0)$ is such that

$$\theta_k'(t) \beta(t) \leq -\lambda_k(t) \quad \text{a.e. on } [T, \infty), \tag{9}$$

where λ_k is given for $t \geq T$ by

$$\lambda_k(t) = \kappa_k(t) + \left(|A_k(t)| + |B_k(t)| \right) \frac{\gamma(t) + |c(t)|}{\gamma(\theta_k(t)) - |c(\theta_k(t))|}. \tag{10}$$

(iii$_n$) The function $\beta_n \in AC_{\text{loc}}([T, \infty), \mathbb{R}_-^0)$ is such that

$$\theta_k'(t) \beta_n(t) \leq -\lambda_{nk}(t) \quad \text{a.e. on } [\tau_n, \infty), \tag{11}$$

where λ_{nk} is given for $t \geq T$ by

$$\lambda_{nk}(t) = \kappa_{nk}(t) + \left(|A_k(t)| + |B_k(t)| \right) \frac{\gamma(t) + |c(t)|}{\gamma(\theta_k(t)) - |c(\theta_k(t))|}. \tag{12}$$

(iv$_n$) The function Λ_n is real locally Lebesgue integrable and the inequalities $\beta_n'(t) \geq \Lambda_n(t) \beta_n(t)$, $\Theta_n(t) \geq \Lambda_n(t)$ are satisfied for almost all $t \in [\tau_n, \infty)$, where Θ_n is given by $\Theta_n(t) = \alpha(t) \operatorname{Re} a(t) + \vartheta(t) - \varkappa_n(t) + m \beta_n(t)$.

Furthermore, denote

$$\Theta(t) = \alpha(t) \operatorname{Re} a(t) + \vartheta(t) - \varkappa(t). \tag{13}$$

3 Main results

First of all, we recall the two results from [14].

Lemma 1 *Let the assumptions* (i), (ii$_0$), (iii$_0$), (iv$_0$) *be fulfilled for some* $\tau_0 \geq T$. *Suppose there exist* $t_1 \geq \tau_0$ *and* $\nu \in (-\infty, \infty)$ *such that*

$$\inf_{t \geq t_1} \left[\int_{t_1}^{t} \Lambda_0(s) \, ds - \ln \left(\gamma(t) + |c(t)| \right) \right] \geq \nu. \tag{14}$$

If $z(t)$ is any solution of (2) *satisfying*

$$\min_{\theta(t_1)\leq s\leq t_1}\left|z(s)\right| > R_0, \qquad \Delta(t_1) > R_0 e^{-\nu}, \tag{15}$$

where

$$\theta(t) = \min_{k=1,\dots,m}\theta_k(t),$$

$$\Delta(t) = \left(\gamma(t) - \left|c(t)\right|\right)\left|z(t)\right| + \beta_0(t)\max_{\theta(t)\leq s\leq t}\left|z(s)\right|\sum_{k=1}^{m}\int_{\theta_k(t)}^{t}\left(\gamma(s)+\left|c(s)\right|\right)ds,$$

then

$$\left|z(t)\right| \geq \frac{\Delta(t_1)}{\gamma(t)+\left|c(t)\right|}\exp\left[\int_{t_1}^{t}\Lambda_0(s)\,ds\right] \tag{16}$$

for all $t \geq t_1$ for which $z(t)$ is defined.

Lemma 2 *Let the conditions* (i), (ii), (iii) *be fulfilled and Λ, θ_k' $(k = 1,\dots,m)$ be continuous functions such that inequality $\Lambda(t) \leq \Theta(t)$ holds a.e. on $[T,\infty)$, where Θ is defined by* (13). *Suppose that $\xi : [T - r,\infty) \to \mathbb{R}$ is a continuous function such that*

$$\Lambda(t) + \beta(t)\sum_{k=1}^{m}\theta_k'(t)\exp\left[-\int_{\theta_k(t)}^{t}\xi(s)\,ds\right] - \xi(t) > \varrho(t)C^{-1}\exp\left[-\int_{T}^{t}\xi(s)\,ds\right] \tag{17}$$

for $t \in [T,\infty]$ and some constant $C > 0$. Then there exists a $t_2 > T$ and a solution $z_0(t)$ of (2) *satisfying*

$$\left|z_0(t)\right| \leq \frac{C}{\gamma(t)-\left|c(t)\right|}\exp\left[\int_{T}^{t}\xi(s)\,ds\right] \tag{18}$$

for $t \geq t_2$.

If we combine the previous two results, we are able to prove the following theorem, which is the fundamental result of this paper.

