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Outer synchronization of stochastic complex networks with time-varying delay

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Abstract

In this paper, the outer synchronization of stochastic complex networks with time-varying delay is investigated. A systematic method that allows one to construct global Lyapunov functions for these systems is provided by employing results from graph theory and Lyapunov method. By the construction of the Lyapunov function, some sufficient conditions of *p*th moment exponential outer synchronization are given. The theoretic result is also applied to an investigation of the outer synchronization of stochastic time-varying delayed coupled oscillators on networks. Finally, a numerical example is examined to illustrate the effectiveness of the results developed.

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Keywords: stochastic complex networks; time-varying delay; outer synchronization

1 Introduction

In the real world, the structure of many coupled systems can be described by complex networks [1, 2]. The complex networks, as a fundamental tool in understanding dynamical behavior of real systems, have been gaining increasing recognition. Networks exhibit complexities in the overall topological properties and dynamical properties of the network nodes. A series of important research problems have resulted from the complex nature of networks. Synchronization is one of the typical problems of complex networks, which has attracted lots of attention. Synchronization of complex networks is widely applied to many fields, such as neural networks, biological systems, and so on [3–6].

In fact, there are two kinds of network synchronization. One network synchronization is concerned with the synchronization among the nodes within a network, which is called inner synchronization. The other is outer synchronization, which is a synchronization between two or more complex networks. Li *et al.* were first to study the complete outer synchronization problem for two complex networks with identical topological structure [7]. Then in [8], the outer synchronization for two complex networks was extended to the discrete time case. Later on, an increasing number of researchers devoted themselves to the work about outer synchronization. It is noticed that the outer synchronization between two nonidentical networks with circumstance noise was investigated in [9], which analyzed both the inner and the outer synchronization between two coupled discrete-time networks with time delays. In [10], the finite-time stochastic outer synchronization between two complex dynamical networks with different topological structure was studied. The authors of [11] studied the effect of noise on the outer synchronization of two unidirectionally coupled complex dynamical networks. Reference [12] obtained several sufficient conditions for the generalized outer synchronization between complex dynamical networks. More research on outer synchronization can be found in [13–18] and the references therein.

In real networks, time delays cannot be neglected due to the finite information transmission and processing speeds among the network nodes. Hence, it is reasonable and necessary to study the effect of time delays which may affect the dynamical behavior of complex networks. For instance, in [13], outer synchronization of uncertain complex delayed networks with adaptive coupling was studied. References [16] and [17] investigated mixed outer synchronization and finite-time outer synchronization of complex networks with coupling time-varying delay, respectively. On the other hand, another addition which exists in nature and man-made systems should also be considered, that is, noise, which is commonly regarded as a random and persistent disturbance obscuring or reducing the clarity of the signal. Hence, noise is another important factor affecting the behavior for dynamical networks. For this reason, synchronization criteria for stochastic systems have been obtained in the literature [9, 11, 14, 19] by various approaches. Furthermore, it is a significant task to understand how noise influences the dynamics of complex networks. For example, [19] studied the synchronization between two coupled chaotic systems determined by the distribution of white noise and analytically proved that synchronizability of coupled chaotic systems could be promoted by white noise. In [9], the synchronization criteria have been established for two nonidentical networks with circumstance noise.

Over the years, lots of efficient synchronization methods for complex networks have been developed by many researchers. In most of the studies above, the criteria ensuring synchronization on networks have been derived mainly based on the Lyapunov method. However, how to construct a global Lyapunov function for the system on networks by means of the Lyapunov method is a challenge for researchers. Fortunately, [20] provided a systematic method which is on the base of graph theory, which allows us to construct a global Lyapunov function for large-scale coupled systems. Based on graph theory, a network can be mathematically described by a directed graph consisting of vertices and directed arcs connecting them.

Motivated by the above discussion, outer synchronization of stochastic complex networks with time-varying delay (SCNVD) is studied in this paper. Based on graph theory, SCNVD can be described as a directed graph in which a system of stochastic time-varying delayed differential equations is assigned at each vertex, and the directed arcs indicate interconnections and interactions among vertex systems. A systematic method to construct a global Lyapunov function for the SCNVD is provided. Then several sufficient criteria for pth moment exponential outer synchronization of SCNVD will be established by constructing a global Lyapunov function and utilizing stochastic stability theory. Moreover, the theoretic result will be applied to an investigation of the outer synchronization of stochastic time-varying delayed coupled oscillators on networks.

The rest of the paper is organized as follows. In the next section, some basic concepts and results on graph theory, our mathematical model of the SCNVD and some preliminary results are introduced. In Section 3, a systematic method to construct a global Lyapunov function for the SCNVD based on the Lyapunov method and graph theory is given, and the sufficient criteria for the outer synchronization are derived. In Section 4, we will introduce stochastic time-varying delayed coupled oscillators on networks and present the sufficient conditions of the outer synchronization. In Section 5, a numerical simulation is given to show the effectiveness of the theoretical results.

Notation Throughout the paper, unless otherwise specified, we will employ the following notations. Let $\mathbb{R}$ and $\mathbb{R}^n$ be the set of real numbers and n-dimensional Euclidean space, respectively. We have $\mathbb{R}^1_+ = [0, +\infty)$, $\mathbb{L} = \{1, 2, \ldots, L\}$. Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete probability space with filtration $\mathbb{F} = \mathcal{F}_{t\geq 0}$ satisfying the usual conditions, and $B(\cdot)$ be the one-dimensional Brownian motion defined on the space. The mathematical expectation with respect to the given probability measure $\mathbb{P}$ is denoted $\mathbb{E}(\cdot)$. Let $|\cdot|$ denote the Euclidean norm for vectors or the trace norm for matrices. Denote by $C([-\tau, 0], \mathbb{R}^n)$ the space of continuous functions $x : [-\tau, 0] \to \mathbb{R}^n$ with norm $|x| = \sup_{-\tau\leq t\leq 0} |x(t)|$. Write $C^{2,1}(\mathbb{R}^n \times \mathbb{R}^1_+; \mathbb{R}^1_+)$ for the family of all nonnegative functions $V(x, t)$ on $\mathbb{R}^n \times \mathbb{R}^1_+$ that are continuously twice differentiable in x and once in t. If A is a vector or matrix, its transpose is denoted by A^T. I stands for the identity matrix of appropriate dimensions.

2 Preliminaries

In this section, we will give some useful preliminaries and a model description.

2.1 Basic concepts on graph theory

Since the complex network considered in this paper is built on a directed graph, it is necessary to recall some concepts and results on graph theory in this subsection.

