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Solutions of the *n*th-order Cauchy difference equation on groups

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Abstract

Let $(G,\cdot)$ be a group, $(H,+)$ be an Abelian group, and $f: G \to H$ be a function. In this paper, for a positive integer n, we first give a representation of nth-order Cauchy difference of f via the function as

$$C^{(n)}f(x_1,x_2,\ldots,x_n,x_{n+1}) = \sum_{1\le m\le n+1}(-1)^{n+1-m}\sum_{1\le i_1<i_2<\cdots<i_m\le n+1} f(x_{i_1}x_{i_2}\cdots x_{i_m}),$$

where $x_1,x_2,\ldots,x_{n+1}\in G$. Then, based on the representation, we get some special solutions of $C^{(n)}f=0$ on free groups. Moreover, sufficient and necessary conditions on symmetric groups and finite cyclic groups are also obtained.

MSC: 39B52; 39A70

Keywords: Cauchy difference; free group; symmetric group; cyclic group

1 Introduction

It is well known that the solutions to Jensen's functional equation

$$f(x+y)+f(x-y)=2f(x) \tag{1.1}$$

are just all linear functions $f(x)=cx+d$ if we assume that f are continuous, and they are the set of all homomorphisms (when $d=0$) on real line $\mathbb{R}$ as an additive group. Let $(G,\cdot)$ be a group, and $(H,+)$ be an Abelian group. Denote by $e\in G$ and $0\in H$ the identity elements, respectively. In [1], it was pointed out that the set of solutions is not equivalent to all homomorphisms on a general group. Therefore, finding out the solutions to equation (1.1) on groups becomes an interesting problem (see [1, 2] and references therein). Note that these solutions are related to their Cauchy differences [3–6]. For a function $f: G\to H$, its Cauchy differences $C^{(m)}f$ are defined by

$$C^{(0)}f=f, \tag{1.2}$$

$$C^{(1)}f(x_1,x_2)=f(x_1x_2)-f(x_1)-f(x_2),$$

$$\begin{aligned} &C^{(m+1)}f(x_1,x_2,\ldots,x_{m+2}) \\ &\quad = C^{(m)}f(x_1x_2,x_3,\ldots,x_{m+2})-C^{(m)}f(x_1,x_3,\ldots,x_{m+2})-C^{(m)}f(x_2,x_3,\ldots,x_{m+2}). \end{aligned} \tag{1.3}$$

The first-order Cauchy difference $C^{(1)}f$ will be abbreviated as Cf. In general, $C^{(m+1)}f = 0$ if and only if $C^{(m)}f$ is m-additive, and $C^{(m)}f = 0$ implies $C^{(n)}f = 0$ for $n \geq m$ [7]. Fischer and Heuvers [7–9] mentioned that any generalized polynomial f with $C^{(m)}f = 0$ has a unique representation of the form $f(x) = f_1(x) + \cdots + f_m(x)$, where each $f_j(x)$, $j = 1, \ldots, m$, is a j-form. Besides the properties of solutions satisfying (1.3), we are also concerned with its general solutions. In [10, 11], by using the reduction formulas and relations given in [1, 2], Ng provided the general solution of the second-and third-order Cauchy difference equations on free groups. Some previous results on second order were extended to high order in [11]. Moreover, for the general solution of the third-order Cauchy difference equation on symmetric groups and finite cyclic groups, see reference [12].

Plentiful results were also devoted to the generalized Cauchy difference functional equations of the form

$$f(x) + f(y) - f(x + y) = g\big(H(x, y)\big), \tag{1.4}$$

where H is given, and f, g are unknown. Under regularity assumptions on H and a particular solution f_0, g_0, Ebanks [5, 6] investigated the general solution (f, g) of equation (1.4). For particular forms of $g(H(x, y))$, the existence and stability of solutions to equation (1.4) are studied extensively in, *e.g.*, [13–23].

Note that equations (1.3) and (1.4) are concerning functions of one variable. Also, for a map $F : G^n \to H$, we define the mth partial Cauchy difference with respect to the ith variable ($1 \leq i \leq n$), which is connected to the representation of the generalized polynomial F (see [7–9, 24]).

It is natural to consider the general expression of nth-order Cauchy difference $C^{(n)}f$ and determine the solutions to equation

$$C^{(n)}f = 0, \tag{1.5}$$

which becomes the motivation of this paper. Remark that equation (1.5) for free group with just one generator has been solved in [11]. In our paper, the purpose is to determine all solutions to equation (1.5) on some given groups. For simplicity, the general solution to equation (1.5) will be denoted by

$$\operatorname{Ker} C^{(n)}(G, H) = \big\{f : G \to H \mid C^{(n)}f = 0\big\}. \tag{1.6}$$

Obviously, $\operatorname{Ker} C^{(n)}(G, H)$ is an Abelian group under the pointwise addition of functions, and $\operatorname{Hom}(G, H) \subseteq \operatorname{Ker} C^{(n)}(G, H)$.

2 Some properties on solutions

In this section, we study properties of solutions for nth-order Cauchy differences.

Proposition 1 *For a positive integer n, the nth-order Cauchy difference $C^{(n)}f$ can be expressed in terms of f as*

$$C^{(n)}f(x_1, x_2, \ldots, x_n, x_{n+1}) = \sum_{m=1}^{n+1} (-1)^{n+1-m} \sum_{1 \leq i_1 < i_2 < \cdots < i_m \leq n+1} f(x_{i_1} x_{i_2} \cdots x_{i_m}). \tag{2.1}$$

Proof Claim (2.1) is true for $n = 1, 2$ from (1.2)-(1.3). Suppose that (2.1) holds for $n - 1$. Note that

$$\begin{aligned} &C^{(n)}f(x_1, x_2, \ldots, x_n, x_{n+1}) \\ &\quad = C^{(n-1)}f(x_1x_2, x_3, \ldots, x_n, x_{n+1}) \\ &\qquad - C^{(n-1)}f(x_1, x_3, \ldots, x_n, x_{n+1}) - C^{(n-1)}f(x_2, x_3, \ldots, x_n, x_{n+1}) \end{aligned}$$

by (1.3). Then, we have

$$\begin{aligned} &C^{(n-1)}f(x_1x_2, x_3, \ldots, x_n, x_{n+1}) \\ &\quad = (-1)^{n-1}f(x_1x_2) + (-1)^{n-1}\sum_{3\le i_1\le n+1} f(x_{i_1}) \\ &\qquad + (-1)^{n-2}\sum_{3\le i_1\le n+1} f(x_1x_2x_{i_1}) + (-1)^{n-2}\sum_{3\le i_1<i_2\le n+1} f(x_{i_1}x_{i_2}) \\ &\qquad + \cdots + (-1)^2\sum_{3\le i_1<i_2<\cdots<i_{n-3}\le n+1} f(x_1x_2x_{i_1}x_{i_2}\cdots x_{i_{n-3}}) \\ &\qquad + (-1)^2\sum_{3\le i_1<i_2<\cdots<i_{n-2}\le n+1} f(x_{i_1}x_{i_2}\cdots x_{i_{n-2}}) \\ &\qquad + (-1)^1\sum_{3\le i_1<i_2<\cdots<i_{n-2}\le n+1} f(x_1x_2x_{i_1}x_{i_2}\cdots x_{i_{n-2}}) + (-1)^1 f(x_3x_4\cdots x_{n+1}) \\ &\qquad + f(x_1x_2x_3\cdots x_{n+1}), \\ &C^{(n-1)}f(x_1, x_3, \ldots, x_n, x_{n+1}) \\ &\quad = (-1)^{n-1}f(x_1) + (-1)^{n-1}\sum_{3\le i_1\le n+1} f(x_{i_1}) \\ &\qquad + (-1)^{n-2}\sum_{3\le i_1\le n+1} f(x_1x_{i_1}) + (-1)^{n-2}\sum_{3\le i_1<i_2\le n+1} f(x_{i_1}x_{i_2}) \\ &\qquad + \cdots + (-1)^2\sum_{3\le i_1<i_2<\cdots<i_{n-3}\le n+1} f(x_1x_{i_1}x_{i_2}\cdots x_{i_{n-3}}) \\ &\qquad + (-1)^2\sum_{3\le i_1<i_2<\cdots<i_{n-2}\le n+1} f(x_{i_1}x_{i_2}\cdots x_{i_{n-2}}) \\ &\qquad + (-1)^1\sum_{3\le i_1<i_2<\cdots<i_{n-2}\le n+1} f(x_1x_{i_1}x_{i_2}\cdots x_{i_{n-2}}) + (-1)^1 f(x_3x_4\cdots x_{n+1}) \\ &\qquad + f(x_1x_3\cdots x_{n+1}), \end{aligned}$$