Theorem 1 *Assume that the hypotheses* (i), (ii), (ii$_n$), (iii), (iii$_n$), (iv$_n$) *are valid for $T \leq \tau_n$, where $0 < R_n$, $n \in \mathbb{N}$, $\inf_{n\in\mathbb{N}} R_n = 0$. Suppose that θ_k', Λ are continuous functions such that inequality $\Theta(t) \geq \Lambda(t)$ is satisfied almost everywhere on $[T,\infty)$, where $\Theta(t) = \alpha(t)\operatorname{Re}a(t) + \vartheta(t) - \varkappa(t)$. Let $\xi : [T - r,\infty) \to \mathbb{R}$ be a continuous function satisfying the inequality*

$$\varrho(t)C^{-1}\exp\left(-\int_{T}^{t}\xi(s)\,ds\right) < \Lambda(t) + \beta(t)\sum_{k=1}^{m}\theta_k'(t)\exp\left[-\int_{\theta_k(t)}^{t}\xi(s)\,ds\right] - \xi(t) \tag{19}$$

for some constant $C > 0$ and $t \in [T,\infty)$. Assume

$$\inf_{\tau_n\leq s\leq t<\infty}\left[\int_{s}^{t}\Lambda_n(\sigma)\,d\sigma - \ln\left(\gamma(t)+\left|c(t)\right|\right)\right] \geq \nu, \tag{20}$$

$$\limsup_{t\to\infty}\left[\int_T^t\left(\Lambda_n(s)-\xi(s)\right)ds+\ln\frac{\gamma(t)-|c(t)|}{\gamma(t)+|c(t)|}\right]=\infty,\tag{21}$$

$$\lim_{t\to\infty}\left[\beta_n(t)\max_{\theta(t)\le s\le t}\frac{\exp[\int_T^s\xi(\sigma)\,d\sigma]}{\gamma(s)-|c(s)|}\sum_{k=1}^m\int_{\theta_k(t)}^t\left(\gamma(s)+|c(s)|\right)ds\right]=0,\tag{22}$$

for $n\in\mathbb{N}$, where $v\in(-\infty,\infty)$ and $\theta(t)=\min_{k=1,\dots,m}\theta_k(t)$. Then there is a solution $z_0(t)$ of (2) *with the property*

$$\lim_{t\to\infty}\min_{\theta(t)\le s\le t}|z_0(s)|=0.\tag{23}$$

Proof Using Lemma 2 we obtain the existence of $T\le t_2$ and a solution $z_0(t)$ of (2) satisfying for $t\ge t_2$ the inequality

$$|z_0(t)|\le\frac{C}{\gamma(t)-|c(t)|}\exp\left[\int_T^t\xi(s)\,ds\right].\tag{24}$$

From (20) we get

$$\inf_{\tau\le t<\infty}\left[\int_\tau^t\Lambda_N(s)\,ds-\ln\left(\gamma(t)+|c(t)|\right)\right]\ge v>-\infty.$$

Lemma 1 yields

$$\frac{\Psi(\tau)}{\gamma(t)+|c(t)|}\exp\left[\int_\tau^t\Lambda_N(s)\,ds\right]\le|z_0(t)|\tag{25}$$

for $\tau\le t$, where Ψ is given by

$$\Psi(\tau)=\left(\gamma(\tau)-|c(\tau)|\right)|z_0(\tau)|+\beta_N(\tau)\max_{\theta(\tau)\le s\le\tau}|z_0(s)|\sum_{k=1}^m\int_{\theta_k(\tau)}^\tau\left(\gamma(s)+|c(s)|\right)ds.$$

Assume that (23) does not hold. This implies the existence of $\varepsilon_0>0$ satisfying $\limsup_{t\to\infty}\min_{\theta(t)\le s\le t}|z_0(s)|>\varepsilon_0$. We take $N\in\mathbb{N}$ such that $\max\{R_N,\frac{2}{\mu}R_Ne^{-v}\}<\varepsilon_0$. Then

$$\max\left\{R_N,\frac{2}{\mu}R_Ne^{-v}\right\}<\min_{\theta(\tau)\le s\le\tau}|z_0(s)|\tag{26}$$

holds for some $\tau>\max\{T,\tau_N,t_2\}$. Taking (22) into account we may assume that

$$|\beta_N(\tau)|C\max_{\theta(\tau)\le s\le\tau}\frac{\exp[\int_T^s\xi(\sigma)\,d\sigma]}{\gamma(s)-|c(s)|}\sum_{k=1}^m\int_{\theta_k(\tau)}^\tau\left(\gamma(s)+|c(s)|\right)ds<\frac{1}{2}R_Ne^{-v}.\tag{27}$$