A directed graph $\mathcal{G} = (V, E)$ contains a set $V = \{1, 2, \ldots, N\}$ of vertices and a set of E of arcs (i, j) leading from initial vertex i to terminal vertex j. A subgraph $\mathcal{H}$ of $\mathcal{G}$ is said to be spanning if $\mathcal{H}$ and $\mathcal{G}$ have the same vertex set. A graph $\mathcal{G}$ is weighted if each arc (j, i) is assigned a positive weight a_{ij}. In our convention, $a_{ij} > 0$ if and only if there exists an arc from vertex j to vertex i in $\mathcal{G}$, and we call $A = (a_{ij})_{N\times N}$ as the weight matrix. The weight $W(\mathcal{G})$ of $\mathcal{G}$ is the product of the weights on all its arcs. A directed path $\mathcal{P}$ in $\mathcal{G}$ is a subgraph with distinct vertices $\{i_1, i_2, \ldots, i_s\}$ such that its set of arcs is $\{(i_k, i_{k+1}) : k = 1, 2, \ldots, s-1\}$. If $i_s = i_1$, we call $\mathcal{P}$ a directed cycle. A connected subgraph $\mathcal{T}$ is a tree if it contains no cycles. A tree $\mathcal{T}$ is rooted at vertex i, called the root, if i is not a terminal vertex of any arcs, and each of the remaining vertices is a terminal vertex of exactly one arc. A subgraph $\mathcal{Q}$ is unicyclic if it is a disjoint union of rooted trees whose roots form a directed cycle. A digraph $\mathcal{G}$ is strongly connected if, for any pair of distinct vertices, there exists a directed path from one to the other. Denote the digraph with weight matrix A as $(\mathcal{G}, A)$. A weighted digraph $(\mathcal{G}, A)$ is said to be balanced if $W(\mathcal{C}) = W(-\mathcal{C})$ for all directed cycles $\mathcal{C}$. Here, $-\mathcal{C}$ denotes the reverse of $\mathcal{C}$ and is constructed by reversing the direction of all arcs in $\mathcal{C}$. For a unicyclic graph $\mathcal{Q}$ with cycle $\mathcal{C}_{\mathcal{Q}}$, let $\tilde{\mathcal{Q}}$ be the unicyclic graph obtained by replacing $\mathcal{C}_{\mathcal{Q}}$ with $-\mathcal{C}_{\mathcal{Q}}$. Suppose that $(\mathcal{G}, A)$ is balanced, then $W(\mathcal{Q}) = W(\tilde{\mathcal{Q}})$.

Here we show a lemma which will be used in the proofs of our main results.

Lemma 2.1 [20] *Assume $N \geq 2$. Let c_i denote the cofactor of the ith diagonal element of L. Then the following identity holds*:

$$\sum_{i,j=1}^{N} c_i a_{ij} F_{ij}(x_i, x_j) = \sum_{\mathcal{Q}\in\mathbb{Q}} w(\mathcal{Q}) \sum_{(s,r)\in E(\mathcal{C}_{\mathcal{Q}})} F_{rs}(x_r, x_s),$$

where $F_{ij}(x_i, x_j)$, $1 \le i, j \le N$, are arbitrary functions, $\mathbb{Q}$ is the set of all spanning unicyclic graph of $(\mathcal{G}, A)$, $w(\mathcal{Q})$ is the weight of $\mathcal{Q}$, and $\mathcal{C}_{\mathcal{Q}}$ denotes the directed cycle of $\mathcal{Q}$. Here $c_i = \sum_{\mathcal{T}\in\mathbb{T}_i} w(\mathcal{T})$, $i = 1, 2, \ldots, N$, where $\mathbb{T}_i$ is the set of all spanning trees $\mathcal{T}$ of $(\mathcal{G}, A)$ that are rooted at vertex i and $w(\mathcal{T})$ is the weight of $\mathcal{T}$. In particular, if $(\mathcal{G}, A)$ is strongly connected, then $c_i > 0$ for $1 \le i \le n$.

2.2 Model formulation

In this paper, we describe the SCNVD as a weighted diagraph $(\mathcal{G}, A)$. We first have L (≥ 2) individual networks and describe them on a digraph $\mathcal{G}$.

We first assume that dynamical system in the kth vertex is described by

$$\mathrm{d}x_k(t) = \bigl[f_k\bigl(x_k(t), t\bigr) + g_k\bigl(x_k\bigl(t - \tau_k(t)\bigr), t\bigr)\bigr]\,\mathrm{d}t, \tag{2.1}$$

where $x_k(t) = (x_k^{(1)}, x_k^{(2)}, \ldots, x_k^{(m_k)})^T \in \mathbb{R}^{m_k}$ is the state of the kth vertex, f_k and $g_k : \mathbb{R}^{m_k} \times \mathbb{R}^1_+ \to \mathbb{R}^{m_k}$ are continuously differentiable nonlinear vector functions, and $\tau_k(t) : \mathbb{R}^1_+ \to [0, \tau]$ stands for the time delay of the kth vertex system.

Second, consider the dispersal factor and the time delays result from dispersal. The dispersal from the hth vertex to the kth vertex is represented by the function $H_{kh} : \mathbb{R}^{m_k} \times \mathbb{R}^1_+ \to \mathbb{R}^{m_k}$. Here $H_{kh} \equiv 0$ if and only if there exists no dispersal from the hth vertex to the kth vertex in $\mathcal{G}$. Also, $\tau^{(kh)}(t) : \mathbb{R}^1_+ \to [0, \tau]$ reflects the transmissible time from vertex h to vertex k. Thus time-varying delayed complex networks can be described as follows:

$$\begin{aligned} \mathrm{d}x_k(t) = \Biggl[& f_k\bigl(x_k(t), t\bigr) + g_k\bigl(x_k\bigl(t - \tau_k(t)\bigr), t\bigr) \\ & + \sum_{h=1}^{L} H_{kh}\bigl(x_h\bigl(t - \tau^{(kh)}(t)\bigr), t\bigr)\Biggr]\,\mathrm{d}t, \quad k \in \mathbb{L}. \end{aligned} \tag{2.2}$$

To realize the outer synchronization between two complex networks, we refer to the system (2.2) as the drive network, and the response network is given by the following:

$$\begin{aligned} \mathrm{d}y_k(t) = & \Biggl[f_k\bigl(y_k(t), t\bigr) + g_k\bigl(y_k\bigl(t - \tau_k(t)\bigr), t\bigr) + \sum_{h=1}^{L} H_{kh}\bigl(y_h\bigl(t - \tau^{(kh)}(t)\bigr), t\bigr)\Biggr]\,\mathrm{d}t \\ & + \sigma\bigl(y_k(t) - x_k(t)\bigr)\,\mathrm{d}B(t), \quad k \in \mathbb{L}. \end{aligned} \tag{2.3}$$

Define $e_k = y_k - x_k$ to be the kth error state vector between networks (2.2) and (2.3); then one gets the following error system:

$$\begin{aligned} \mathrm{d}e_k(t) = \Biggl[& f_k\bigl(y_k(t), t\bigr) - f_k\bigl(x_k(t), t\bigr) + g_k\bigl(y_k\bigl(t - \tau_k(t)\bigr), t\bigr) \\ & - g_k\bigl(x_k\bigl(t - \tau_k(t)\bigr), t\bigr) + \sum_{h=1}^{L} \bigl(H_{kh}\bigl(y_h\bigl(t - \tau^{(kh)}(t)\bigr), t\bigr) \\ & - H_{kh}\bigl(x_h\bigl(t - \tau^{(kh)}(t)\bigr), t\bigr)\bigr)\Biggr]\,\mathrm{d}t + \sigma e_k(t)\,\mathrm{d}B(t). \end{aligned} \tag{2.4}$$

Here the functions f_k, g_k, and H_{kh} satisfy the global Lipschitz condition. Based on the theory of stochastic differential equation, the error system (2.4) possesses a unique solution. Assume that, for any initial condition $e_k(0) = y_k(0) - x_k(0)$, the error system (2.4) has a unique solution $e_k(t, e_k(0)) = \phi(t)$, where $\phi(t)$ is a continuous function on $[-\tau, 0]$. Obviously, $e_k(t, 0) \equiv 0$ is a trivial solution for the error system (2.4). Throughout this paper, the following hypothesis and definitions are needed to derive our main results.

