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Homoclinic solutions for second order Hamiltonian systems near the origin

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Abstract

Under some local superquadratic conditions on $W(t,u)$ with respect to u, the existence of infinitely many homoclinic solutions is obtained for the nonperiodic second order Hamiltonian systems $\ddot{u}(t) - L(t)u(t) + \nabla W(t,u(t)) = 0$, $\forall t \in \mathbb{R}$, where $L(t)$ is unnecessarily coercive.

Keywords: homoclinic solutions; second order Hamiltonian systems; local conditions

1 Introduction and main results

Let us consider the second order Hamiltonian systems

$$\ddot{u}(t) - L(t)u + \nabla W\bigl(t,u(t)\bigr) = 0, \quad \forall t \in \mathbb{R}, \tag{1}$$

where $L \in C(\mathbb{R},\mathbb{R}^{N^2})$ is a symmetric matrix-valued function and $\nabla W(t,x) = \frac{\partial}{\partial x}W(t,x)$. As usual, we say that u is a nontrivial homoclinic solution (to 0) if $u \in C^2(\mathbb{R},\mathbb{R}^N)$, $u \not\equiv 0$, $u(t) \to 0$ as $|t| \to \infty$. In the following, $(\cdot,\cdot) : \mathbb{R}^N \times \mathbb{R}^N \mapsto \mathbb{R}$ denotes the standard inner product in $\mathbb{R}^N$ and $|\cdot|$ is the induced norm, and let I_N be the identity matrix of order N.

With the variational methods, the existence and multiplicity of homoclinic orbits of problem (1) have been obtained in many papers (see [1–16]), mainly in the case that W satisfies some global assumptions for all t and u. Among other results, under some local conditions on W, Lv and Jiang [6] investigated the existence of one nontrivial homoclinic solution for the second order Hamiltonian systems

$$\ddot{u} - L(t)u(t) + \nabla W\bigl(t,u(t)\bigr) = f(t), \quad \forall t \in \mathbb{R}$$

as a limit of periodic solutions of a certain sequence of boundary-value problems. Later, it was proved in [16] that if $L(t)$ is coercive and $W(t,u)$ is subquadratic near the origin with respect to u, then problem (1) has a sequence of homoclinic solutions converging to zero in L^∞ norm. There were no conditions assumed on W for u large. More precisely, one presented the following assumptions:

(A_1) There exists a constant $\alpha < 2$ such that $l(t)|t|^{\alpha-2} \to \infty$ as $|t| \to \infty$, where

$$l(t) = \inf_{x\in\mathbb{R}^N,|x|=1} \bigl(L(t)x,x\bigr).$$

(A_2) There are constants $c_0 > 0$ and $\frac{1}{2} \leq \nu \in (\frac{1}{3-\alpha}, 1)$ such that

$$\left|\nabla W(t,x)\right| \leq c_0|x|^\nu, \quad \forall (t,x) \in \mathbb{R} \times B_\delta(0),$$

where $B_\delta(0)$ denotes the ball in $\mathbb{R}^N$ centered at 0 with radius $\delta > 0$.

(A_3)

$$\lim_{|x|\to 0} \frac{W(t,x)}{|x|^2} = \infty \quad \text{uniformly for } t \in \mathbb{R}.$$

(A_4) $2W(t,x) - (\nabla W(t,x), x) > 0$ for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^N \setminus \{0\}$.

In this note, we will consider problem (1) where $L(t)$ is unnecessarily coercive, and $W(t,u)$ is superquadratic near the origin. The exact assumptions on L and W are as follows.

Theorem 1 *Assume the following conditions hold*:

(L_1) *There exists* $l_0 \geq 0$ *such that*

$$l(t) := \inf_{x\in\mathbb{R}^N, |x|=1} \bigl(L(t)x, x\bigr) \geq -l_0, \quad \forall t \in \mathbb{R}.$$

(L_2) *There exists a constant* $\xi > 1$ *such that*

$$\operatorname{meas}\bigl(t \in \mathbb{R} \mid |t|^{-\xi} L(t) \ngeq M_0 I_N\bigr) < +\infty, \quad \forall M_0 > 0.$$

(W_1) $W \in C^1(\mathbb{R} \times B_\delta(0), \mathbb{R})$ *is even in* u *and* $W(t,0) = 0$, *where* $B_\delta(0)$ *denotes the ball in* $\mathbb{R}^N$ *centered at* 0 *with radius* $\delta > 0$.