and

$$\begin{aligned} &C^{(n-1)}f(x_2, x_3, \ldots, x_n, x_{n+1}) \\ &\quad = (-1)^{n-1}f(x_2) + (-1)^{n-1}\sum_{3\le i_1\le n+1} f(x_{i_1}) \\ &\qquad + (-1)^{n-2}\sum_{3\le i_1\le n+1} f(x_2x_{i_1}) + (-1)^{n-2}\sum_{3\le i_1<i_2\le n+1} f(x_{i_1}x_{i_2}) \\ &\qquad + \cdots + (-1)^2\sum_{3\le i_1<i_2<\cdots<i_{n-3}\le n+1} f(x_2x_{i_1}x_{i_2}\cdots x_{i_{n-3}}) \end{aligned}$$

$$+(-1)^2 \sum_{3\le i_1<i_2<\cdots<i_{n-2}\le n+1} f(x_{i_1}x_{i_2}\cdots x_{i_{n-2}})$$
$$+(-1)^1 \sum_{3\le i_1<i_2<\cdots<i_{n-2}\le n+1} f(x_2x_{i_1}x_{i_2}\cdots x_{i_{n-2}}) + (-1)^1 f(x_3x_4\cdots x_{n+1})$$
$$+f(x_2x_3\cdots x_{n+1}).$$

Summing over these three equalities, we get

$$C^{(n)}f(x_1,x_2,\ldots,x_n,x_{n+1}) = \sum_{m=1}^{n+1}(-1)^{n+1-m} \sum_{1\le i_1<i_2<\cdots<i_m\le n+1} f(x_{i_1}x_{i_2}\cdots x_{i_m}),$$

and this gives (2.1). □

Proposition 2 *If $f \in \operatorname{Ker} C^{(n)}(G,H)$, then the following properties are valid.*

(i) *For $i = 1,2,\ldots,n-1$ and $j = 1,2,\ldots,i+1$, we have*

$$C^{(i)}f(x_1,x_2,\ldots,x_{j-1},e,x_{j+1},\ldots,x_{i+1}) = 0. \tag{2.2}$$

In particular,

$$f(e) = 0. \tag{2.3}$$

(ii) *$C^{(n-1)}f$ is a homomorphism with respect to each variable.*

Proof We first check (2.3). For $f \in \operatorname{Ker} C^{(n)}(G,H)$, we take $x_1 = e$ in (2.1). Then, it follows from (1.3) that

$$\begin{aligned}
0 &= C^{(n)}f(e,x_2,\ldots,x_{n+1})\\
&= C^{(n-1)}f(e,x_2,x_3,\ldots,x_{n+1}) - C^{(n-1)}f(x_2,x_3,\ldots,x_{n+1}) - C^{(n-1)}f(e,x_3,\ldots,x_{n+1})\\
&= (-1)C^{(n-1)}f(e,x_3,\ldots,x_{n+1}) = (-1)^2C^{(n-2)}f(e,x_4,\ldots,x_{n+1}) = \cdots\\
&= (-1)^{n-1}Cf(e,x_{n+1}) = (-1)^n f(e),
\end{aligned}$$

which gives (2.3).

Obviously, (2.2) is true for $i = 1,2$ by (1.3) and (2.3). Assume that (2.2) holds for all numbers smaller than $i \ge 3$. By induction, for $j = 1$, we have

$$\begin{aligned}
&C^{(i)}f(e,x_2,x_3,\ldots,x_{i+1})\\
&\quad= C^{(i-1)}f(x_2,x_3,\ldots,x_{i+1}) - C^{(i-1)}f(x_2,x_3,\ldots,x_{i+1}) - C^{(i-1)}f(e,x_3,\ldots,x_{i+1})\\
&\quad= -C^{(i-1)}f(e,x_3,\ldots,x_{i+1}) = 0;
\end{aligned}$$

for $j = 2$,

$$\begin{aligned}
&C^{(i)}f(x_1,e,x_3,\ldots,x_{i+1})\\
&\quad= C^{(i-1)}f(x_1,x_3,\ldots,x_{i+1}) - C^{(i-1)}f(e,x_3,\ldots,x_{i+1}) - C^{(i-1)}f(x_1,x_3,\ldots,x_{i+1})\\
&\quad= -C^{(i-1)}f(e,x_3,\ldots,x_{i+1}) = 0;
\end{aligned}$$

and

$$\begin{aligned}
&C^{(i)}f(x_1,x_2,\dots,x_{j-1},e,x_{j+1},\dots,x_{i+1})\\
&\quad= C^{(i-1)}f(x_1x_2,x_3,\dots,x_{j-1},e,x_{j+1},\dots,x_{i+1})\\
&\qquad- C^{(i-1)}f(x_1,x_3,\dots,x_{j-1},e,x_{j+1},\dots,x_{i+1})\\
&\qquad- C^{(i-1)}f(x_2,x_3,\dots,x_{j-1},e,x_{j+1},\dots,x_{i+1})\\
&\quad= 0
\end{aligned}$$

in the case $j \geq 3$. This confirms (2.2).

We infer from (1.3) and (1.6) that $C^{(n-1)}f$ is a homomorphism with respect to the first variable. Then by the symmetry among the variables the Cauchy difference of $C^{(n-1)}f$ in its first variable is equivalent to the other variable, and therefore, (ii) is proved. □

Remark 1 For any function $f : G \to H$, the following statements are equivalent:

(i) The function $f \in \operatorname{Ker} C^{(n)}(G,H)$.

(ii) $C^{(n-1)}f$ is a homomorphism with respect to the j-variable, $j \in \{1,2,\dots,n+1\}$.

Next, we give two useful lemmas.

Lemma 1 (Lemma 2.4 in [10]) *The following identity is valid for any function $f : G \to H$ and $\ell \in \mathbb{N}$:*

$$f(x_1x_2\cdots x_\ell) = \sum_{m\leq \ell}\sum_{1\leq i_1<i_2<\cdots<i_m\leq \ell} C^{(m-1)}f(x_{i_1},x_{i_2},\dots,x_{i_m}). \tag{2.4}$$

Lemma 2 (Proposition 2.4 in [11]) *Let n be a positive integer. If $f \in \operatorname{Ker} C^{(n)}(G,H)$, then for all $x \in G$ and $p \in \mathbb{Z}$, we have*

$$f\bigl(x^p\bigr) = \begin{cases} \sum_{j=0}^{n-1}\frac{p(p-1)\cdots(p-j)}{(j+1)!}C^{(j)}f(\underbrace{x,x,\dots,x}_{j+1}), & p\leq 0 \text{ or } p\geq n,\\ \sum_{j=0}^{p-1}\frac{p(p-1)\cdots(p-j)}{(j+1)!}C^{(j)}f(\underbrace{x,x,\dots,x}_{j+1}), & 0<p<n. \end{cases} \tag{2.5}$$

The following statement is a vision of Lemma 1 under the restriction $f \in \operatorname{Ker} C^{(n)}(G,H)$.