Assumption 2.2 Assume that functions $\tau_k(t)$ and $\tau^{(kh)}(t)$, $k, h \in \mathbb{L}$ are differentiable and theirs derivative are bounded by a constant $\bar{\tau} \in [0, 1)$.

Definition 2.3 The network (2.2) is said to reach pth moment exponentially outer synchronization (ME-synchronization) with network (2.3), if for any initial state ϕ, we have

$$\limsup_{t\to\infty} \frac{1}{t} \ln\bigl(\mathbb{E}\bigl|e(t,\phi)\bigr|^p\bigr) < 0, \quad i \in \mathbb{L}. \tag{2.5}$$

Definition 2.4 Function $V_k(e_k, t) \in C^{2,1}(\mathbb{R}^{m_k} \times \mathbb{R}^1_+; \mathbb{R}^1_+)$ is called vertex-Lyapunov function for the system (2.4), if the following assumptions hold:

(A1) There exist positive constants $p \geq 2$, α_k, β_k, such that

$$\alpha_k |e_k|^p \leq V_k(e_k, t) \leq \beta_k |e_k|^p. \tag{2.6}$$

(A2) There exist positive constants σ_k, η_k, ϵ_{kh}, a_{kh}, and a function $F_{kh}(e_k, e_h)$ such that

$$\begin{aligned}
\mathcal{L}V_k(e_k, t) \leq{}& -\sigma_k |e_k|^p + \eta_k \bigl|e_k\bigl(t - \tau_k(t)\bigr)\bigr|^p + \sum_{h=1}^{L} \epsilon_{kh} \bigl|e_k\bigl(t - \tau^{(hk)}(t)\bigr)\bigr|^p \\
&+ \sum_{h=1}^{L} a_{kh} F_{kh}\bigl(e_k\bigl(t - \tau^{(hk)}(t)\bigr), e_h\bigl(t - \tau^{(kh)}(t)\bigr)\bigr),
\end{aligned} \tag{2.7}$$

where the differential operator $\mathcal{L}V_k(e_k, t)$ associated with the kth vertex system is defined by

$$\begin{aligned}
\mathcal{L}V_k(e_k, t) ={}& \frac{\partial V_k(e_k, t)}{\partial t} + \frac{\partial V_k(e_k, t)}{\partial e_k} \Biggl[f_k\bigl(y_k(t), t\bigr) - f_k\bigl(x_k(t), t\bigr) \\
&+ g_k\bigl(y_k\bigl(t - \tau_k(t)\bigr), t\bigr) - g_k\bigl(x_k\bigl(t - \tau_k(t)\bigr), t\bigr) \\
&+ \sum_{h=1}^{L} \bigl(H_{kh}\bigl(y_h\bigl(t - \tau^{(kh)}(t)\bigr), t\bigr) - H_{kh}\bigl(x_h\bigl(t - \tau^{(kh)}(t)\bigr), t\bigr)\bigr) \Biggr] \\
&+ \frac{1}{2} \operatorname{trace}\bigl(\sigma e_k(t)\bigr)^T V''_{ee} \sigma e_k(t).
\end{aligned} \tag{2.8}$$

(A3) Along each directed cycle $\mathcal{C}_Q$ of the weighted digraph $(\mathcal{G}, A)$, we have

$$\sum_{(h,k)\in E(\mathcal{C}_Q)} F_{kh}(e_k, e_h) \leq 0 \tag{2.9}$$

for all $e_k \in \mathbb{R}^{m_k}$, $e_h \in \mathbb{R}^{m_h}$.

3 Main results

We investigate outer synchronization of SCNVD in this section, and our main results for the outer synchronization criteria of SCNVD are presented.

3.1 Lyapunov-type theorem

Theorem 3.1 *Suppose that the error system* (2.4) *admits vertex-Lyapunov functions* $V_k(e_k,t)$, *and digraph* $(\mathcal{G},A)$ *is strongly connected, in which* $A = (a_{kh})_{L\times L}$. *If*

$$\eta_k + \sum_{h=1}^{L} \epsilon_{kh} < \sigma_k(1-\bar{\tau}), \tag{3.1}$$

then the network (2.2) *and* (2.3) *will reach ME-synchronization.*

Proof Let the construction of the Lyapunov function for the error system (2.4) be in the form of

$$V(e,t) = \sum_{k=1}^{L} c_k V_k(e_k,t), \tag{3.2}$$

in which c_k is the cofactor of the kth diagonal element of Laplacian matrix of digraph $(\mathcal{G},A)$. It is obvious from Lemma 2.1 that $c_k > 0$, because the digraph $(\mathcal{G},A)$ is strongly connected.

Using (2.6), we obtain

$$V(e,t) = \sum_{k=1}^{L} c_k V_k(e_k,t) \le \sum_{k=1}^{L} c_k\beta_k|e_k|^p \le \left(\sum_{k=1}^{L} c_k\beta_k\right)|e|^p$$

and

$$\begin{aligned} V(e,t) &\ge \sum_{k=1}^{L} c_k\alpha_k|e_k|^p \\ &= \sum_{j=1}^{L} c_j\alpha_j \sum_{k=1}^{L}\left[\frac{c_k\alpha_k}{\sum_{i=1}^{L} c_i\alpha_i}\left(|e_k|^2\right)^{\frac{p}{2}}\right] \\ &\ge \sum_{j=1}^{L} c_j\alpha_j \left[\sum_{k=1}^{L}\frac{c_k\alpha_k}{\sum_{i=1}^{L} c_i\alpha_i}\left(|e_k|^2\right)\right]^{\frac{p}{2}} \\ &\ge \left(\sum_{k=1}^{L} c_k\alpha_k\right)^{1-\frac{p}{2}}\left(\min_{1\le k\le L}\{c_k\alpha_k\}\right)^{\frac{p}{2}}|e|^p. \end{aligned}$$