(W_2) *There are constants* $c_1 > 0$ *and* $0 < \theta < 1$ *such that*

$$\left|\nabla W(t,x)\right| \leq c_1|x|^\theta, \quad \forall (t,x) \in \mathbb{R} \times B_\delta(0).$$

(W_3) *There exists a constant* $p > 2$ *such that*

$$\lim_{|x|\to 0} \frac{W(t,x)}{|x|^p} = 0 \quad \textit{uniformly for } t \in \mathbb{R}.$$

(W_4) $2W(t,x) - (\nabla W(t,x), x) < 0$ *for all* $t \in \mathbb{R}$ *and* $x \in \mathbb{R}^N \setminus \{0\}$.

(W_5) *There exists a constant* $\mu > 2$ *such that*

$$\lim_{|x|\to 0} \frac{W(t,x)}{|x|^\mu} = \infty \quad \textit{uniformly for } t \in \mathbb{R}.$$

Then problem (1) *has a sequence of homoclinic solutions* $\{u_k\}$ *such that* $\max_{t\in\mathbb{R}} |u_k(t)| \to 0$ *as* $k \to \infty$.

Remark 1 There exist L and W that satisfy all assumptions in Theorem 1. For example, let $L(t) = (t^4 \sin^2 t + 1)I_N$ and $W(t,x) = |x|^4$ for $|x| < 1$ with $\theta = \frac{1}{2}$, $p = 3$, $\mu = 5$. Note that since L does not satisfy the coercive condition (A_1) and W is superquadratic near the origin, Theorem 1 is different from Theorem 1.1 in [16]. As far as the authors know, there is little research concerning the multiplicity of homoclinic solutions for problem (1) simultaneously under local conditions and noncoercive conditions, so our result is different from the previous results in the literature.

The proof is motivated by the argument in [16]. We will modify and extend W to an appropriate $\widetilde{W}$ and show for the associated modified functional I the existence of a sequence of homoclinic solutions converging to zero in L^∞ norm, and therefore we obtain infinitely many homoclinic solutions for the original problem.

2 Proof of theorems

First of all, we introduce the Sobolev space that we study. Let A be the self-adjoint extension of the operator $-(\frac{d^2}{dt^2}) + L(t)$ with the domain $\mathcal{D}(A) \subset L^2 \equiv L^2(\mathbb{R},\mathbb{R}^N)$. Denote by $\{E(\lambda) \mid -\infty < \lambda < \infty\}$ and $|A|$ the spectral resolution and the absolute value of A, respectively. Define $U = I - E(0) - E(-0)$. U commutes with A, $|A|$ and $|A|^{1/2}$, and $A = U|A|$ is the polar decomposition of A (see [17]). Let $E = \mathcal{D}(|A|^{1/2})$, the domain of $|A|^{1/2}$. Define on E the inner product and the corresponding norm:

$$(u,v)_0 = \big(|A|^{1/2}u, |A|^{1/2}v\big)_2 + (u,v)_2,$$
$$\|u\|_0 = (u,u)_0^{1/2},$$

where $u, v \in E$ and $(\cdot,\cdot)_2$ denotes the inner product of L^2. Then E is a Hilbert space, and it is easy to verify that E is continuously embedded in $H^1(\mathbb{R},\mathbb{R}^N)$.

Lemma 1 (see [10]) *Suppose that $L(t)$ satisfies* (L_1) *and* (L_2). *Then E is compactly embedded in $L^q(\mathbb{R},\mathbb{R}^N)$ for $q \in [1,\infty]$.*

Remark 2 By Lemma 1 it is easy to prove that the spectrum $\sigma(A)$ consists of eigenvalues numbered by $\lambda_1 \leq \lambda_2 \leq \cdots \to +\infty$, and a corresponding system of eigenfunctions, $(e_j)(Ae_j = \lambda_j e_j)$, forms an orthogonal basis in L^2. Let $j^- = \#\{j \mid \lambda_j < 0\}$, $j^0 = \#\{j \mid \lambda_j = 0\}$ and $\bar{j} = j^- + j^0$, where $\#\mathcal{A}$ denotes the number of elements of the set $\mathcal{A}$. Set $E^- = \operatorname{span}\{e_1,\dots,e_{j^-}\}$, $E^0 = \operatorname{span}\{e_{j^-+1},\dots,e_{\bar{j}}\} = \ker A$, and $E^+ = \overline{\operatorname{span}\{e_{\bar{j}+1},\dots\}}$. Then one has $E = E^- \oplus E^0 \oplus E^+$. We introduce on E the following inner product and the corresponding norm:

$$(u,v) = \big(|A|^{\frac{1}{2}}u, |A|^{\frac{1}{2}}v\big)_2 + \big(u^0, v^0\big)_2,$$
$$\|u\|^2 = (u,u) = \big\||A|^{\frac{1}{2}}u\big\|_2^2 + \big\|u^0\big\|_2^2,$$

where $u = u^- + u^0 + u^+$ and $v = v^- + v^0 + v^+ \in E = E^- \oplus E^0 \oplus E^+$. Obviously the norms $\|\cdot\|$ and $\|\cdot\|_0$ are equivalent and so the norm $\|\cdot\|$ on E will always be used. By Lemma 1 we see that there exists a constant $\gamma_q > 0$ such that

$$\|u\|_q \leq \gamma_q\|u\|, \quad \forall u \in E, \forall q \in [1,\infty]. \tag{2}$$