Theorem 1 *Suppose that $f \in \operatorname{Ker} C^{(n)}(G,H)$. Then the following identities are valid.*

(i) *If $l \geq n$, then, for $m_k \in \mathbb{Z}$ and $x_k \in G$, $i = 1,2,\dots,l$, such that $x_k \neq x_{k+1}$, we have*

$$\begin{aligned}
&f\bigl(x_1^{m_1}x_2^{m_2}\cdots x_l^{m_l}\bigr)\\
&\quad= \sum_{k=1}^{l}\left(\sum_{j=0}^{n-1}\frac{m_k(m_k-1)\cdots(m_k-j)}{(j+1)!}C^{(j)}f(\underbrace{x_k,x_k,\dots,x_k}_{j+1})\right)\\
&\qquad+ \sum_{i=2}^{n-1}\sum_{1\leq k_1<k_2<\cdots<k_i\leq l} C^{(i-1)}f\bigl(x_{k_1}^{m_{k_1}},x_{k_2}^{m_{k_2}},\dots,x_{k_i}^{m_{k_i}}\bigr)\\
&\qquad+ m_{k_1}m_{k_2}\cdots m_{k_n}\sum_{1\leq k_1<k_2<\cdots<k_n\leq l} C^{(n-1)}f(x_{k_1},x_{k_2},\dots,x_{k_n}).
\end{aligned} \tag{2.6}$$

(ii) *If $l < n$, then, for $m_k \in \mathbb{Z}$ and all $x_k \in G$, $k = 1, 2, \ldots, l$, such that $x_k \neq x_{k+1}$, we have*

$$\begin{aligned} f\bigl(x_1^{m_1} x_2^{m_2} \cdots x_l^{m_l}\bigr) &= \sum_{k=1}^{l} \left(\sum_{j=0}^{n-1} \frac{m_k(m_k-1)\cdots(m_k-j)}{(j+1)!} C^{(j)} f(\underbrace{x_k, x_k, \ldots, x_k}_{j+1}) \right) \\ &\quad + \sum_{i=2}^{l} \sum_{1\le k_1<k_2<\cdots<k_i\le l} C^{(i-1)} f\bigl(x_{k_1}^{m_{k_1}}, x_{k_2}^{m_{k_2}}, \ldots, x_{k_i}^{m_{k_i}}\bigr). \end{aligned} \tag{2.7}$$

Proof Replacing x_k in (2.4) by $x_k^{m_k}$, we have

$$f\bigl(x_1^{m_1} x_2^{m_2} \cdots x_l^{m_l}\bigr) = \sum_{i=1}^{l} \sum_{1\le k_1<k_2<\cdots<k_i\le l} C^{(i-1)} f\bigl(x_{k_1}^{m_{k_1}}, x_{k_2}^{m_{k_2}}, \ldots, x_{k_i}^{m_{k_i}}\bigr). \tag{2.8}$$

We first consider the case $l \ge n$. Vanishing of $C^{(i-1)}f$ for $i > n$ yields

$$\begin{aligned} f\bigl(x_1^{m_1} x_2^{m_2} \cdots x_l^{m_l}\bigr) &= \sum_{k=1}^{l} f\bigl(x_k^{m_k}\bigr) + \sum_{i=2}^{n-2} \sum_{1\le k_1<k_2<\cdots<k_i\le l} C^{(i-1)} f(x_{k_1}, x_{k_2}, \ldots, x_{k_i}) \\ &\quad + \sum_{1\le k_1<k_2<\cdots<k_n\le l} C^{(n-1)} f\bigl(x_{k_1}^{m_{k_1}}, x_{k_2}^{m_{k_2}}, \ldots, x_{k_n}^{m_{k_n}}\bigr). \end{aligned}$$

Therefore, by (2.5) in Lemma 2 and (ii) of Proposition 2 we have

$$f\bigl(x_k^{m_k}\bigr) = \sum_{j=0}^{n-1} \frac{m_k(m_k-1)\cdots(m_k-j)}{(j+1)!} C^{(j)} f(\underbrace{x_k, x_k, \ldots, x_k}_{j+1}), \tag{2.9}$$

$$C^{(n-1)} f\bigl(x_{k_1}^{m_{k_1}}, x_{k_2}^{m_{k_2}}, \ldots, x_{k_n}^{m_{k_n}}\bigr) = m_{k_1} m_{k_2} \cdots m_{k_n} C^{(n-1)} f(x_{k_1}, x_{k_2}, \ldots, x_{k_n}), \tag{2.10}$$

which is formula (2.6).

In the case $l < n$, (2.7) is obtained by (2.8)-(2.9) directly. This completes the whole proof. □

3 Solutions on free groups

In this section, we discuss some special solutions of $C^{(n)}f = 0$ for the free group on an alphabet $\langle \mathscr{A} \rangle$ with $|\mathscr{A}| \ge 2$.

An element $x \in \mathscr{A}$ can be written in the form

$$x = a_1^{n_1} a_2^{n_2} \cdots a_l^{n_l}, \quad \text{where } a_i \in \mathscr{A}, n_i \in \mathbb{Z}. \tag{3.1}$$

For each fixed $a \in \mathscr{A}$ and fixed pair of distinct $a, b \in \mathscr{A}$, define the functions W, W_2, W_3 by

$$W(x; a) = \sum_{a_i = a} n_i, \tag{3.2}$$

$$W_2(x; a, b) = \sum_{i<j, a_i = a, a_j = b} n_i n_j, \tag{3.3}$$

$$W_3(x;a,b) = \sum_{i>j,a_i=a,a_j=b} n_i n_j \tag{3.4}$$

along with (3.1). Referred from [1, 2], these functions are well defined. Furthermore, they satisfy the following relations:

$$W(xy;a) = W(x;a) + W(y;a), \tag{3.5}$$

$$W_2(x;a,b) = W_3(x;b,a). \tag{3.6}$$

Proposition 3 *For any fixed $a \in \mathscr{A}$ and fixed pair of distinct a, b in $\mathscr{A}$, the following assertions hold*:

(i) $W(\cdot;a)$ *belongs to* $\operatorname{Ker} C^{(n)}(\langle\mathscr{A}\rangle,\mathbb{Z})$;
(ii) $W_2(\cdot;a,b)$ *belongs to* $\operatorname{Ker} C^{(n)}(\langle\mathscr{A}\rangle,\mathbb{Z})$;
(iii) $W_3(\cdot;a,b)$ *belongs to* $\operatorname{Ker} C^{(n)}(\langle\mathscr{A}\rangle,\mathbb{Z})$.

Proof Statement (i) follows from the fact that $x \mapsto W(x;a)$ is a morphism from $\langle\mathscr{A}\rangle$ to $\mathbb{Z}$ by (3.5).

Now we consider statement (ii). Let $x_1, x_2, \ldots, x_{n+1}$ in the free group $\langle\mathscr{A}\rangle$ be written as

$$\begin{aligned} &x_1 = a_{11}^{t_{11}} a_{12}^{t_{12}} \cdots a_{1r_1}^{t_{1r_1}}, \qquad x_2 = a_{21}^{t_{21}} a_{22}^{t_{22}} \cdots a_{2r_2}^{t_{2r_2}}, \qquad \ldots, \\ &x_{n+1} = a_{n+1,1}^{t_{n+1,1}} a_{n+1,2}^{t_{n+1,2}} \cdots a_{n+1,r_{n+1}}^{t_{n+1,r_{n+1}}}. \end{aligned}$$