Denote $\beta = \sum_{k=1}^{L} c_k\beta_k$ and $\alpha = (\sum_{k=1}^{L} c_k\alpha_k)^{1-\frac{p}{2}}(\min_{k\in\mathbb{L}}\{c_k\alpha_k\})^{\frac{p}{2}}$, then we have

$$\alpha|e|^p \le V(e,t) \le \beta|e|^p.$$

By (2.6), (2.7), (2.8), and Lemma 2.1, we can see that

$$\begin{aligned}
\mathcal{L}V(e,t) &= \sum_{k=1}^{L} c_k \mathcal{L}V_k(e_k,t) \\
&\leq -\sum_{k=1}^{L} c_k\sigma_k|e_k|^p + \sum_{k=1}^{L} c_k\eta_k\big|e_k\big(t-\tau_k(t)\big)\big|^p \\
&\quad + \sum_{k,h=1}^{L} c_k\epsilon_{kh}\big|e_k\big(t-\tau^{(hk)}(t)\big)\big|^p \\
&\quad + \sum_{k,h=1}^{L} c_k a_{kh} F_{kh}\big(e_k\big(t-\tau^{(hk)}(t)\big), e_h\big(t-\tau^{(kh)}(t)\big)\big) \\
&= -\sum_{k=1}^{L} c_k\sigma_k|e_k|^p + \sum_{k=1}^{L} c_k\eta_k\big|e_k\big(t-\tau_k(t)\big)\big|^p \\
&\quad + \sum_{k,h=1}^{L} c_k\epsilon_{kh}\big|e_k\big(t-\tau^{(hk)}(t)\big)\big|^p \\
&\quad + \sum_{\mathcal{Q}\in\mathbb{Q}} w(Q) \sum_{(h,k)\in E(\mathcal{C}_{\mathcal{Q}})} F_{kh}\big(e_k\big(t-\tau^{(hk)}(t)\big), e_h\big(t-\tau^{(kh)}(t)\big)\big) \\
&\leq -\sum_{k=1}^{L} c_k\sigma_k|e_k|^p + \sum_{k=1}^{L} c_k\eta_k\big|e_k\big(t-\tau_k(t)\big)\big|^p \\
&\quad + \sum_{k,h=1}^{L} c_k\epsilon_{kh}\big|e_k\big(t-\tau^{(hk)}(t)\big)\big|^p.
\end{aligned}$$

It is not difficult to see from (3.1) that there exists a constant $\gamma > 0$ which is sufficiently small, such that

$$\gamma\beta_k - \sigma_k + \frac{e^{\gamma\tau}}{1-\bar{\tau}}\left(\eta_k + \sum_{h=1}^{L}\epsilon_{kh}\right) < 0. \tag{3.3}$$

Using the Itô formula and the condition (A2), one shows that

$$\begin{aligned}
\mathbb{E}\big[e^{\gamma t}V\big(e(t),t\big)\big] &= \mathbb{E}V\big(\phi(0),0\big) + \mathbb{E}\int_0^t e^{\gamma s}\big[\mathcal{L}V\big(e(s),s\big) + \gamma V\big(e(s),s\big)\big]\,\mathrm{d}s \\
&\leq \mathbb{E}V\big(\phi(0),0\big) + \sum_{k=1}^{L} c_k(\gamma\beta_k - \sigma_k)\mathbb{E}\int_0^t e^{\gamma s}\big|e_k(s)\big|^p\,\mathrm{d}s \\
&\quad + \sum_{k=1}^{L} c_k\eta_k\mathbb{E}\int_0^t e^{\gamma s}\big|e_k\big(s-\tau_k(s)\big)\big|^p\,\mathrm{d}s \\
&\quad + \sum_{h,k=1}^{L} c_k\epsilon_{kh}\mathbb{E}\int_0^t e^{\gamma s}\big|e_k\big(s-\tau^{(hk)}(s)\big)\big|^p\,\mathrm{d}s.
\end{aligned}$$

By virtue of an integration by substitution and (3.1), one can obtain

$$\begin{aligned}
&\mathbb{E}\bigl[e^{\gamma t}V\bigl(e(t),t\bigr)\bigr] \\
&\quad\le \mathbb{E}V\bigl(\phi(0),0\bigr)+\sum_{k=1}^{L}c_k\left(\gamma\beta_k-\sigma_k+\frac{\eta_k e^{\gamma\tau}}{1-\bar{\tau}}+\sum_{h=1}^{L}\epsilon_{kh}\frac{e^{\gamma\tau}}{1-\bar{\tau}}\right)\mathbb{E}\int_0^t e^{\gamma s}\bigl|e_k(s)\bigr|^p \\
&\qquad+\frac{e^{\gamma\tau}}{1-\bar{\tau}}\sum_{k=1}^{L}c_k\left(\eta_k+\sum_{h=1}^{L}\epsilon_{kh}\right)\mathbb{E}\int_{-\tau}^{0}e^{\gamma t}\max_{-\tau\le s\le 0}\bigl|e_k(s)\bigr|^p\,\mathrm{d}s \\
&\quad\le \mathbb{E}V\bigl(\phi(0),0\bigr)+\frac{e^{\gamma\tau}}{\gamma(1-\bar{\tau})}\left(\sum_{k=1}^{L}c_k\eta_k+\sum_{h,k=1}^{L}c_k\epsilon_{kh}\right)\mathbb{E}|\phi|^p \\
&\quad\le \left[\frac{e^{\gamma\tau}}{\gamma(1-\bar{\tau})}\left(\sum_{k=1}^{L}c_k\eta_k+\sum_{h,k=1}^{L}c_k\epsilon_{kh}\right)+\beta\right]\mathbb{E}|\phi|^p. \qquad (3.4)
\end{aligned}$$

Then we have

$$\mathbb{E}\bigl[e^{\gamma t}\alpha\bigl|e(t)\bigr|^p\bigr]\le\mathbb{E}\bigl[e^{\gamma t}V\bigl(e(t),t\bigr)\bigr]\le\left[\frac{e^{\gamma\tau}}{\gamma(1-\bar{\tau})}\left(\sum_{k=1}^{L}c_k\eta_k+\sum_{h,k=1}^{L}c_k\epsilon_{kh}\right)+\beta\right]\mathbb{E}|\phi|^p.$$

So, we can easily compute that

$$\mathbb{E}\bigl[\bigl|e(t)\bigr|^p\bigr]\le\frac{1}{\alpha}\left[\frac{e^{\gamma\tau}}{\gamma(1-\bar{\tau})}\left(\sum_{k=1}^{L}c_k\eta_k+\sum_{h,k=1}^{L}c_k\epsilon_{kh}\right)+\beta\right]e^{-\gamma t}\mathbb{E}|\phi|^p.$$

Hence,

$$\limsup_{t\to\infty}\frac{1}{t}\ln\mathbb{E}\bigl|e(t)\bigr|^p\le-\gamma.$$

That is, the trivial solution of (2.4) is ME-stable and the pth moment Lyapunov exponent is not greater than $-\gamma$, which means the networks (2.2) and (2.3) will reach ME-synchronization. □

Remark 3.2 Theorem 3.1 shows that the Lyapunov function $V(e,t)$ for the system (2.4) can be obtained by the weighted sum of $V_k(e_k,t)$. Hence finding the vertex-Lyapunov function $V_k(e_k,t)$ is a key point in the study of the stability for the system (2.4). In the practical applications, complex networks are coupled together in regular ways by the simple and nearly identical dynamical systems whose Lyapunov functions have been obtained. The Lyapunov function of these systems can be chosen as the Lyapunov function of vertex systems.