Lemma 2 *Assume that* (W_1)-(W_4) *are satisfied. There is $0 < r < \frac{\delta}{2}$ and $\widetilde{W} \in C^1(\mathbb{R},\mathbb{R}^N)$ such that*

(i)

$$\left|\nabla \widetilde{W}(t,x)\right| \le c_2\left(|x|^\theta + |x|^{p-1}\right), \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N, \tag{3}$$

where c_2 is a constant;

(ii)

$$\widehat{W}(t,x) := 2\widetilde{W}(t,x) - \left(\nabla \widetilde{W}(t,x), x\right) \le 0, \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N \tag{4}$$

and

$$\widehat{W}(t,x) = 0 \quad \textit{iff} \quad |x| = 0. \tag{5}$$

Proof By the mean value theorem, (W_1) and (W_2) imply that

$$\left|W(t,x)\right| \le c_1 |x|^{\theta+1}, \quad \forall (t,x) \in \mathbb{R} \times B_\delta(0). \tag{6}$$

Next we modify $W(t,x)$ for x outside a neighborhood of the origin 0. Choose

$$0 < \beta < \frac{1}{4\gamma_p^p},$$

where γ_p is the constant given in (2). By (W_3), there is a constant $r \in (0, \frac{\delta}{2})$ such that

$$W(t,x) \le \beta |x|^p, \quad \forall t \in \mathbb{R} \text{ and } |x| \le 2r. \tag{7}$$

Define a cut-off function $\rho \in C^1(\mathbb{R}, \mathbb{R})$ satisfying

$$\rho(t) = \begin{cases} 1, & 0 \le t \le r, \\ 0, & t \ge 2r, \end{cases}$$

and $-\frac{2}{r} \le \rho'(t) < 0$ for $r < t < 2r$. Using ρ, we define

$$\widetilde{W}(t,x) := \rho\left(|x|\right) W(t,x) + \left(1 - \rho\left(|x|\right)\right) W_\infty(x), \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N, \tag{8}$$

where $W_\infty(x) = \beta |x|^p$. Then by direct computation we get

$$\nabla \widetilde{W}(t,x) = \rho\left(|x|\right) \nabla W(t,x) + \rho'\left(|x|\right) W(t,x) + \left(1 - \rho\left(|x|\right)\right) W'_\infty(x) - \rho'\left(|x|\right) W_\infty(x), \tag{9}$$

$$\begin{aligned} \widehat{W}(t,x) = {} & \rho\left(|x|\right)\left(2W(t,x) - \left(\nabla W(t,x), x\right)\right) + (2-p)\left(1 - \rho\left(|x|\right)\right) W_\infty(x) \\ & - \rho'\left(|x|\right)\left(W(t,x) - W_\infty(x)\right)|x| \end{aligned} \tag{10}$$

for $(t,x) \in \mathbb{R} \times \mathbb{R}^N$. It follows from (W_1) and (W_2) that

$$\nabla \widetilde{W}(t,0) = \widehat{W}(t,0) = 0, \quad \forall t \in \mathbb{R}. \tag{11}$$

Then by (6), (9), (W_2), and the choice of the cut-off function ρ, we have

$$\left|\nabla \widetilde{W}(t,x)\right| \le \beta p |x|^{p-1}, \quad \forall t \in \mathbb{R}, |x| \ge 2r$$

and

$$\begin{aligned}
\big|\nabla \widetilde{W}(t,x)\big| &\leq \big|\nabla W(t,x)\big| + \frac{2}{r}\big|W(t,x)\big| + W'_\infty(x) + \frac{2}{r}W_\infty(x) \\
&\leq c_1|x|^\theta + 4c_1|x|^\theta + \beta p|x|^{p-1} + 4\beta|x|^{p-1} \\
&= 5c_1|x|^\theta + (4+p)\beta|x|^{p-1}, \quad \forall t \in \mathbb{R}, |x| < 2r.
\end{aligned}$$

Therefore, (3) is satisfied if $c_2 = \max\{5c_1, (4+p)\beta\}$.