Then by (1.6) we have

$$\begin{aligned} &C^{(n)} W_2(x_1, x_2, \ldots, x_{n+1}; a, b) \\ &\quad= \sum_{m=1}^{n+1} (-1)^{n+1-m} \sum_{1\le l_1<l_2<\cdots<l_m\le n+1} W_2(x_{l_1} x_{l_2} \cdots x_{l_m}; a, b) \\ &\quad= (-1)^n \sum_{l=1}^{n+1} W_2(x_l; a, b) + (-1)^{n-1} \sum_{1\le l_1<l_2\le n+1} W_2(x_{l_1} x_{l_2}; a, b) \\ &\qquad+ (-1)^{n-2} \sum_{1\le l_1<l_2<l_3\le n+1} W_2(x_{l_1} x_{l_2} x_{l_3}; a, b) + \cdots \\ &\qquad+ (-1)^1 \sum_{1\le l_1<l_2<\cdots<l_n\le n+1} W_2(x_{l_1} x_{l_2} \cdots x_{l_n}; a, b) \\ &\qquad+ (-1)^0 W_2(x_1 x_2 \cdots x_{n+1}; a, b) \\ &\quad= (-1)^n \sum_{k=1}^{n+1} \sum_{i<j,a_{ki}=a,a_{kj}=b} t_{ki} t_{kj} \\ &\qquad+ (-1)^{n-1} \sum_{1\le l_1<l_2\le n+1} \Biggl(\sum_{1\le k\le 2} \sum_{i<j,a_{l_k i}=a,a_{l_k j}=b} t_{l_k i} t_{l_k j} + \sum_{a_{l_1 i}=a,a_{l_2 j}=b} t_{l_1 i} t_{l_2 j} \Biggr) \\ &\qquad+ (-1)^{n-2} \sum_{1\le l_1<l_2<l_3\le n+1} \Biggl(\sum_{1\le k\le 3} \sum_{i<j,a_{l_k i}=a,a_{l_k j}=b} t_{l_k i} t_{l_k j} \end{aligned}$$

$$
\begin{aligned}
&+ \sum_{1\le p<q\le 3} \sum_{a_{l_p i}=a, a_{l_q j}=b} t_{l_p i} t_{l_q j}\Bigg) + \cdots \\
&+ (-1)^1 \sum_{1\le l_1<l_2\cdots<l_n\le n+1} \Bigg(\sum_{1\le k\le n} \sum_{i<j, a_{l_k i}=a, a_{l_k j}=b} t_{l_k i} t_{l_k j} \\
&+ \sum_{1\le p<q\le n} \sum_{a_{l_p i}=a, a_{l_q j}=b} t_{l_p i} t_{l_q j}\Bigg) \\
&+ (-1)^0 \Bigg(\sum_{1\le k\le n+1} \sum_{i<j, a_{ki}=a, a_{kj}=b} t_{ki} t_{kj} + \sum_{1\le p<q\le n+1} \sum_{a_{pi}=a, a_{qj}=b} t_{pi} t_{qj}\Bigg) \\
= &\Bigg((-1)^n \sum_{1\le k\le n+1} \sum_{i<j, a_{ki}=a, a_{kj}=b} t_{ki} t_{kj} \\
&+ (-1)^{n-1} \sum_{1\le l_1<l_2\le n+1} \sum_{1\le k\le 2} \sum_{i<j, a_{l_k i}=a, a_{l_k j}=b} t_{l_k i} t_{l_k j} \\
&+ (-1)^{n-2} \sum_{1\le l_1<l_2<l_3\le n+1} \sum_{1\le k\le 3} \sum_{i<j, a_{l_k i}=a, a_{l_k j}=b} t_{l_k i} t_{l_k j} + \cdots \\
&+ (-1)^1 \sum_{1\le l_1<l_2<\cdots<l_n\le n+1} \sum_{1\le k\le n} \sum_{i<j, a_{l_k i}=a, a_{l_k j}=b} t_{l_k i} t_{l_k j} \\
&+ (-1)^0 \sum_{1\le k\le n+1} \sum_{i<j, a_{ki}=a, a_{kj}=b} t_{ki} t_{kj}\Bigg) \\
&+ \Bigg((-1)^{n-1} \sum_{1\le l_1<l_2\le n+1} \sum_{a_{l_1 i}=a, a_{l_2 j}=b} t_{l_1 i} t_{l_2 j} \\
&+ (-1)^{n-2} \sum_{1\le l_1<l_2<l_3\le n+1} \sum_{1\le p<q\le 3} \sum_{a_{l_p i}=a, a_{l_q j}=b} t_{l_p i} t_{l_q j} \\
&+ \cdots + (-1)^1 \sum_{1\le l_1<l_2<\cdots<l_n\le n+1} \sum_{1\le p<q\le n} \sum_{a_{l_p i}=a, a_{l_q j}=b} t_{l_p i} t_{l_q j} \\
&+ (-1)^0 \sum_{1\le p<q\le n+1} t_{pi} t_{qj}\Bigg) \\
\triangleq &\, I_1 + I_2,
\end{aligned}
$$

where

$$
\begin{aligned}
I_1 = &(-1)^n \sum_{1\le k\le n+1} \sum_{i<j, a_{ki}=a, a_{kj}=b} t_{ki} t_{kj} \\
&+ (-1)^{n-1} \sum_{1\le l_1<l_2\le n+1} \sum_{1\le k\le 2} \sum_{i<j, a_{l_k i}=a, a_{l_k j}=b} t_{l_k i} t_{l_k j} \\
&+ (-1)^{n-2} \sum_{1\le l_1<l_2<l_3\le n+1} \sum_{1\le k\le 3} \sum_{i<j, a_{l_k i}=a, a_{l_k j}=b} t_{l_k i} t_{l_k j} + \cdots \\
&+ (-1)^1 \sum_{1\le l_1<l_2<\cdots<l_n\le n+1} \sum_{1\le k\le n} \sum_{i<j, a_{l_k i}=a, a_{l_k j}=b} t_{l_k i} t_{l_k j} \\
&+ (-1)^0 \sum_{1\le k\le n+1} \sum_{i<j, a_{ki}=a, a_{kj}=b} t_{ki} t_{kj}.
\end{aligned}
\tag{3.7}
$$

By the symmetry we can see that for any $1 \le k \le n+1$ and $i < j$, the coefficient of the item $t_{ki}t_{kj}$ in (3.7) is identical and equals

$$\begin{aligned}
&(-1)^n\binom{n}{0} + (-1)^{n-1}\binom{n}{1} + (-1)^{n-2}\binom{n}{2} \\
&\quad + \cdots + (-1)^1\binom{n}{n-1} + (-1)^0\binom{n}{n} \\
&= (-1+1)^n = 0,
\end{aligned}$$

which gives $I_1 = 0$. Now compute

$$\begin{aligned}
I_2 &= (-1)^{n-1} \sum_{1\le l_1<l_2\le n+1} \sum_{a_{l_1 i}=a, a_{l_2 j}=b} t_{l_1 i}t_{l_2 j} \\
&\quad + (-1)^{n-2} \sum_{1\le l_1<l_2<l_3\le n+1} \sum_{1\le p<q\le 3} \sum_{a_{l_p i}=a, a_{l_q j}=b} t_{l_p i}t_{l_q j} \\
&\quad + \cdots + (-1)^1 \sum_{1\le l_1<l_2<\cdots<l_n\le n+1} \sum_{1\le p<q\le n} \sum_{a_{l_p i}=a, a_{l_q j}=b} t_{l_p i}t_{l_q j} \\
&\quad + (-1)^0 \sum_{1\le p<q\le n+1} t_{pi}t_{qj}.
\end{aligned} \tag{3.8}$$

Obviously, for any $1 \le p < q \le n+1$, the coefficient of the item $t_{pi}t_{qj}$ in (3.8) is identical and equals

$$\begin{aligned}
&(-1)^{n-1}\binom{n-1}{0} + (-1)^{n-2}\binom{n-1}{1} + (-1)^{n-3}\binom{n-1}{2} \\
&\quad + \cdots + (-1)^1\binom{n-1}{n-2} + (-1)^0\binom{n-1}{n-1} \\
&= (-1+1)^{n-1} = 0,
\end{aligned}$$

which gives $I_2 = 0$. This concludes assertion (ii).

Statement (iii) follows from (3.6) directly. □

4 Solutions on symmetric groups

The symmetric group on a finite set X is a group whose elements are all bijective maps from X to itself and whose group operation is that of the map composition. If $X = \{1, 2, \ldots, m\}$, then it is called a symmetric group of degree m, denoted by S_m. In this section, we consider $C^{(n)}f = 0$ for $G = S_m$.