With the help of some properties in graph theory, now some other simple conditions are discussed. Note that if the digraph $(\mathcal{G},A)$ is balanced, then $W(\mathcal{Q})=W(-\mathcal{Q})$. We can obtain

$$\sum_{k,h=1}^{L}c_k a_{kh}F_{kh}(x_k,x_h)=\frac{1}{2}\sum_{\mathcal{Q}\in\mathbb{Q}}w(\mathcal{Q})\sum_{(k,h)\in E(\mathcal{C}_{\mathcal{Q}})}\bigl[F_{kh}(x_k,x_h)+F_{hk}(x_h,x_k)\bigr].$$

In this case, the condition (A3) is replaced by the following:

$$\sum_{(h,k)\in E(\mathcal{C}_Q)} \left[F_{kh}(e_k,e_h) + F_{hk}(e_h,e_k\right] \leq 0. \tag{3.5}$$

Consequently, we get the following corollary.

Corollary 3.3 *Suppose that digraph* $(\mathcal{G},A)$ *is balanced. Then the conclusion of Theorem* 3.1 *holds if* (2.9) *is replaced by* (3.5).

3.2 Coefficients-type theorem

In this subsection, based on the Lyapunov method, a sufficient criterion of exponential outer synchronization which is easily verifiable is established in the form of coefficients.

Theorem 3.4 *Suppose that the following conditions hold*:

(B1) *Let* $p \geq 2$ *and the digraph* $(\mathcal{G},A)$ *is strongly connected. Suppose there are positive constants* α_k, β_k, *such that*

$$e_k^T\bigl[f_k\bigl(y_k(t)\bigr) - f_k\bigl(x_k(t)\bigr)\bigr] \leq -\alpha_k\bigl|e_k(t)\bigr|^2 \tag{3.6}$$

and

$$\bigl|g_k\bigl(y_k\bigl(t-\tau_k(t)\bigr)\bigr) - g_k(x_k\bigl(t-\tau_k(t)\bigr)\bigr| \leq \beta_k\bigl|e_k\bigl(t-\tau_k(t)\bigr)\bigr|. \tag{3.7}$$

(B2) *There are constants* A_{kh}, *such that*

$$\bigl|H_{kh}\bigl(y_k\bigl(t-\tau^{(kh)}(t)\bigr)\bigr) - H_{kh}\bigl(x_k\bigl(t-\tau^{(kh)}(t)\bigr)\bigr)\bigr| \leq A_{kh}\bigl|e_h\bigl(t-\tau^{(kh)}(t)\bigr)\bigr|. \tag{3.8}$$

(B3) *Suppose that*

$$\beta_k + \sum_{h=1}^{L} A_{kh} < \sigma_k(1-\bar{\tau}) \tag{3.9}$$

holds, where $\sigma_k = p\alpha_k - (p-1)\beta_k - (p-1)\sum_{h=1}^{L} A_{kh} - p(p-1)|\sigma|^2$.

Then the networks (2.2) *and* (2.3) *can reach ME-synchronization*.

Proof Take the following function as the Lyapunov function of the kth vertex:

$$V_k(e_k) = |e_k|^p. \tag{3.10}$$

Then it follows easily by (2.8) and the condition (B1) that

$$\begin{aligned}
&\mathcal{L}V_k(e_k) \\
&\quad = p\bigl|e_k(t)\bigr|^{p-2}e_k^T(t)\bigl[f_k\bigl(y_k(t),t\bigr) - f_k\bigl(x_k(t),t\bigr) + g_k\bigl(y_k\bigl(t-\tau_k(t)\bigr),t\bigr) \\
&\qquad - g_k\bigl(x_k\bigl(t-\tau_k(t)\bigr),t\bigr)\bigr] + p\bigl|e_k(t)\bigr|^{p-2}e_k^T(t)
\end{aligned}$$

$$
\begin{aligned}
&\times \sum_{h=1}^{L}\bigl[\bigl(H_{kh}\bigl(y_h\bigl(t-\tau^{(kh)}(t)\bigr),t\bigr)-H_{kh}\bigl(x_h\bigl(t-\tau^{(kh)}(t)\bigr),t\bigr)\bigr)\bigr] \\
&+\frac{1}{2}\operatorname{trace}\bigl(\sigma e_k(t)\bigr)^T\bigl(p\bigl|e_k(t)\bigr|^{p-2}I+p(p-2)\bigl|e_k(t)\bigr|^{p-4}e_k^T(t)e_k(t)\bigr)\sigma e_k(t) \\
\le{}& -p\bigl|e_k(t)\bigr|^{p-2}\alpha_k\bigl|e_k(t)\bigr|^2+p\bigl|e_k(t)\bigr|^{p-2}e_k^T(t)\beta_k\bigl|e_k\bigl(t-\tau_k(t)\bigr)\bigr| \\
&+p\bigl|e_k(t)\bigr|^{p-2}e_k^T(t)\sum_{h=1}^{L}\bigl[\bigl(H_{kh}\bigl(y_h\bigl(t-\tau^{(kh)}(t)\bigr),t\bigr)-H_{kh}\bigl(x_h\bigl(t-\tau^{(kh)}(t)\bigr),t\bigr)\bigr)\bigr] \\
&+p(p-1)|\sigma|^2\bigl|e_k(t)\bigr|^p \\
={}&\bigl[-\alpha_k p+p(p-1)|\sigma|^2\bigr]\bigl|e_k(t)\bigr|^p+\beta_k p\bigl|e_k(t)\bigr|^{p-1}\bigl|e_k\bigl(t-\tau_k(t)\bigr)\bigr| \\
&+p\bigl|e_k(t)\bigr|^{p-2}e_k^T(t)\sum_{h=1}^{L}\bigl[\bigl(H_{kh}\bigl(y_h\bigl(t-\tau^{(kh)}(t)\bigr),t\bigr)-H_{kh}\bigl(x_h\bigl(t-\tau^{(kh)}(t)\bigr),t\bigr)\bigr)\bigr]. \qquad (3.11)
\end{aligned}
$$