Finally, we prove (4) and (5). On one hand, using (11) we know that $\widehat{W}(t,x) = 0$ whenever $x = 0$. On the other hand, assume that $r < |x| < 2r$. By (10), (W_4), (7), and the choice of the cut-off function ρ, we obtain

$$\begin{aligned}
&\rho\big(|x|\big)\big(2W(t,x) - \big(\nabla W(t,x), x\big)\big) < 0, \\
&(2-p)\big(1-\rho\big(|x|\big)\big)W_\infty(x) \leq 0,
\end{aligned}$$

and

$$-\rho'\big(|x|\big)\big(W(t,x) - W_\infty(x)\big)|x| \leq 0.$$

The above estimates imply that $\widehat{W}(t,x) < 0$ if $r < |x| < 2r$. Besides, when $|x| \geq 2r$, by (10) we have

$$\widehat{W}(t,x) = (2-p)W_\infty(x) < 0.$$

When $0 < |x| \leq r$, by (W_4) we get

$$\widehat{W}(t,x) = 2W(t,x) - \big(\nabla W(t,x), x\big) < 0.$$

Thus (4) and (5) are verified. The proof is completed. □

We now consider the modified problem

$$\ddot{u}(t) - L(t)u + \nabla \widetilde{W}\big(t, u(t)\big) = 0, \quad \forall t \in \mathbb{R}, \tag{12}$$

whose solutions correspond to critical points of the functional

$$\begin{aligned}
I(u) &= \frac{1}{2}\int_{\mathbb{R}} \big(|\dot{u}|^2 + \big(L(t)u, u\big)\big)\,dt - \int_{\mathbb{R}} \widetilde{W}(t,u)\,dt \\
&= \frac{1}{2}\big\|u^+\big\|^2 - \frac{1}{2}\big\|u^-\big\|^2 - \int_{\mathbb{R}} \widetilde{W}(t,u)\,dt
\end{aligned}$$

for all $u = u^- + u^0 + u^+ \in E = E^- + E^0 + E^+$. By (6) and (8) we have

$$\big|\widetilde{W}(t,u)\big| \leq c_1|u|^{\theta+1} + \beta|u|^p, \quad \forall (t,u) \in \mathbb{R} \times \mathbb{R}^N. \tag{13}$$

Thus, I is well defined.

Rewrite I as follows:

$$I = I_1 - I_2,$$

where

$$I_1 = \frac{1}{2}\int_{\mathbb{R}} \big(|\dot{u}|^2 + \big(L(t)u, u\big)\big)\,dt \quad \text{and} \quad I_2 = \int_{\mathbb{R}} \widetilde{W}(t,u)\,dt.$$

In the following, c will be used to denote various positive constants where the exact values are different.

Lemma 3 *Let* (L_1), (L_2), (W_1) *and* (W_2) *be satisfied. Then* $I_2 \in C^1(E,\mathbb{R})$ *and* $I_2' : E \to E^*$ *is compact, and hence* $I \in C^1(E,\mathbb{R})$. *Moreover,*

$$\begin{aligned}
\langle I_2'(u), v\rangle &= \int_{\mathbb{R}} \big(\nabla \widetilde{W}(t,u), v\big)\,dt, \\
\langle I'(u), v\rangle &= \big(u^+, v^+\big) - \big(u^-, v^-\big) - \int_{\mathbb{R}} \big(\nabla \widetilde{W}(t,u), v\big)\,dt
\end{aligned}$$

for $u, v \in E = E^- \oplus E^0 \oplus E^+$, *and nontrivial critical points of* I *on* E *belong to* $C^2(\mathbb{R},\mathbb{R}^N)$ *and are homoclinic solutions of problem* (12).

Proof By (3), for any $\eta \in [0,1]$, $u, h \in \mathbb{R}^N$ we have

$$\big|\big(\nabla \widetilde{W}(t, u + \eta h), h\big)\big| \le c\big(|u|^{\theta}|h| + |h|^{\theta+1} + |u|^{p-1}|h| + |h|^p\big),$$

where c is independent of η. Hence, for any $u, h \in E$, by the mean value theorem and Lebesgue's dominated convergence theorem, we get

$$\begin{aligned}
\lim_{s\to 0} \frac{I_2(u+sh) - I_2(h)}{s} &= \lim_{s\to 0} \int_{\mathbb{R}} \big(\nabla \widetilde{W}\big(t, u + \tau(t)sh\big), h\big)\,dt \\
&= \int_{\mathbb{R}} \big(\nabla \widetilde{W}(t,u), h\big)\,dt \\
&:= W_0(u,h),
\end{aligned}$$

where $\tau(t) \in [0,1]$ depends on u, h, s. Moreover, it follows from (2) and (3) that

$$\begin{aligned}
\big|W_0(u,h)\big| &\le \int_{\mathbb{R}} \big|\big(\nabla \widetilde{W}(t,u), h\big)\big|\,dt \\
&\le c\big(\|u\|_{\theta+1}^{\theta}\|h\|_{\theta+1} + \|u\|_p^{p-1}\|h\|_p\big) \\
&\le c\big(\|u\|^{\theta} + \|u\|^{p-1}\big)\|h\|.
\end{aligned}$$

Therefore, $W_0(u,\cdot)$ is linear and bounded in h, and $dI_2(u) = W_0(u,\cdot) \in E^*$ is the Gateaux derivative of I_2 at u.