Proposition 4 *If $f \in \operatorname{Ker} C^{(n)}(S_m, H)$, then for any $i = 1, 2, \ldots, n$, we have*

$$\begin{aligned}
&C^{(n-1)}f(x_1, x_2, \ldots, x_{i-1}, y_1y_2\cdots y_p, x_{i+1}, \ldots, x_n) \\
&\quad = C^{(n-1)}f(x_1, x_2, \ldots, x_{i-1}, y_{\pi(1)}y_{\pi(2)}\cdots y_{\pi(p)}, x_{i+1}, \ldots, x_n)
\end{aligned} \tag{4.1}$$

for all $x_i, y_j \in S_n$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, p$, and all rearrangements π.

Proof Note that $C^{(n-1)}f$ is a homomorphism with respect to each variable and H is an Abelian group, which yields

$$\begin{aligned} &C^{(n-1)}f(x_1,x_2,\ldots,x_{i-1},y_1y_2\cdots y_p,x_{i+1},\ldots,x_n)\\ &\quad= C^{(n-1)}f(x_1,x_2,\ldots,x_{i-1},y_1,x_{i+1},\ldots,x_n)\\ &\qquad+ C^{(n-1)}f(x_1,x_2,\ldots,x_{i-1},y_2,x_{i+1},\ldots,x_n)\\ &\qquad+\cdots+ C^{(n-1)}f(x_1,x_2,\ldots,x_{i-1},y_p,x_{i+1},\ldots,x_n)\\ &\quad= C^{(n-1)}f(x_1,x_2,\ldots,x_{i-1},y_{\pi(1)}y_{\pi(2)}\cdots y_{\pi(p)},x_{i+1},\ldots,x_n). \end{aligned}$$

This proves (4.1). □

Proposition 5 *Let τ be an arbitrary 2-cycle in S_m, and $f \in \operatorname{Ker} C^{(n)}(S_m,H)$. Then we have*

$$f(\tau^2) = 0, \tag{4.2}$$

$$C^{(n)}f(\tau,\tau,\ldots,\tau) = (-2)^n f(\tau) = \begin{cases} 2^n f(\tau) & \text{if } n \text{ is even},\\ -2^n f(\tau) & \text{if } n \text{ is odd}, \end{cases} \tag{4.3}$$

$$2^n f(\tau) = 0. \tag{4.4}$$

Proof We only need to prove (4.3). To this end, we need the following facts.

(i) If n is even, then

$$\begin{aligned} &\binom{n+1}{1}+\binom{n+1}{3}+\binom{n+1}{5}+\cdots+\binom{n+1}{n+1}\\ &\quad=\binom{n+1}{0}+\binom{n+1}{2}+\binom{n+1}{4}+\cdots+\binom{n+1}{n}. \end{aligned} \tag{4.5}$$

(ii) If n is odd, then

$$\begin{aligned} &\binom{n+1}{1}+\binom{n+1}{3}+\binom{n+1}{5}+\cdots+\binom{n+1}{n}\\ &\quad=\binom{n+1}{0}+\binom{n+1}{2}+\binom{n+1}{4}+\cdots+\binom{n+1}{n+1}. \end{aligned} \tag{4.6}$$

Note that by (2.1)

$$\begin{aligned} &C^{(n)}f(\tau,\tau,\ldots,\tau)\\ &\quad=(-1)^n\binom{n+1}{1}f(\tau)+(-1)^{n-1}\binom{n+1}{2}f(\tau^2)+(-1)^{n-2}\binom{n+1}{3}f(\tau^3)\\ &\qquad+\cdots+(-1)^1\binom{n+1}{n}f(\tau^n)+\binom{n+1}{n+1}f(\tau^{n+1}). \end{aligned} \tag{4.7}$$

We first prove the even case of (4.3). When n is even, by (4.5), (4.7) becomes

$$\begin{aligned} &C^{(n)}f(\tau,\tau,\ldots,\tau)\\ &\quad=\binom{n+1}{1}f(\tau)+\binom{n+1}{3}f(\tau)+\binom{n+1}{5}f(\tau)+\cdots+\binom{n+1}{n+1}f(\tau) \end{aligned}$$

$$= \left(\binom{n+1}{1} + \binom{n+1}{3} + \binom{n+1}{5} + \cdots + \binom{n+1}{n+1} \right) f(\tau)$$
$$= \frac{1}{2} \sum_{j=0}^{n+1} \binom{n+1}{j} f(\tau) = \frac{1}{2}(1+1)^{n+1} f(\tau)$$
$$= 2^n f(\tau),$$

which confirms the even case. When n is odd, by (4.6), (4.7) becomes

$$C^{(n)} f(\tau, \tau, \ldots, \tau)$$
$$= -\binom{n+1}{1} f(\tau) - \binom{n+1}{3} f(\tau) - \binom{n+1}{5} f(\tau) - \cdots - \binom{n+1}{n} f(\tau)$$
$$= \left(-\binom{n+1}{1} - \binom{n+1}{3} - \binom{n+1}{5} - \cdots - \binom{n+1}{n} \right) f(\tau)$$
$$= -\frac{1}{2} \sum_{j=0}^{n+1} \binom{n+1}{j} f(\tau) = -\frac{1}{2}(1+1)^{n+1} f(\tau)$$
$$= -2^n f(\tau).$$

This completes the proof. □

Proposition 6 *For any* 2-*cycle* $\sigma_1, \sigma_2, \ldots, \sigma_n$ *and* $f \in \operatorname{Ker} C^{(n)}(S_m, H)$, *we have*

$$C^{(n-1)} f(\sigma_1, \sigma_2, \ldots, \sigma_n) = C^{(n-1)} f\big((12), (12), \ldots, (12)\big). \tag{4.8}$$

Proof For any 2-cycle σ_1, there exists $z \in S_m$ such that $\sigma_1 = z(12)z^{-1}$. Hence, for any $x_2, x_3, \ldots, x_n \in S_m$, by (4.1) we have

$$\begin{aligned} &C^{(n-1)} f(\sigma_1, x_2, x_3, \ldots, x_n) \\ &\quad = C^{(n-1)} f\big(z(12)z^{-1}, x_2, x_3, \ldots, x_n\big) \\ &\quad = C^{(n-1)} f\big((12)zz^{-1}, x_2, x_3, \ldots, x_n\big) \\ &\quad = C^{(n-1)} f\big((12), x_2, x_3, \ldots, x_n\big). \end{aligned} \tag{4.9}$$

Similarly, for any $2 \le i \le n$,

$$\begin{aligned} &C^{(n-1)} f(x_1, x_2, \ldots, x_{i-1}, \sigma_i, x_{i+1}, \ldots, x_n) \\ &\quad = C^{(n-1)} f\big(x_1, x_2, \ldots, x_{i-1}, z(12)z^{-1}, x_{i+1}, \ldots, x_n\big), \\ &\quad = C^{(n-1)} f\big(x_1, x_2, \ldots, x_{i-1}, (12)zz^{-1}, x_{i+1}, \ldots, x_n\big), \\ &\quad = C^{(n-1)} f\big(x_1, x_2, \ldots, x_{i-1}, (12), x_{i+1}, \ldots, x_n\big). \end{aligned} \tag{4.10}$$

In particular, (4.8) follows from (4.9)-(4.10). □

Lemma 3 *Assume that*

$$C^{(j)} f(\sigma_1, \sigma_2, \ldots, \sigma_{j+1}) = C^{(j)} f\big((12), (12), \ldots, (12)\big) \tag{4.11}$$

for every 2-cycle $\sigma_1, \sigma_2, \ldots, \sigma_{j+1} \in S_m$, $j = 1, 2, \ldots, n-2$. *Then for any* $x, y, \beta, \tau_i \in S_m$ *and rearrangements* π, *where* β, τ_i *are 2-cycles, we have*

$$f(\tau_1\tau_2\cdots\tau_p) = f(\tau_{\pi(1)}\tau_{\pi(2)}\cdots\tau_{\pi(p)}), \tag{4.12}$$

$$f(x\beta y) = f\big(x(12)y\big), \tag{4.13}$$

$$f(\beta) = f\big((12)\big) \tag{4.14}$$

for every $f \in \operatorname{Ker} C^{(n)}(S_m, H)$.