By using the Young inequality

$$
|a|^p|b|^q \le \varepsilon|a|^{p+q}+\frac{q}{p+q}\left[\frac{p}{\varepsilon(p+q)}\right]^{\frac{p}{q}}|b|^{p+q}
$$

for any $a,b\in\mathbb{R}$ and any constants $p,q,\varepsilon>0$, we can compute that

$$
\bigl|e_k(t)\bigr|^{p-1}\bigl|e_k\bigl(t-\tau_k(t)\bigr)\bigr| \le \frac{p-1}{p}\bigl|e_k(t)\bigr|^p+\frac{1}{p}\bigl|e_k\bigl(t-\tau_k(t)\bigr)\bigr|^p, \qquad (3.12)
$$

and combining with (3.8), the following can be obtained:

$$
\begin{aligned}
&p\bigl|e_k(t)\bigr|^{p-2}e_k^T(t)\sum_{h=1}^{L}\bigl[\bigl(H_{kh}\bigl(y_h\bigl(t-\tau^{(kh)}(t)\bigr),t\bigr)-H_{kh}\bigl(x_h\bigl(t-\tau^{(kh)}(t)\bigr),t\bigr)\bigr)\bigr] \\
&\quad\le p\bigl|e_k(t)\bigr|^{p-1}\sum_{h=1}^{L}A_{kh}\bigl|e_h\bigl(t-\tau^{(kh)}(t)\bigr),t\bigr| \\
&\quad\le (p-1)\sum_{h=1}^{L}A_{kh}\bigl|e_k(t)\bigr|^p+\sum_{h=1}^{L}A_{kh}\bigl|e_k\bigl(t-\tau^{(kh)}(t)\bigr)\bigr|^p. \qquad (3.13)
\end{aligned}
$$

Then we have

$$
\begin{aligned}
\mathcal{L}V_k(e_k) \le{}& \bigl[-\alpha_k p+p(p-1)|\sigma|^2\bigr]\bigl|e_k(t)\bigr|^p+(p-1)\beta_k\bigl|e_k(t)\bigr|^p+\beta_k\bigl|e_k\bigl(t-\tau_k(t)\bigr)\bigr|^p \\
&+(p-1)\sum_{h=1}^{L}A_{kh}\bigl|e_k(t)\bigr|^p+\sum_{h=1}^{L}A_{kh}\bigl|e_k\bigl(t-\tau^{(kh)}(t)\bigr)\bigr|^p \\
={}&\left[-\alpha_k p+p(p-1)|\sigma|^2+(p-1)\beta_k+(p-1)\sum_{h=1}^{L}A_{kh}\right]\bigl|e_k(t)\bigr|^p \\
&+\beta_k\bigl|e_k\bigl(t-\tau_k(t)\bigr)\bigr|^p+\sum_{h=1}^{L}A_{kh}\bigl|e_k\bigl(t-\tau^{(hk)}(t)\bigr)\bigr|^p \\
&+\sum_{h=1}^{L}A_{kh}\bigl(\bigl|e_k\bigl(t-\tau^{(kh)}(t)\bigr)\bigr|^p-\bigl|e_k\bigl(t-\tau^{(hk)}(t)\bigr)\bigr|^p\bigr).
\end{aligned}
$$

Denote $\eta_k = \beta_k$, $\epsilon_{kh} = A_{kh}$, $a_{kh} = 1$, and

$$F_{kh}\big(e_k\big(t-\tau^{(hk)}(t)\big), e_h\big(t-\tau^{(kh)}(t)\big)\big) = \sum_{h=1}^{L} A_{kh}\big(\big|e_h\big(t-\tau^{(kh)(t)}\big)\big|^p - \big|e_h\big(t-\tau^{(hk)}(t)\big)\big|^p\big).$$

It is easy to see that the conditions in Definition 2.4 are satisfied, which means $V_k(e_k, t)$ is the vertex-Lyapunov function. Therefore, by Theorem 3.1, the networks (2.2) and (2.3) will reach ME-synchronization. □

4 Synchronization of stochastic time-varying delayed coupled oscillators on networks

The phenomenon of synchronization of oscillatory dynamics is observed in a wide variety of natural systems. In some physical and biological applications, the system of coupling oscillators with time-varying delay will be studied in this part by the analytical results established in the previous section.

Now, we give a digraph with L vertices. In the kth vertex there is assigned an oscillator with time-varying delay described by

$$\ddot{x}_k(t) + \alpha_k \dot{x}_k(t) + x_k(t) + \varepsilon_k x_k\big(t-\tau_k(t)\big) = 0. \tag{4.1}$$

Suppose that $H_{kh} : \mathbb{R} \times \mathbb{R}^1_+ \to \mathbb{R}^1$, $k, h \in \mathbb{L}$, represents the influence of vertex h on vertex k. We have $H_{kh} \equiv 0$ if and only if there exists no dispersal from vertex h to vertex k in $\mathcal{G}$. Hence, we get the following system:

$$\ddot{x}_k(t) + \alpha_k \dot{x}_k(t) + x_k(t) + \varepsilon_k x_k\big(t-\tau_k(t)\big) + \sum_{k=1}^{L} H_{kh}\big(x_h\big(t-\tau^{(kh)}(t)\big), t\big) = 0, \tag{4.2}$$

which is called the drive system. The response system is described by

$$\ddot{y}_k(t) + \alpha_k \dot{y}(t) + y_k(t) + \varepsilon_k y_k\big(t-\tau_k(t)\big) + \sum_{h=1}^{L} H_{kh}\big(y_h\big(t-\tau^{(kh)}(t)\big), t\big) = \sigma\big(y_k(t) - x_k(t)\big)\dot{B}(t). \tag{4.3}$$

For the convenience of analysis and calculation, let $\bar{x}_k(t) = \dot{x}_k(t) + \eta_k x_k(t)$, where $\eta_k > 0$. Then the system (4.2) can be rewritten in the form of a first order system:

$$\begin{aligned} \dot{x}_k(t) &= \bar{x}_k(t) - \eta_k x_k(t), \\ \dot{\bar{x}}_k(t) &= (-\alpha_k + \eta_k)\bar{x}_k(t) + \big(\alpha_k\eta_k - \eta_k^2 - 1\big)x_k(t) - \varepsilon_k x_k\big(t-\tau(t)\big) \\ &\quad - \sum_{k=1}^{L} H_{kh}\big(x_h\big(t-\tau^{(kh)}(t)\big), t\big). \end{aligned}$$

Denote $X_k(t) = (x_k(t), \bar{x}_k(t))^T$, $X_k(t-\tau_k(t)) = (x_k(t-\tau_k(t)), 0)^T$, $F_k(X_k(t)) = (\bar{x}_k(t) - \eta x_k(t), (-\alpha_k + \eta_k)\bar{x}_k(t) + (\alpha_k\eta_k - \eta_k^2 - 1)x_k)^T$, $G_k(X_k(t-\tau_k(t))) = (0, \varepsilon_k x_k(t-\tau_k(t)))^T$, $\bar{H}_{kh}(X_k(t)) =$

$(0, -H_{kh}(x_h(t-\tau^{(kh)}(t)), t))^T$. Then the system (4.2) is rewritten as

$$\frac{\mathrm{d}X_k(t)}{\mathrm{d}t} = F_k\big(X_k(t)\big) - G_k\big(X_k\big(t-\tau_k(t)\big)\big) + \sum_{h=1}^{L} \bar{H}_{kh}\big(X_h\big(t-\tau^{(kh)}(t)\big), t\big). \tag{4.4}$$