Next we prove that $dI_2(u)$ is weakly continuous. Set $Bu := \nabla \widetilde{W}(t,u)$. There exist B_1, B_2 such that $B = B_1 + B_2$, where B_1 is bounded and continuous from $L^{\theta+1}(\mathbb{R})$ to $L^{\frac{\theta+1}{\theta}}(\mathbb{R})$ and

B_2 is bounded and continuous from $L^p(\mathbb{R})$ to $L^{\frac{p}{p-1}}(\mathbb{R})$. For any $v, h \in E$,

$$\begin{aligned}
\left|\langle dI_2(u) - dI_2(v), h\rangle\right| &= \left|\int_{\mathbb{R}} (Bu - Bv, h)\, dt\right| \\
&= \left|\int_{\mathbb{R}} (B_1u + B_2u - B_1v - B_2v, h)\, dt\right| \\
&\le \int_{\mathbb{R}} |B_1u - B_1v||h|\, dt + \int_{\mathbb{R}} |B_2u - B_2v||h|\, dt \\
&\le c\|B_1u - B_1v\|_{\frac{\theta+1}{\theta}} \|h\| + c\|B_2u - B_2v\|_{\frac{p}{p-1}} \|h\|,
\end{aligned}$$

which implies that

$$\left\|dI_2(u) - dI_2(v)\right\|_{E^*} \le c\|B_1u - B_2v\|_{\frac{\theta+1}{\theta}} + c\|B_2u - B_2v\|_{\frac{p}{p-1}}.$$

Now suppose $u_n \rightharpoonup u$ in E, then by Lemma 1, $u_n \to u$ in $L^{(\theta+1)}(\mathbb{R})$ and $L^p(\mathbb{R})$. Combining the above arguments, we see that $dI_2(u)$ is weakly continuous. Therefore, $I_2(u) \in C^1(E, \mathbb{R})$ and $I_2' : E \to E^*$ is compact.

Finally, we show that nontrivial critical points of I on E are homoclinic solutions of problem (12). Let $u \in E$ be a nontrivial critical point of I. A standard argument shows that $u \in C^2(\mathbb{R}, \mathbb{R}^N)$ and satisfies problem (12) (see [18] for more details). Since E is continuously embedded in $H^1(\mathbb{R}, \mathbb{R}^N)$, $u(t) \to 0$ as $|t| \to \infty$. The proof is completed. □

Lemma 4 *Assume that* (L_1), (L_2), (W_1)-(W_4) *are satisfied. Then* 0 *is the only critical point of* I *such that* $I(u) = 0$.

Proof By (W_1), (W_2), and Lemma 3, we know that 0 is a critical point of I with $I(0) = 0$. Now let $u \in E$ be a critical point of I with $I(u) = 0$. Then we have

$$0 = 2I(u) - \langle I'(u), u\rangle = -\int_{\mathbb{R}} \widehat{W}(t, u)\, dt,$$

where $\widehat{W}$ is defined in (4). This together with (ii) of Lemma 2 implies that $|u(t)| = 0$ for all $t \in \mathbb{R}$. The proof is completed. □

The following lemma is due to Bartsch and Willem [19] and we quote it from [16].

Lemma 5 *Let* E *be a Banach space with a finite-dimensional approximation in the sense that* $E = \bigoplus_{j\in\mathbb{N}} E(j)$, *where* $E(j)$ *are all finite-dimensional subspaces. Let* $I \in C^1(E, \mathbb{R})$ *be an even functional and satisfy*:

- (F_1) *For every* $k \ge k_0$, *there exists* $R_k > 0$ *such that* $I(u) \ge 0$ *for every* $u \in E_k := \bigoplus_{j\ge k} E(j)$ *with* $\|u\| = R_k$, *and* $b_k := \inf_{u\in B_k} I(u) \to 0$ *as* $k \to \infty$. *Here* $B_k := \{u \in E_k \mid \|u\| \le R_k\}$.
- (F_2) *For every* $k \in \mathbb{N}$, *there exist* $r_k \in (0, R_k)$ *and* $d_k < 0$ *such that* $I(u) \le d_k$ *for every* $u \in E^k := \bigoplus_{j\le k} E(j)$ *with* $\|u\| = r_k$.
- (F_3) I *satisfies* $(PS)^*$ *condition with respect to* $\{E^m \mid m \in \mathbb{N}\}$, *i.e., every sequence* $u_m \in E^m$ *with* $I(u_m) < 0$ *bounded and* $(I|_{E^m})'(u_m) \to 0$ *as* $m \to \infty$ *has a subsequence which converges to a critical point of* I.