Proof First, for any 2-cycle $\tau_i \in S_m$, $i = 1, 2, \ldots, p$, and rearrangement π, it follows from (4.11), (2.4), and (4.8) that when $p \geq n$,

$$\begin{aligned}
&f(\tau_1\tau_2\cdots\tau_p) \\
&\quad = \sum_{i=1}^{p} f(\tau_i) + \sum_{1\leq i_1<i_2\leq p} Cf(\tau_{i_1}, \tau_{i_2}) + \sum_{1\leq i_1<i_2<i_3\leq p} C^{(2)}f(\tau_{i_1}, \tau_{i_2}, \tau_{i_3}) \\
&\qquad + \cdots + \sum_{1\leq i_1<i_2<\cdots<i_n\leq p} C^{(n-1)}f(\tau_{i_1}, \tau_{i_2}, \ldots, \tau_{i_n}) \\
&\quad = \sum_{i=1}^{p} f(\tau_{\pi(i)}) + \frac{p(p-1)}{2} Cf\big((12), (12)\big) + \frac{p(p-1)(p-2)}{6} C^{(2)}f\big((12), (12), (12)\big) \\
&\qquad + \cdots + \frac{p(p-1)(p-2)\cdots(p-n+1)}{(n)!} C^{(n-1)}f\big((12), (12), \ldots, (12)\big) \\
&\quad = f(\tau_{\pi(1)}\tau_{\pi(2)}\cdots\tau_{\pi(p)}),
\end{aligned}$$

which gives the case of $p \geq n$ in (4.12). The case of $p < n$ is similar to verify. This confirms the proof of (4.12).

On the other hand, for all $x, y, \beta \in S_m$, there exist 2-cycles $\sigma_i, \tau_j, z \in S_m$, $i = 1, 2, \ldots, p$, $j = 1, 2, \ldots, q$, such that $x = \sigma_1\sigma_2\cdots\sigma_p$, $y = \tau_1\tau_2\cdots\tau_q$, and $\beta = z(12)z^{-1}$. Noting that $z = \delta_1\delta_2\cdots\delta_r$ for some 2-cycles $\delta_1, \delta_2, \ldots, \delta_r \in S_m$, we obtain

$$\begin{aligned}
f(x\beta y) &= f\big(\sigma_1\sigma_2\cdots\sigma_p\delta_1\delta_2\cdots\delta_r(12)\delta_r^{-1}\delta_{r-1}^{-1}\cdots\delta_1^{-1}\tau_1\tau_2\cdots\tau_q\big) \\
&= f\big(\sigma_1\sigma_2\cdots\sigma_p(12)\delta_1\delta_2\cdots\delta_r\delta_r^{-1}\delta_{r-1}^{-1}\tau_1\tau_2\cdots\tau_q\big) \\
&= f\big(x(12)y\big)
\end{aligned}$$

by (4.12). In particular, taking $x = y = e$ in (4.13), we get formula (4.14). This completes the proof. □

According to Lemma 3, we give the following main result in this section.

Theorem 2 *Assume that* (4.11) *holds. Then* f *is a solution to the equation* $C^{(n)}(S_m, H) = 0$ *if and only if there is* $h_0 \in H$ *such that* $2^n h_0 = 0$ *and*

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is even,} \\ h_0 & \text{if } x \text{ is odd.} \end{cases} \tag{4.15}$$

Proof We first deal with the necessity. Let $f \in \operatorname{Ker} C^{(n)}(S_m, H)$. Then, for any $x \in S_m$, there exist 2-cycles $\alpha_i \in S_m$, $i = 1, 2, \ldots, p$, such that $x = \alpha_1\alpha_2 \cdots \alpha_p$. In view of (4.1), (4.3), and (4.14), we get that for $p \geq n$,

$$\begin{aligned}
f(x) &= f(\alpha_1\alpha_2\cdots\alpha_p) \\
&= \sum_{i=1}^{p} f(\alpha_i) + \sum_{1\le i_1<i_2\le p} Cf(\alpha_{i_1}, \alpha_{i_2}) + \sum_{1\le i_1<i_2<i_3\le p} C^{(2)}f(\alpha_{i_1}, \alpha_{i_2}, \alpha_{i_3}) \\
&\quad + \cdots + \sum_{1\le i_1<i_2<\cdots<i_n\le p} C^{(n-1)}f(\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_n}) \\
&= pf\big((12)\big) + \frac{p(p-1)}{2} Cf\big((12),(12)\big) + \frac{p(p-1)(p-2)}{6} C^{(2)}f\big((12),(12),(12)\big) \\
&\quad + \cdots + \frac{p(p-1)(p-2)\cdots(p-n+1)}{n!} C^{(n-1)}f\big((12),(12),\ldots,(12)\big) \\
&= \binom{p}{1} f\big((12)\big) + \binom{p}{2}\big((-2)^1 f\big((12)\big)\big) + \binom{p}{3}\big((-2)^2 f\big((12)\big)\big) \\
&\quad + \cdots + \binom{p}{n}\big((-2)^{n-1} f\big((12)\big)\big) \\
&= \left(\binom{p}{1} + (-2)^1\binom{p}{2} + (-2)^2\binom{p}{3} + \cdots + (-2)^{n-1}\binom{p}{n}\right) f\big((12)\big).
\end{aligned} \tag{4.16}$$

Let

$$g(p) = \binom{p}{1} + (-2)^1\binom{p}{2} + (-2)^2\binom{p}{3} + \cdots + (-2)^{n-1}\binom{p}{n}.$$

We claim that

$$g(p) \in \begin{cases} 2^n\mathbb{Z} & \text{if } p \text{ is even,} \\ 2^n\mathbb{Z} + 1 & \text{if } p \text{ is odd.} \end{cases} \tag{4.17}$$

We first prove the even case. Obviously, (4.17) is true for $p = 2$ since $g(2) = 0$. For an inductive proof, suppose that (4.17) holds for $p = 2t$, $t \in \mathbb{N}$. Then we have

$$\begin{aligned}
&g(2t+2) - g(2t) \\
&\quad = \left(\binom{2t+2}{1} - \binom{2t}{1}\right) + (-2)^1\left(\binom{2t+2}{2} - \binom{2t}{2}\right) \\
&\qquad + (-2)^2\left(\binom{2t+2}{3} - \binom{2t}{3}\right) + \cdots + (-2)^{n-1}\left(\binom{2t+2}{n} - \binom{2t}{n}\right) \\
&\quad = 2 + (-2)^1\left(2\binom{2t}{1} + \binom{2t}{0}\right) + (-2)^2\left(2\binom{2t}{2} + \binom{2t}{1}\right) + \cdots \\
&\qquad + (-2)^{n-1}\left(2\binom{2t}{n-1} + \binom{2t}{n-2}\right) \\
&\quad = \left(2 + (-2)^1\binom{2t}{0} + (-2)^2\binom{2t}{1} + \cdots + (-2)^{n-1}\binom{2t}{n-2}\right)
\end{aligned}$$

$$+\left((-2)^1\cdot 2\binom{2t}{1}+(-2)^2\cdot 2\binom{2t}{2}+\cdots+(-2)^{n-1}\cdot 2\binom{2t}{n-1}\right)$$
$$\triangleq J_1+J_2,$$

where

$$\begin{aligned}J_1&=2+(-2)^1\binom{2t}{0}+(-2)^2\binom{2t}{1}+\cdots+(-2)^{n-1}\binom{2t}{n-2}\\&=(-2)^2\left(\binom{2t}{1}+(-2)^1\binom{2t}{2}+\cdots+(-2)^{n-3}\binom{2t}{n-2}\right)\\&=(-2)^2\left(g(2t)-(-2)^{n-2}\binom{2t}{n-1}-(-2)^{n-1}\binom{2t}{n}\right)\\&=4g(2t)-(-2)^n\binom{2t}{n-1}-(-2)^{n+1}\binom{2t}{n}\end{aligned}$$