Hence, with respect to (5), (7), (24), (26), (27), and the nonpositiveness of β_N, we obtain

$$\left(\gamma(\tau)-|c(\tau)|\right)|z_0(\tau)|+\beta_N(\tau)\max_{\theta(\tau)\le s\le\tau}|z_0(s)|\sum_{k=1}^m\int_{\theta_k(\tau)}^\tau\left(\gamma(s)+|c(s)|\right)ds$$

$$\ge\left(\gamma(\tau)-|c(\tau)|\right)|z_0(\tau)|+\beta_N(\tau)C\max_{\theta(\tau)\le s\le\tau}\frac{\exp[\int_T^s\xi(\sigma)\,d\sigma]}{\gamma(s)-|c(s)|}$$

$$\times\sum_{k=1}^m\int_{\theta_k(\tau)}^\tau\left(\gamma(s)+|c(s)|\right)ds\ge\mu\frac{2}{\mu}R_Ne^{-v}-\frac{1}{2}R_Ne^{-v}>R_Ne^{-v}.$$

The inequalities (24) and (25) give the estimation

$$\frac{\Psi(\tau)}{\gamma(t) + |c(t)|} \exp\left[\int_\tau^t \Lambda_N(s)\,ds\right] \le \frac{C}{\gamma(t) - |c(t)|} \exp\left[\int_T^t \xi(s)\,ds\right],$$

which means that

$$\int_T^t \left[\Lambda_N(s) - \xi(s)\right]ds + \ln\frac{\gamma(t) - |c(t)|}{\gamma(t) + |c(t)|} \le \int_T^\tau \Lambda_N(s)\,ds - \ln\left[C^{-1}\Psi(\tau)\right]$$

for $\tau \le t$, which is in contradiction to (21). The proof is complete. $\qquad\square$

Remark 1 Theorem 1 covers more general situations than Theorem 3 in [7], where the different fundamental assumption $\liminf_{t\to\infty}(|\operatorname{Im}a(t)| - |b(t)|) > 0$ is supposed to hold. Indeed, if we take for example $a(t) \equiv 7 + i$ and $b(t) \equiv 2i$, then condition (5) in this paper is satisfied but the condition $\liminf_{t\to\infty}(|\operatorname{Im}a(t)| - |b(t)|) > 0$ is not valid.

The following nontrivial example was constructed to illustrate an application of the theoretical result presented in Theorem 1.

Example 1 Consider the two-dimensional system of the nonlinear delayed differential equations

$$x_1'(t) = 4x_1(t) + \frac{1}{t^2}x_1(t) - 2x_2(t) + \sum_{k=1}^m \frac{1}{4m}e^{-2t}x_1\left(t - e^{-kt}\right) + e^{-t}, \tag{28}$$

$$x_2'(t) = 4x_1(t) + 4x_2(t) + \frac{1}{t^2}x_2(t) - \sum_{k=1}^m \frac{1}{4m}e^{-2t}x_2\left(t - e^{-kt}\right). \tag{29}$$

This system can be written in matrix form (1), where

$$\mathsf{A}(t) = \begin{pmatrix} 4 + \frac{1}{t^2} & -2 \\ 4 & 4 + \frac{1}{t^2} \end{pmatrix}, \qquad \mathsf{B}_k(t) = \frac{1}{4m}\begin{pmatrix} e^{-2t} & 0 \\ 0 & e^{-2t} \end{pmatrix}, \quad \text{and}$$

$$h\left(t, x(t), x(\theta_1(t)), \ldots, x(\theta_m(t))\right) = \begin{pmatrix} e^{-t} \\ 0 \end{pmatrix}.$$

Following our approach, we use a transformation into the complex plane and obtain the delayed differential equation (2) with complex-valued coefficients,

$$z'(t) = (4 + 3i)z(t) + i\bar{z}(t) + \frac{1}{t^2}z(t) + \sum_{k=1}^m \frac{1}{4m}e^{-2t}\bar{z}\left(t - e^{-kt}\right) + e^{-t}, \tag{30}$$

where $a(t) \equiv 4 + 3i$, $b(t) \equiv i$, $A_k(t) \equiv 0$, $B_k(t) \equiv 0$, $\theta_k(t) = t - e^{-kt}$ for $k = 1,\ldots,m$, $g(t, z, w_1, \ldots, w_m) = \frac{1}{t^2}z + \sum_{k=1}^m \frac{1}{4m}e^{-2t}w_k + e^{-t}$.