In a similar way, let $\bar{y}_k(t) = \dot{y}(t) + \eta y_k(t)$, then the system (4.3) is rewritten as

$$\begin{aligned}\frac{\mathrm{d}Y_k(t)}{\mathrm{d}t} &= F_k\big(Y_k(t)\big) - G_k\big(Y_k\big(t-\tau_k(t)\big)\big) + \sum_{h=1}^{L} \bar{H}_{kh}\big(Y_h\big(t-\tau^{(kh)}(t)\big), t\big) \\ &\quad + \sigma\big(Y_k(t) - X_k(t)\big)\mathrm{d}B(t), \end{aligned} \tag{4.5}$$

where $Y_k(t) = (y_k(t), \bar{y}_k(t))^T$, $Y_k(t-\tau_k(t)) = (y_k(t-\tau_k(t)), 0)^T$, $F_k(Y_k(t)) = (\bar{y}_k(t) - \eta y_k(t), (-\alpha_k + \eta_k)\bar{y}_k(t) + (\alpha_k\eta_k - \eta_k^2 - 1)y_k)^T$, $G_k(Y_k(t-\tau_k(t))) = (0, \varepsilon_k y_k(t-\tau_k(t)))^T$, $\bar{H}_{kh}(Y_k(t)) = (0, -H_{kh}(y_h(t-\tau^{(kh)}(t)), t))^T$.

Theorem 4.1 *Let the digraph* $(\mathcal{G}, A)$ *be strongly connected. Suppose that the following conditions hold*:

(C1) *There are constants* $A_{kh} > 0$, *such that*

$$\begin{aligned}&\big|H_{kh}\big(y_h\big(t-\tau^{(kh)}(t)\big)\big) - H_{kh}\big(x_h\big(t-\tau^{(kh)}(t)\big)\big)\big| \\ &\quad \le A_{kh}\big|y_h\big(t-\tau^{(kh)}(t)\big) - x_h\big(t-\tau^{(kh)}(t)\big)\big|.\end{aligned} \tag{4.6}$$

(C2) *The drive system* (4.2) *and the response system* (4.3) *satisfy*

$$\eta_k(\alpha_k - \eta_k) \le 1, \qquad \eta_k \ge \frac{1}{2}\alpha_k, \qquad \varepsilon_k + \sum_{h=1}^{L} A_{kh} < \sigma_k(1-\bar{\tau}). \tag{4.7}$$

Then the drive system (4.2) *and the response system* (4.3) *will reach ME-synchronization.*

Proof Let $E_k(t) = Y_k(t) - X_k(t) = (y_k(t) - x_k(t), \bar{y}_k(t) - \bar{x}_k(t))^T = (e_k(t), \bar{e}_k(t))^T$. Then the error system of the drive system (4.2) and the response system (4.3) can be rewritten as

$$\begin{aligned}\mathrm{d}E_k(t)\,\mathrm{d}t = \Bigg[&F_k\big(Y_k(t)\big) - F_k\big(X_k(t)\big) - G_k\big(Y_k\big(t-\tau_k(t)\big)\big) + G_k\big(X_k\big(t-\tau_k(t)\big)\big) \\ &+ \sum_{h=1}^{L} \bar{H}_{kh}\big(Y_h\big(t-\tau^{(kh)}(t)\big), t\big) - \sum_{h=1}^{L} \bar{H}_{kh}\big(X_h\big(t-\tau^{(kh)}(t)\big), t\big)\Bigg]\mathrm{d}t \\ &+ \sigma\big(E_k(t)\big)\,\mathrm{d}B(t).\end{aligned}$$

Then we will verify that the error system satisfies Theorem 3.4.

From the condition (C2), we obtain

$$\begin{aligned}E_k^T(t)F_k\big(E_k(t), t\big) &= e_k(t)\bar{e}_k(t) - \eta_k e_k^2(t) + (-\alpha_k + \eta_k)\bar{e}_k^2(t) \\ &\quad + \big(\alpha_k\eta_k - \eta_k^2 - 1\big)\bar{e}_k(t)e_k(t) - \varepsilon_k\bar{e}_k(t)e_k\big(t-\tau(t)\big)\end{aligned}$$

$$
\begin{aligned}
&\le -\left(\eta_k - \frac{1}{2}(\alpha_k\eta_k - \eta_k^2)\varepsilon^2\right)e_k^2(t) \\
&\quad - \left(\alpha_k - \eta_k - \frac{1}{2\varepsilon^2}(\alpha_k\eta_k - \eta_k^2) + \frac{\varepsilon_k}{2}\right)\bar{e}_k^2(t) - \frac{\varepsilon_k}{2}e_k^2\big(t-\tau(t)\big) \\
&\le -\left(\eta_k - \frac{1}{2}(\alpha_k\eta_k - \eta_k^2)\varepsilon^2\right)e_k^2(t) \\
&\quad - \left(\alpha_k - \eta_k - \frac{1}{2\varepsilon^2}(\alpha_k\eta_k - \eta_k^2) + \frac{\varepsilon_k}{2}\right)\bar{e}_k^2(t) \\
&\le -\frac{1}{2}\eta_k e_k^2(t) - \frac{1}{2}\eta_k\bar{e}_k^2(t) \\
&= -\frac{1}{2}\eta_k\big|e_k(t)\big|^2.
\end{aligned}
\tag{4.8}
$$

From this, it is obvious that (3.6) holds.

Also we have

$$
\big|G_k\big(Y_k\big(t-\tau_k(t)\big)\big) - G_k\big(X_k\big(t-\tau_k(t)\big)\big)\big| = \varepsilon_k e_k\big(t-\tau_k(t)\big),
$$

which means (3.7) holds.

Hence, from Theorem 3.4, the drive system (4.2), and the response system (4.3) we obtain ME-synchronization. □

5 Numerical test

In this section, we will consider a complex network with four vertices as a numerical example. The dynamic nature of each vertex can be described as coupled oscillators with time-varying delay,

$$
\begin{aligned}
&\ddot{x}_k(t) + \alpha_k\dot{x}_k(t) + x_k(t) + \varepsilon_k x_k\big(t-\tau_k(t)\big) + \sum_{k=1}^{L} H_{kh}\big(x_h\big(t-\tau^{(kh)}(t)\big),t\big) = 0, \\
&\quad k = 1,2,3,4,
\end{aligned}
\tag{5.1}
$$

in which

$$
H_{kh}\big(x_h\big(t-\tau^{(kh)}(t)\big),t\big) = \theta x_h(t - \bar{\tau}\arctan t), \qquad \bar{\tau} = 0.5.
$$