and

$$\begin{aligned}J_2&=(-2)^1\cdot 2\binom{2t}{1}+(-2)^2\cdot 2\binom{2t}{2}+\cdots+(-2)^{n-1}\cdot 2\binom{2t}{n-1}\\&=-2^2\left(\binom{2t}{1}+(-2)^1\binom{2t}{2}+\cdots+(-2)^{n-2}\cdot\binom{2t}{n-1}\right)\\&=-2^2\left(g(2t)-(-2)^{n-1}\binom{2t}{n}\right)\\&=-4g(2t)+(-2)^{n+1}\binom{2t}{n},\end{aligned}$$

which yields $g(2t+2)-g(2t)=-(-2)^n\binom{2t}{n-1}$ and $g(2t+2)\in 2^n\mathbb{Z}$. This confirms the even case of (4.17). When p is odd, (4.17) is true for $p=1$ because of $g(1)=1$. Suppose that (4.17) holds for $p=2t-1$, and then we get

$$\begin{aligned}&g(2t+1)-g(2t-1)\\&\quad=\left(\binom{2t+1}{1}-\binom{2t-1}{1}\right)+(-2)^1\left(\binom{2t+1}{2}-\binom{2t-1}{2}\right)\\&\qquad+(-2)^2\left(\binom{2t+1}{3}-\binom{2t-1}{3}\right)+\cdots+(-2)^{n-1}\left(\binom{2t+1}{n}-\binom{2t-1}{n}\right)\\&\quad=2+(-2)^1\left(2\binom{2t-1}{1}+\binom{2t-1}{0}\right)+(-2)^2\left(2\binom{2t-1}{2}+\binom{2t-1}{1}\right)+\cdots\\&\qquad+(-2)^{n-1}\left(2\binom{2t-1}{n-1}+\binom{2t-1}{n-2}\right)\\&\quad=\left(2+(-2)^1\binom{2t-1}{0}+(-2)^2\binom{2t-1}{1}+\cdots+(-2)^{n-1}\binom{2t-1}{n-2}\right)\\&\qquad+\left((-2)^1\cdot 2\binom{2t-1}{1}+(-2)^2\cdot 2\binom{2t-1}{2}+\cdots+(-2)^{n-1}\cdot 2\binom{2t-1}{n-1}\right)\\&\quad\triangleq J_3+J_4,\end{aligned}$$

where

$$\begin{aligned} J_3 &= 2 + (-2)^1\binom{2t-1}{0} + (-2)^2\binom{2t-1}{1} + \cdots + (-2)^{n-1}\binom{2t-1}{n-2} \\ &= (-2)^2\left(\binom{2t-1}{1} + (-2)^1\binom{2t-1}{2} + \cdots + (-2)^{n-3}\binom{2t-1}{n-2}\right) \\ &= (-2)^2\left(g(2t-1) - (-2)^{n-2}\binom{2t-1}{n-1} - (-2)^{n-1}\binom{2t-1}{n}\right) \\ &= 4g(2t-1) - (-2)^n\binom{2t-1}{n-1} - (-2)^{n+1}\binom{2t-1}{n} \end{aligned}$$

and

$$\begin{aligned} J_4 &= (-2)^1\cdot 2\binom{2t-1}{1} + (-2)^2\cdot 2\binom{2t-1}{2} + \cdots + (-2)^{n-1}\cdot 2\binom{2t-1}{n-1} \\ &= -2^2\left(\binom{2t-1}{1} + (-2)^1\binom{2t-1}{2} + \cdots + (-2)^{n-2}\cdot\binom{2t-1}{n-1}\right) \\ &= -2^2\left(g(2t-1) - (-2)^{n-1}\binom{2t-1}{n}\right) \\ &= -4g(2t-1) + (-2)^{n+1}\binom{2t-1}{n}, \end{aligned}$$

which yields $g(2t+1) - g(2t-1) = -(-2)^n\binom{2t-1}{n-1}$ and $g(2t-1) \in 2^n\mathbb{Z} + 1$. This completes the proof of (4.17). According to (4.17) and (4.4), equation (4.16) becomes

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is even,} \\ f((12)) & \text{if } x \text{ is odd.} \end{cases}$$

This proves that when $p \geq n$, f must be of the form (4.15) with $h_0 = f((12))$.

When $p < n$,

$$\begin{aligned} f(x) &= f(\alpha_1\alpha_2\cdots\alpha_p) \\ &= \sum_{i=1}^{p} f(\alpha_i) + \sum_{1\leq i_1<i_2\leq p} Cf(\alpha_{i_1},\alpha_{i_2}) + \sum_{1\leq i_1<i_2<i_3\leq p} C^{(2)}f(\alpha_{i_1},\alpha_{i_2},\alpha_{i_3}) \\ &\quad + \cdots + \sum_{1\leq i_1<i_2<\cdots<i_{p-1}\leq p} C^{(p-2)}f(\alpha_{i_1},\alpha_{i_2},\ldots,\alpha_{i_{p-1}}) \\ &\quad + C^{(p-1)}f(\alpha_1,\alpha_2,\ldots,\alpha_p) \\ &= pf\big((12)\big) + \frac{p(p-1)}{2}Cf\big((12),(12)\big) + \frac{p(p-1)(p-2)}{6}C^{(2)}f\big((12),(12),(12)\big) \\ &\quad + \cdots + \frac{p(p-1)(p-2)\cdots 2}{(p-1)!}C^{(p-2)}f\big((12),(12),\ldots,(12)\big) \\ &\quad + C^{(p-1)}f\big((12),(12),\ldots,(12)\big) \\ &= \binom{p}{1}f\big((12)\big) + \binom{p}{2}\big((-2)^1 f\big((12)\big)\big) + \binom{p}{3}\big((-2)^2 f\big((12)\big)\big) + \cdots \end{aligned}$$

$$
\begin{aligned}
&+\binom{p}{p-1}\big((-2)^{p-2}f\big((12)\big)\big)+(-2)^{p-1}f\big((12)\big)\\
&=\left(\binom{p}{1}+(-2)^1\binom{p}{2}+(-2)^2\binom{p}{3}+\cdots+(-2)^{p-2}\binom{p}{p-1}+(-2)^{p-1}\right)f\big((12)\big)\\
&=\left(-\frac{1}{2}\left((-2)^0\binom{p}{0}+(-2)^1\binom{p}{1}+(-2)^2\binom{p}{2}+(-2)^3\binom{p}{3}\right.\right.\\
&\quad\left.\left.+\cdots+(-2)^{p-1}\binom{p}{p-1}+(-2)^p\right)+\frac{1}{2}\right)f\big((12)\big)\\
&=\left(-\frac{1}{2}(-2+1)^p+\frac{1}{2}\right)f\big((12)\big)\\
&=\begin{cases}0 & \text{if } p \text{ is even},\\ f((12)) & \text{if } p \text{ is odd}.\end{cases}
\end{aligned}
$$

This implies that when $p < n, f$ must be of the form (4.15) with $h_0 = f((12))$.

Now we turn to the sufficiency. Let $f : S_m \to H$ be defined by (4.15), where h_0 is a constant with $2^n h_0 = 0$. In order to prove $C^{(n)}f = 0$, by the symmetry of $x_1, x_2, \ldots, x_{n+1}$, it suffices to verify that it holds when $x_1, x_2, \ldots, x_k$ are odd and $x_{k+1}, x_{k+2}, \ldots, x_{n+1}$ are even for any $k = 1, 2, \ldots, n+1$. To this end, we only need to verify the following two cases: (i) k is odd; (ii) k is even. In fact, for case (i), by (2.1) we have