Obviously $t - 1 \le \theta_k(t) \le t$ and $\theta_k'(t) = 1 + ke^{-kt} \ge 1 > 0$ for $t \ge 0$.

Suppose $t_0 = 0$ and $T \ge 1$. Then $\gamma(t) = |a(t)| + \sqrt{|a(t)|^2 - |b(t)|^2} \equiv 5 + 2\sqrt{6}$, $c(t) = \bar{a}(t)b(t)/|a(t)| \equiv \frac{3 + 4i}{5}$.

Further,

$$
\left| \gamma(t)g(t,z,w_1,\ldots,w_m) + c(t)\bar{g}(t,z,w_1,\ldots,w_m) \right|
$$

$$
\leq \frac{\gamma + |c|}{\gamma - |c|} \frac{1}{t^2} \left| \gamma(t)z + c(t)\bar{z} \right| + \frac{\gamma + |c|}{\gamma - |c|} \sum_{k=1}^{m} \left[\frac{1}{4m} e^{-2t} \left| \gamma(\theta_k(t))w_k + c(\theta_k(t))\bar{w}_k \right| \right] + e^{-t}
$$

$$
= \frac{\sqrt{3}}{\sqrt{2}} \frac{1}{t^2} \left| \gamma(t)z + c(t)\bar{z} \right| + \frac{\sqrt{3}}{\sqrt{2}} \frac{e^{-2t}}{4m} \sum_{k=1}^{m} \left| \gamma(\theta_k(t))w_k + c(\theta_k(t))\bar{w}_k \right| + e^{-t}.
$$

Thus conditions (i) and (ii) are fulfilled with $\varkappa(t) \equiv \frac{\sqrt{3}}{t^2\sqrt{2}}$, $\kappa_k(t) = \frac{e^{-2t}\sqrt{3}}{4m\sqrt{2}}$, and $\varrho(t) = e^{-t}$. To meet condition (ii$_n$), we estimate for $R_n = \frac{1}{n}$, $\tau_n = n$, and $|z| > R_n$

$$
\left| \gamma(t)g(t,z,w_1,\ldots,w_m) + c(t)\bar{g}(t,z,w_1,\ldots,w_m) \right|
$$

$$
\leq \frac{\gamma + |c|}{\gamma - |c|} \left[\frac{1}{t^2} + \frac{\varrho(t)}{R_n} \right] \left| \gamma(t)z + c(t)\bar{z} \right|
$$

$$
+ \frac{\gamma + |c|}{\gamma - |c|} \sum_{k=1}^{m} \left[\frac{1}{4m} e^{-2t} \left| \gamma(\theta_k(t))w_k + c(\theta_k(t))\bar{w}_k \right| \right]
$$

$$
= \frac{\sqrt{3}}{\sqrt{2}} \left[\frac{1}{t^2} + ne^{-t} \right] \left| \gamma(t)z + c(t)\bar{z} \right| + \frac{\sqrt{3}}{\sqrt{2}} \frac{e^{-2t}}{4m} \sum_{k=1}^{m} \left| \gamma(\theta_k(t))w_k + c(\theta_k(t))\bar{w}_k \right|,
$$

where $\varkappa_n(t) = \frac{\sqrt{3}}{\sqrt{2}}\left[\frac{1}{t^2} + ne^{-t}\right]$ and $\kappa_{nk}(t) = \frac{\sqrt{3}}{\sqrt{2}}\frac{e^{-2t}}{4m}$.

Condition (iii) holds with

$$
-\lambda_k(t)\left(\theta_k'(t)\right)^{-1} = -\frac{e^{-2t}\sqrt{3}}{4m\sqrt{2}} \frac{1}{1 + ke^{-kt}} \geq -\frac{e^{-2t}\sqrt{3}}{4m\sqrt{2}} = \beta(t).
$$

Condition (iii$_n$) is satisfied for

$$
\beta_n(t) = -\frac{e^{-2(t-1)}}{m} \leq -\frac{e^{-2t}\sqrt{3}}{4m\sqrt{2}} = -\lambda_{nk}(t)\left(\theta_k'(t)\right)^{-1}.
$$

We get condition (iv$_n$) by setting

$$
\Lambda_n(t) = \Theta_n(t) = \frac{16}{5} - \frac{\sqrt{3}}{\sqrt{2}}\left[\frac{1}{t^2} + ne^{-t}\right] - e^{-2(t-1)} > 0.
$$

Further, we put $\Lambda(t) = \Theta(t) = \frac{16}{5} - \frac{\sqrt{3}}{t^2\sqrt{2}}$.