It is not difficult to see that $H_{kh}(x_h(t-\tau^{(kh)}(t)),t)$ satisfies the condition (C1) in Theorem 4.1 when $\theta \in (0, 0.7)$. So the drive system can be represented as

$$
\begin{cases}
\ddot{x}_1(t) + \alpha_1\dot{x}_1(t) + x_1(t) + \varepsilon_1 x_1(t - 0.5(\sin t + 1)) + \theta x_2(t - 0.5\arctan t) = 0, \\
\ddot{x}_2(t) + \alpha_2\dot{x}_2(t) + x_2(t) + \varepsilon_2 x_2(t - 0.5(\cos t + 1)) + \theta x_1(t - 0.3\arctan t) = 0, \\
\ddot{x}_3(t) + \alpha_3\dot{x}_3(t) + x_3(t) + \varepsilon_3 x_3(t - 0.6(\sin t + 1)) + \theta x_4(t - 0.5\arctan t) = 0, \\
\ddot{x}_4(t) + \alpha_4\dot{x}_4(t) + x_4(t) + \varepsilon_4 x_4(t - 0.6(\cos t + 1)) + \theta x_3(t - 0.2\arctan t) = 0,
\end{cases}
\tag{5.2}
$$

where $\varepsilon_k = 0.1$, $k = 1,2,3,4$. Set $\alpha_1 = 1.25$, $\alpha_2 = \alpha_3 = 1.5$, $\alpha_4 = 1.75$, respectively.

The response system under the effect of white noise can be represented as

$$\begin{cases} \ddot{y}_1(t) + \alpha_1\dot{y}_1(t) + y_1(t) + \varepsilon_1 y_1(t - 0.5(\sin t + 1)) + \theta y_2(t - 0.5\arctan t) \\ \quad = \sigma(y_1(t) - x_1(t))\dot{B}(t), \\ \ddot{y}_2(t) + \alpha_2\dot{y}_2(t) + y_2(t) + \varepsilon_2 y_2(t - 0.5(\cos t + 1)) + \theta y_1(t - 0.3\arctan t) \\ \quad = \sigma(y_2(t) - x_2(t))\dot{B}(t), \\ \ddot{y}_3(t) + \alpha_3\dot{y}_3(t) + y_3(t) + \varepsilon_3 y_3(t - 0.6(\sin t + 1)) + \theta y_4(t - 0.5\arctan t) \\ \quad = \sigma(y_3(t) - x_3(t))\dot{B}(t), \\ \ddot{y}_4(t) + \alpha_4\dot{y}_4(t) + y_4(t) + \varepsilon_4 y_4(t - 0.6(\cos t + 1)) + \theta y_3(t - 0.25\arctan t) \\ \quad = \sigma(y_4(t) - x_4(t))\dot{B}(t), \end{cases} \tag{5.3}$$

where ε_k and α_k $(k = 1, 2, 3, 4)$ are the same as the parameters in the drive system. $\sigma = 0.1$ is the strength of the coupling. To get a feeling of what is going on, the dynamical behaviors of the systems (5.2) and (5.3) can be seen Figures 1 and 2.

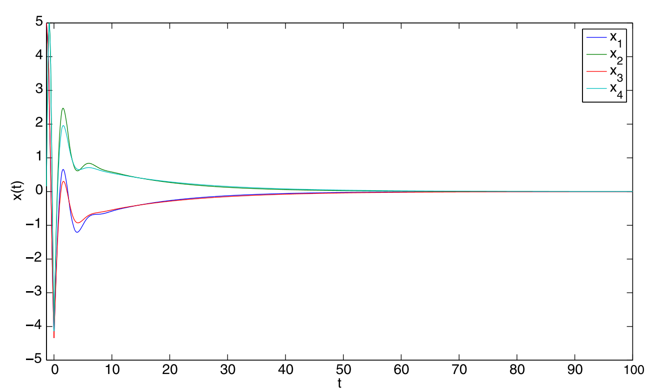

Figure 1 The dynamical behavior of the system (5.2).

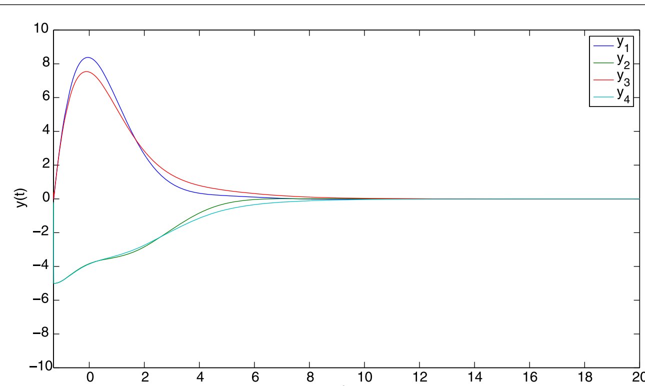

Figure 2 The dynamical behavior of the system (5.3).

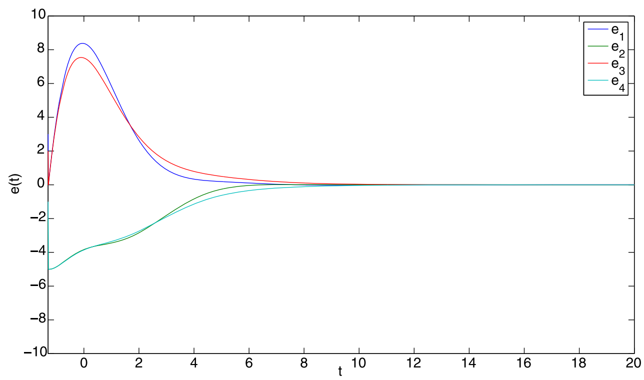

Figure 3 The dynamical behavior of the system (5.4).

Let $e_k = y_k - x_k$, $k = 1, 2, 3, 4$. Then the error system is

$$\begin{cases} \ddot{e}_1(t) + 1.25\dot{e}_1(t) + e_1(t) + 0.1e_1(t - 0.5(\sin t + 1)) + \theta e_2(t - 0.5\arctan t) \\ \quad = 0.1e_1(t)\dot{B}(t), \\ \ddot{e}_2(t) + 1.5\dot{e}_2(t) + e_2(t) + 0.1e_2(t - 0.5(\cos t + 1)) + \theta e_1(t - 0.3\arctan t) \\ \quad = 0.1e_2(t)\dot{B}(t), \\ \ddot{e}_3(t) + 1.5\dot{e}_3(t) + e_3(t) + 0.1e_3(t - 0.6(\sin t + 1)) + \theta e_4(t - 0.5\arctan t) \\ \quad = 0.1e_3(t)\dot{B}(t), \\ \ddot{e}_4(t) + 1.75\dot{e}_4(t) + e_4(t) + 0.1e_4(t - 0.6(\cos t + 1)) + \theta e_3(t - 0.25\arctan t) \\ \quad = 0.1e_4(t)\dot{B}(t). \end{cases} \tag{5.4}$$

The trivial solution of the system (5.4) is presented in Figure 3. In fact, we can see clearly from Figure 3 that the trivial solution of the system (5.4) is exponentially stable. The numerical results show the effectiveness and feasibility of the developed results.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
WL proposed the model and completed the main part of this manuscript, TJ checked all the theorems and polished the language, and JF enhanced the revised version. All the authors read and approved the manuscript.

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