Then, for each $k \geq k_0$, *I has a critical value* $\xi_k \in [b_k, d_k]$, *hence* $\xi_k < 0$ *and* $\xi_k \to 0$ *as* $k \to \infty$.

Let $E(j) = \operatorname{span}\{e_j\}$ for each $j \in \mathbb{N}$, where $\{e_j : j \in \mathbb{N}\}$ is the system of eigenfunctions given in Remark 2. Now we show that the functional I has the geometric property of Lemma 5 under the conditions of Theorem 1.

Lemma 6 *Assume that* (L_1), (L_2), (W_1), *and* (W_2) *hold. Then there exist a positive integer* k_0 *and a sequence* $R_k \to 0^+$ *as* $k \to \infty$ *such that*

$$\inf_{u\in E_k, \|u\|=R_k} I(u) \geq 0, \quad \forall k \geq k_0$$

and

$$b_k := \inf_{u\in B_k} I(u) \to 0 \quad \text{as } k \to \infty,$$

where $E_k := \bigoplus_{j\geq k} E(j)$ *and* $B_k := \{u \in E_k \mid \|u\| \leq R_k\}$ *for all* $k \in \mathbb{N}$.

Proof Note that $E_k \subset E^+$ for all $k \geq \bar{j} + 1$ (see Remark 2 for details). Thus for each $k \geq \bar{j} + 1$, by (13) we obtain

$$\begin{aligned} I(u) &\geq \frac{1}{2}\|u\|^2 - \int_R \widetilde{W}(t,u)\,dt \\ &\geq \frac{1}{2}\|u\|^2 - c_1\|u\|_{\theta+1}^{\theta+1} - \beta\|u\|_p^p, \quad \forall u \in E_k. \end{aligned} \tag{14}$$

Set

$$l_k = \sup_{u\in E_k, \|u\|=1} \|u\|_{\theta+1}, \quad \forall k \geq \bar{j} + 1. \tag{15}$$

Since E is compactly embedded into $L^{\theta+1}$, we have (see [20])

$$l_k \to 0^+ \quad \text{as } k \to \infty. \tag{16}$$

For each $k \geq \bar{j} + 1$, it follows from (2), (14), (15), and the choice of β that

$$\begin{aligned} I(u) &\geq \frac{1}{2}\|u\|^2 - c_1 l_k^{\theta+1}\|u\|^{\theta+1} - \beta\gamma_p^p\|u\|^p \\ &\geq \frac{1}{2}\|u\|^2 - c_1 l_k^{\theta+1}\|u\|^{\theta+1} - \frac{1}{4}\|u\|^p, \quad \forall u \in E_k. \end{aligned} \tag{17}$$

For each $k \geq \bar{j} + 1$, choose

$$R_k = 4c_1 l_k^{\theta+1}, \tag{18}$$

then, by (15),

$$R_k \to 0^+ \quad \text{as } k \to \infty \tag{19}$$

and hence there exists a positive integer $k_0 \geq \bar{j} + 1$ such that

$$R_k < 1, \quad \forall k \geq k_0. \tag{20}$$

Now by (17), (18), and (20), we have

$$\inf_{u \in E_k, \|u\| = R_k} I(u) \geq \frac{1}{2} R_k^2 - \frac{1}{4} R_k^{\theta+2} - \frac{1}{4} R_k^p \geq 0, \quad \forall k \geq k_0.$$

Noting that $I(0) = 0$ and

$$I(u) \geq -c_1 l_k^{\theta+1} \|u\|^{\theta+1} - \frac{1}{4} \|u\|^p, \quad \forall k \geq \bar{j} + 1, u \in E_k,$$

we have

$$0 \geq \inf_{u \in B_k} I(u) \geq -c_1 l_k^{\theta+1} R_k^{\theta+1} - \frac{1}{4} R_k^p, \quad \forall k \geq \bar{j} + 1,$$

which combined with (16) and (19) implies that

$$b_k := \inf_{u \in B_k} I(u) \to 0 \quad \text{as } k \to \infty.$$

The proof is completed. □

Lemma 7 *Assume that* (L_1), (L_2), (W_1), *and* (W_5) *hold. Then for every* $k \in \mathbb{N}$, *there exist* $r_k \in (0, R_k)$ *and* $d_k < 0$ *such that* $I(u) \leq d_k$ *for every* $u \in E^k := \bigoplus_{j \leq k} E(j)$ *with* $\|u\| = r_k$.