Then condition (19) holds for

$$
\xi(t) = \frac{8}{5} - \frac{\sqrt{3}}{2t^2\sqrt{2}} - \frac{\sqrt{3}}{\sqrt{2}}\frac{e^{-2t}}{4m} - e^{-t}
$$

and $\xi(t) > 0$ for $t \geq T = 1$.

Now it is not difficult to verify conditions (21) and (20), since $\Lambda_n(t) - \xi(t) > 0$ and $\Lambda_n(t) > 0$ for $n \in \mathbb{N}$. Investigating the factors of the product in parentheses in (22), we come to the

conclusion that

$$\beta_n = O(e^{-2t}), \qquad \exp\left[\int_T^s \xi(\sigma)\, d\sigma\right] = O(e^{1.6t})$$

and

$$\sum_{k=1}^m \int_{\theta_k(t)}^t \left(\gamma(s) + |c(s)|\right) ds = O\left(\sum_{k=1}^m e^{-kt}\right) = O(e^{-t}).$$

Consequently, the product of these factors is asymptotically equal to $O(e^{-\delta t})$, where $\delta > 0$ and thus condition (22) is satisfied. All assumptions of Theorem 1 are fulfilled and we can conclude that there exist $t_2 > 1$ and a solution $z_0(t)$ of (30) satisfying (23) for $t \geq t_2$.

Remark 2 This result is slightly surprising. However, it is in good agreement with the well-known fact that introducing delay into an unstable system without delay can cause a change of behavior of the system. Such situations are described and corresponding results are formulated *e.g.* in [16].

4 Conclusion

We proved an interesting result about the stability of two-dimensional systems with bounded delays. Sufficient conditions for the stability of an originally unstable system were presented. The result is in perfect agreement with the results stated and proved in the established literature. An example showed how this result can be used in practice.

References
1. Baculíková, B, Džurina, J, Rogovchenko, YV: Oscillation of third order trinomial delay differential equations. Appl. Math. Comput. **218**(13), 7023-7033 (2012)
2. Berezansky, L, Braverman, E: On nonoscillation and stability for systems of differential equations with a distributed delay. Automatica **48**(4), 612-618 (2012)
3. Berezansky, L, Diblík, J, Svoboda, Z, Šmarda, Z: Simple uniform exponential stability conditions for a system of linear delayed differential equations. Appl. Math. Comput. **250**(1), 605-614 (2012)
4. Braverman, E, Berezansky, L: New stability conditions for linear differential equations with several delays. Abstr. Appl. Anal. **2011**, Article ID 178568 (2011)
5. Burton, TA: Stability by Fixed Point Theory for Functional Differential Equations, 348 pp. Dover, New York (2006)
6. Olach, R: Positive periodic solutions of delay differential equations. Appl. Math. Lett. **26**, 1141-1145 (2013)
7. Šmarda, Z, Rebenda, J: Asymptotic behaviour of a two-dimensional differential system with a finite number of nonconstant delays under the conditions of instability. Abstr. Appl. Anal. **2012**, Article ID 952601 (2012)
8. Stević, S: On q-difference asymptotic solutions of a system of nonlinear functional differential equations. Appl. Math. Comput. **219**(15), 8295-8301 (2013)
9. Stević, S: Solutions converging to zero of some systems of nonlinear functional differential equations with iterated deviating argument. Appl. Math. Comput. **219**(8), 4031-4035 (2012)
10. Kalas, J: Asymptotic behaviour of a two-dimensional differential system with delay under the conditions of instability. Nonlinear Anal. **62**, 207-224 (2005)

11. Kalas, J: Asymptotic properties of an unstable two-dimensional differential system with delay. Math. Bohem. **131**, 305-319 (2006)
12. Kalas, J: Asymptotic properties of a two-dimensional differential system with a bounded nonconstant delay under the conditions of instability. Far East J. Math. Sci.: FJMS **29**, 513-532 (2008)
13. Kalas, J, Rebenda, J: Asymptotic behaviour of solutions of real two-dimensional differential system with nonconstant delay in an unstable case. Electron. J. Qual. Theory Differ. Equ. **2012**, 8 (2012)
14. Kalas, J, Rebenda, J: Asymptotic behaviour of a two-dimensional differential system with a nonconstant delay under the conditions of instability. Math. Bohem. **136**(2), 215-224 (2011)
15. Ráb, M, Kalas, J: Stability of dynamical systems in the plane. Differ. Integral Equ. **3**, 127-144 (1990)
16. Kolmanovskii, V, Myshkis, A: Introduction to the Theory and Applications of Functional Differential Equations. Kluwer Academic, Dordrecht (1999)