$$
\begin{aligned}
&C^{(n)}f(x_1,x_2,\ldots,x_{n+1})\\
&\quad=\sum_{1\le m\le n+1}(-1)^{n+1-m}\sum_{1\le i_1<i_2<\cdots<i_m\le n+1}f(x_{i_1}x_{i_2}\cdots x_{i_m})\\
&\quad=\left((-1)^n\binom{k}{1}+(-1)^{n-1}\binom{k}{1}\binom{n+1-k}{1}+(-1)^{n-2}\binom{k}{1}\binom{n+1-k}{2}\right.\\
&\qquad\left.+\cdots+(-1)^{k-1}\binom{k}{1}\binom{n+1-k}{n+1-k}\right)h_0\\
&\qquad+\left((-1)^{n-2}\binom{k}{3}+(-1)^{n-3}\binom{k}{3}\binom{n+1-k}{1}+(-1)^{n-4}\binom{k}{3}\binom{n+1-k}{2}\right.\\
&\qquad\left.+\cdots+(-1)^{k-3}\binom{k}{3}\binom{n+1-k}{n+1-k}\right)h_0\\
&\qquad+\left((-1)^{n-4}\binom{k}{5}+(-1)^{n-5}\binom{k}{5}\binom{n+1-k}{1}+(-1)^{n-6}\binom{k}{5}\binom{n+1-k}{2}\right.\\
&\qquad\left.+\cdots+(-1)^{k-5}\binom{k}{5}\binom{n+1-k}{n+1-k}\right)h_0\\
&\qquad+\cdots\\
&\qquad+\left((-1)^{n-k+1}\binom{k}{k}+(-1)^{n-k}\binom{k}{k}\binom{n+1-k}{1}+(-1)^{n-k-1}\binom{k}{k}\binom{n+1-k}{2}\right.\\
&\qquad\left.+\cdots+(-1)^0\binom{k}{k}\binom{n+1-k}{n+1-k}\right)h_0\\
&\quad=\left(\binom{k}{1}(-1)^{k-1}(-1+1)^{n+1-k}\right)h_0+\left(\binom{k}{3}(-1)^{k-3}(-1+1)^{n+1-k}\right)h_0\\
&\qquad+\left(\binom{k}{5}(-1)^{k-5}(-1+1)^{n+1-k}\right)h_0+\cdots
\end{aligned}
$$

$$+\left(\binom{k}{k}(-1)^{k-k}(-1+1)^{n+1-k}\right)h_0$$
$$= 0,$$

which confirms the odd case. With a similar discussion, the proof of the even case is also obtained. □

5 Solutions on finite cyclic groups

Let $C_m = \langle a \mid a^m = e\rangle$ be a cyclic group of order m with generator a. In this section, we study a general solution on the finite cyclic group C_m.

Theorem 3 *Assume that* $mC^{(k)}f(\underbrace{a,a,\ldots,a}_{k+1}) = 0$, $k = 1,2,\ldots,n-2$, *and* $m = n$, $mf(a) = 0$. *Then* f *is a solution to the equation* $C^{(n)}(C_m,H) = 0$ *if and only if it is given by*

$$f\left(a^p\right) = \begin{cases} \sum_{j=0}^{n-1} \frac{p(p-1)\cdots(p-j)}{(j+1)!} C^{(j)}f(\underbrace{a,a,\ldots,a}_{j+1}) & \text{if } p \le 0 \text{ or } p \ge n, \\ \sum_{j=0}^{p-1} \frac{p(p-1)\cdots(p-j)}{(j+1)!} C^{(j)}f(a,a,\ldots,a) & \text{if } 0 < p < n, \end{cases} \tag{5.1}$$

where $p \in \mathbb{Z}$ *and* $f(a)$, $C^{(n-1)}f(a,a,\ldots,a)$ *satisfy*

$$mC^{(n-1)}f(a,a,\ldots,a) = 0, \tag{5.2}$$

$$mf(a) = 0 \quad \text{if } m < n \text{ or } m > n, \tag{5.3}$$

$$C^{(m-1)}f(a,a,\ldots,a) = 0 \quad \text{if } m = n. \tag{5.4}$$

Proof Let $f : C_m \to H$ be a function satisfying $C^{(n)}f = 0$. Then, by (2.5) we see that f also satisfies (5.1). Now using (2.2), (ii) of Proposition 2, and the fact $a^m = e$, we get

$$mC^{(n-1)}f(a,a,\ldots,a) = C^{(n-1)}f\left(a^m,a,\ldots,a\right) = C^{(n-1)}f(e,a,\ldots,a) = 0,$$

which gives (5.2). Furthermore, let $p = m$ in (5.1), according to the assumptions of $mC^{(k)}f(a,a,\ldots,a) = 0$, $k = 1,2,\ldots,n-2$, $mf(a) = 0$, and $m = n$, (5.3)-(5.4) are obtained. This proves (5.2)-(5.4) and implies the necessity.

To give out the sufficiency, we claim that (5.1)-(5.4) give a well-defined function on C_m. Indeed, when $m > n$, it suffices to verify the following four cases: (i) $p + m \ge n$ and $n - m \le p \le 0$ or $p \ge n$; (ii) $p + m \ge n$ and $0 < p < n$; (iii) $p + m \le 0$ and $p \le -m < 0$; (iv) $0 < p + m < n$ and $-m < p < n - m < 0$.

For case (i), by (5.1) we have

$$\begin{aligned} &f\left(a^{p+m}\right) - f\left(a^p\right) \\ &\quad= \sum_{j=0}^{n-1} \frac{(p+m)(p+m-1)\cdots(p+m-j)}{(j+1)!} C^{(j)}f(a,a,\ldots,a) \\ &\qquad- \sum_{j=0}^{n-1} \frac{p(p-1)\cdots(p-j)}{(j+1)!} C^{(j)}f(a,a,\ldots,a) \end{aligned}$$

$$= mf(a) + \sum_{j=1}^{n-1} \left(\frac{(p+m)(p+m-1)\cdots(p+m-j)}{(j+1)!} - \frac{p(p-1)\cdots(p-j)}{(j+1)!} \right) \times C^{(j)} f(a,a,\ldots,a). \tag{5.5}$$

It is easy to see that the coefficient $C^{(j)}f(a,a,\ldots,a)$ is an integer multiple of m for $j = 1,2,\ldots,n-1$, and therefore, by (5.2)-(5.3) and the assumption of $mC^{(k)}f(a,a,\ldots,a) = 0$, $k = 1,2,\ldots,n-2$, (5.5) equals 0.

For case (ii), we compute that

$$\begin{aligned} &f\left(a^{p+m}\right) - f\left(a^{p}\right) \\ &\quad= \sum_{j=0}^{n-1} \frac{(p+m)(p+m-1)\cdots(p+m-j)}{(j+1)!} C^{(j)} f(\underbrace{a,a,\ldots,a}_{j+1}) \\ &\qquad - \sum_{j=0}^{p-1} \frac{p(p-1)\cdots(p-j)}{(j+1)!} C^{(j)} f(\underbrace{a,a,\ldots,a}_{j+1}) \\ &\quad= mf(a) + \sum_{j=1}^{p-1} \left(\frac{(p+m)(p+m-1)\cdots(p+m-j)}{(j+1)!} \right. \\ &\qquad \left. - \frac{p(p-1)\cdots(p-j)}{(j+1)!} \right) C^{(j)} f(\underbrace{a,a,\ldots,a}_{j+1}) \\ &\qquad + \sum_{j=p}^{n-1} \frac{(p+m)(p+m-1)\cdots(p+m-j)}{(j+1)!} C^{(j)} f(\underbrace{a,a,\ldots,a}_{j+1}). \end{aligned} \tag{5.6}$$

With the same discussion as in case (i), (5.6) equals 0. The proofs of the other two cases are similar to case (i).

When $m < n$, it suffices to verify the following five cases: (i) $p + m \geq n$ and $n - m < p < n$; (ii) $p + m \geq n$ and $p \geq n$; (iii) $p + m \leq 0$ and $p \leq -m$; (iv) $0 < p + m < n$ and $-m < p < 0$; (v) $0 < p + m < n$ and $0 < p < n - m$. Since the proof is similar to the case of $m > n$, we omit the details.

When $m = n$, it suffices to verify the following four cases: (i) $p + m \geq n$ and $0 < p < n$; (ii) $p + m \geq n$ and $p \geq n$; (iii) $p + m \leq 0$ and $p \leq -m$; (iv) $0 < p + m < n$ and $-m < p < 0$. Similarly to the cases of $m > n$ and $m < n$, the proof of $m = n$ is omitted. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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