Proof For a fixed $k \in \mathbb{N}$, since E^k is finite dimensional, there is a constant $C_k > 0$ such that

$$C_k \|u\|^\mu \leq \|u\|_\mu^\mu, \quad \forall u \in E^k. \tag{21}$$

Set $p_k = \min\{R_k, \frac{w_k}{\gamma_\infty}\}$. Then by (W_5), there exists a constant $0 < w_k < r$ such that

$$\widetilde{W}(t, u) = W(t, u) \geq m_k |u|^\mu, \quad \forall t \in \mathbb{R} \text{ and } |u| \leq w_k, \tag{22}$$

where $m_k = \frac{1}{p_k^{\mu-2} C_k}$. Now by (2), (21), (22), and Lemma 1, for $u \in E^k$ with $\|u\| \leq \frac{w_k}{\gamma_\infty}$, we get

$$\begin{aligned} I(u) &= \frac{1}{2} \|u^+\|^2 - \frac{1}{2} \|u^-\|^2 - \int_{\mathbb{R}} \widetilde{W}(t, u)\, dt \\ &\leq \frac{1}{2} \|u\|^2 - m_k \|u\|_\mu^\mu \\ &\leq \frac{1}{2} \|u\|^2 - m_k C_k \|u\|^\mu \\ &= \frac{1}{2} \|u\|^2 \left(1 - \frac{2}{p_k^{\mu-2}} \|u\|^{\mu-2}\right). \end{aligned}$$

Choose

$$0 < r_k = \left(\frac{2}{3}\right)^{\frac{1}{\mu-2}} p_k < p_k$$

and let

$$d_k = -\frac{r_k^2}{6} < 0.$$

If $u \in E^k$ with $\|u\| = r_k$, we have

$$I(u) \leq d_k.$$

The proof is completed. □

Lemma 8 *Assume that* (L_1), (L_2), (W_1), (W_2), *and* (W_4) *hold. Then I satisfies* $(PS)^*$ *condition with respect to* $\{E^m \mid m \in \mathbb{N}\}$.

Proof Let $u_m \in E^m$ be a $(PS)^*$ sequence, that is,

$$I(u_m) \quad \text{is bounded} \quad \text{and} \quad (I|_{E^m})'(u_m) \to 0 \quad \text{as } m \to \infty. \tag{23}$$

Then we claim that $\{u_m\}$ is bounded. If not, passing to a subsequence if necessary, we may assume that

$$\|u_m\| \to \infty \quad \text{as } m \to \infty. \tag{24}$$

From (8), (9), (10), we have

$$\begin{aligned} 2I(u_m) - \langle (I|_{E^m})'(u_m), u_m \rangle &= \int_{\mathbb{R}} \big[\big(\nabla \widetilde{W}(t, u_m), u_m \big) - 2\widetilde{W}(t, u_m) \big] \, dt \\ &\geq (p-2)\beta \int_{\{t \in \mathbb{R} | |u_m(t)| \geq 2r\}} |u_m|^p \, dt \end{aligned} \tag{25}$$

for all $m \in \mathbb{N}$. From (23), (24), and (25), it follows that

$$\frac{\int_{\{t \in \mathbb{R} | |u_m(t)| \geq 2r\}} |u_m|^p \, dt}{\|u_m\|} \to 0 \tag{26}$$

as $m \to \infty$. Let

$$v_m(t) = \begin{cases} u_m(t), & \text{if } |u_m(t)| < 2r, \\ 0, & \text{if } |u_m(t)| \geq 2r \end{cases} \tag{27}$$

and

$$w_m(t) = u_m(t) - v_m(t) \tag{28}$$

for all $m \in \mathbb{N}$ and all $t \in \mathbb{R}$. By (23), (25), and (28),

$$c\big(1 + \|u_m\|\big) \geq \|w_m\|_p^p \tag{29}$$

for all positive integer m. From Hölder's inequality, (27), (28), and the equivalence of the norms on the finite-dimensional subspace $E^- \oplus E^0$, we have

$$\begin{aligned}\|u_m^- + u_m^0\|_2^2 &= (u_m^- + u_m^0, u_m)_2 \\ &= (u_m^- + u_m^0, v_m)_2 + (u_m^- + u_m^0, w_m)_2 \\ &\le \|u_m^- + u_m^0\|_1 \|v_m\|_\infty + \|u_m^- + u_m^0\|_{p'} \|w_m\|_p \\ &\le c\|u_m^- + u_m^0\|_2 (1 + \|w_m\|_p) \end{aligned} \tag{30}$$

for all $m \in \mathbb{N}$, where $\frac{1}{p} + \frac{1}{p'} = 1$. Then, from the equivalence of the norms on the finite-dimensional subspace $E^- \oplus E^0$, (29), and (30) it follows that

$$\begin{aligned}\|u_m^- + u_m^0\| &\le c\|u_m^- + u_m^0\|_2 \\ &\le c(1 + \|w_m\|_p) \\ &\le c(1 + \|u_m\|^{\frac{1}{p}})\end{aligned}$$

for all $m \in \mathbb{N}$, which implies that

$$\frac{\|u_m^- + u_m^0\|}{\|u_m\|} \to 0 \tag{31}$$

as $m \to \infty$. By (3) we get

$$|\nabla \widetilde{W}(t,x)| \le c(1 + |x|^{p-1}), \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N,$$

which combined with (2) implies that

$$\begin{aligned}\langle (I|_{E^m})'(u_m), u_m^+\rangle &\ge \|u_m^+\|^2 - \int_{\mathbb{R}} |\nabla \widetilde{W}(t,u_m)||u_m^+|\,dt \\ &\ge \|u_m^+\|^2 - c\int_{\mathbb{R}} |u_m|^{p-1}|u_m^+|\,dt - c\int_{\mathbb{R}} |u_m^+|\,dt \\ &\ge \|u_m^+\|^2 - c\|u_m^+\|_\infty \int_{\{t\in\mathbb{R}||u_m(t)|\ge 2r\}} |u_m|^{p-1}\,dt \\ &\quad - c(2r)^{p-1}\int_{\{t\in\mathbb{R}||u_m(t)|<2r\}} |u_m^+|\,dt - c\|u_m^+\|_1 \\ &\ge \|u_m^+\|^2 - c\|u_m^+\|_\infty (2r)^{-1}\int_{\{t\in\mathbb{R}||u_m(t)|\ge 2r\}} |u_m|^p\,dt \\ &\quad - c(2r)^{p-1}\|u_m^+\|_1 - c\|u_m^+\|_1 \\ &\ge \|u_m^+\|^2 - c\gamma_\infty\|u_m^+\|(2r)^{-1}\int_{\{t\in\mathbb{R}||u_m(t)|\ge 2r\}} |u_m|^p\,dt \\ &\quad - c(2r)^{p-1}\gamma_1\|u_m^+\| - c\gamma_1\|u_m^+\|.\end{aligned}$$

From this and (26) it follows that

$$\frac{\|u_m^+\|}{\|u_m\|} \to 0 \tag{32}$$

as $m \to \infty$. Combining (31) and (32), one gets

$$1 = \frac{\|u_m\|}{\|u_m\|} \leq \frac{\|u_m^+\| + \|u_m^- + u_m^0\|}{\|u_m\|} \to 0$$

as $m \to \infty$, which is a contradiction. Hence $\{u_m\}$ is bounded. Noting that by Lemma 3, I' is a compact perturbation of the identity in $E \ominus E^0$. Since E^0 has finite dimension, $\{u_m\}$ has a subsequence converging to a critical point of I (see [18]). Hence, I satisfies the $(PS)^*$ condition. The proof is completed. □

Proof of Theorem 1 It follows from Lemmas 6-8 that the functional I satisfies the conditions (F_1)-(F_3) of Lemma 5. Therefore, by Lemma 5, there exists a sequence of critical values $\xi_k < 0$ with $\xi_k \to 0$ as $k \to \infty$. Let $\{u_k\}$ be a sequence of critical points of I corresponding to these critical values, *i.e.*, $I(u_k) = \xi_k$ and $I'(u_k) = 0$ for all k. Then by Lemma 3, $\{u_k\} \subset C^2(\mathbb{R}, \mathbb{R}^N)$ is a sequence of homoclinic solutions of problem (12). Moreover, $\{u_k\}$ forms a (PS) sequence in E. By Lemma 8 and Remark 3.19 in [20], I satisfies the (PS) condition and hence we may assume without loss of generality that $u_k \to u$ in E as $k \to \infty$. Evidently, u is a critical point of I with $I(u) = 0$. Then by Lemma 4, u must be 0. Thus $u_k \to 0$ in E as $k \to \infty$. By (2), we further have $u_k \to 0$ in $L^\infty(\mathbb{R}, \mathbb{R}^N)$ as $k \to \infty$. Therefore, for k large enough, they are homoclinic solutions of problem (1). The proof is completed. □

Competing interests
The author declares that they have no competing interests.

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