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Dynamical analysis in exponential RED algorithm with communication delay

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Abstract

In this paper, an exponential RED algorithm with communication delay is considered. By choosing the delay as a bifurcation parameter, we demonstrate that a Hopf bifurcation would occur when the delay exceeds a critical value. Some explicit formulas are worked out for determining the stability and the direction of the bifurcated periodic solutions by using the normal form theory and center manifold theory. Finally, numerical simulations supporting the theoretical analysis are provided.

Keywords: exponential RED; stability; Hopf bifurcation; congestion control; periodic solution

1 Introduction

It is well known that Internet applications, such as the world wide web, file transfer, Usenet news and remote login, are delivered via the Transmission Control Protocol (TCP). With a spectacular growth of Internet applications, congestion control has become a subject of intense research activity. An uncontrolled network may suffer from severe congestion, which can cause high packet loss rates and increasing delays, the upper formation application systems performance drop, and it can even break the whole system by causing congestion collapse (or Internet meltdown) [1]. Thus congestion control is one of the critical issues for the efficient operation of the Internet. Many researchers have investigated the congestion control problems of the Internet and a lot of excellent and interesting results have been obtained, for example, Raina and Heckmann [2] investigated the stability properties of the congestion avoidance phase of TCP with drop tail. Guo *et al.* [3] investigated the stability of an exponential RED model with heterogeneous delays. Liu *et al.* [1] studied the bifurcation and chaotic behavior of the Transmission Control Protocol (TCP) and User Datagram Protocol (UDP) network with Random Early Detection (RED) queue management. Xu *et al.* [4] studied the Local Hopf bifurcation and global existence of periodic solutions of the following TCP system:

$$\frac{dx(t)}{dt} = x(t-\tau)\left[\frac{M(1-p(x(t-\tau)))}{Nx(t)} - \frac{x(t)p(x(t-\tau))}{2M}\right], \tag{1.1}$$

where M, N are two positive constants, $p(y)$ is an increasing non-negative continuous function in $(0,+\infty)$ and τ is the round-trip propagation delay for each of the TCP connections. Guo *et al.* [5] considered the stability and Hopf bifurcation of the following conges-

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tion control model with two communication delays:

$$\begin{cases} \dot{x}_1(t) = k_1 x_1(t-\tau_1)[\frac{\alpha}{x_1(t)} - \beta x_1(t) p(t-\tau_1)], \\ \dot{x}_2(t) = k_2 x_2(t-\tau_2)[\frac{\alpha}{x_2(t)} - \beta x_2(t) p(t-\tau_2)], \\ \dot{p}(t) = \gamma p(t)[x_1(t-\tau_1) + x_2(t-\tau_2) - c], \end{cases} \tag{1.2}$$

where $x_i(t)$ is the rate at which source i transmits data at time t, α and β are positive real numbers, $p(t)$ is the loss probability function, τ_i is the round-trip delay for source i, c is the link capacity, k and γ are positive gain parameter, $i = 1, 2$. For more related work on congestion control, see [6–12].

Based on [7–9], Guo *et al.* [13] recently studied the following exponential RED algorithm coupled with TCP-Reno with a single source and with two sources:

$$\begin{cases} \dot{x}_1(t) = k_1 x_1(t-\tau)[\frac{1-p(t)}{\alpha^2 x_1(t)} - \beta_1 x_1(t) p(t)], \\ \dot{x}_2(t) = k_2 x_2(t-\tau)[\frac{1-p(t)}{\alpha^2 x_2(t)} - \beta_2 x_2(t) p(t)], \\ \dot{p}(t) = k_3 p(t)[x_1(t-\tau) + x_2(t-\tau) - c], \end{cases} \tag{1.3}$$

where $\dot{x}_i(t) = \frac{dx}{dt}$ ($i = 1, 2, 3$) represent the transmission rate of the i source per second at time t, τ, k_i ($i = 1, 2, 3$) and β_j ($j = 1, 2$) are the round-trip time (RTT), the positive gain parameter, and the decrease factor of the source i, respectively. By choosing the delay as a bifurcation parameter, Guo *et al.* [13] obtained the necessary and sufficient conditions for the existence of Hopf bifurcation and a formula for determining the direction of the Hopf bifurcation and the stability of bifurcating periodic solutions.

Motivated by [13] and considering that, when the number of sources is large, the simplified model can reflect the really exponential RED algorithm more closely, in this paper, we consider an exponential RED algorithm coupled with TCP-Reno with a single source and with three sources, then we have the following system:

$$\begin{cases} \dot{x}_1(t) = k_1 x_1(t-\tau)[\frac{1-p(t)}{\alpha^2 x_1(t)} - \beta_1 x_1(t) p(t)], \\ \dot{x}_2(t) = k_2 x_2(t-\tau)[\frac{1-p(t)}{\alpha^2 x_2(t)} - \beta_2 x_2(t) p(t)], \\ \dot{x}_3(t) = k_3 x_3(t-\tau)[\frac{1-p(t)}{\alpha^2 x_3(t)} - \beta_3 x_3(t) p(t)], \\ \dot{p}(t) = k_4 p(t)[x_1(t-\tau) + x_2(t-\tau) + x_3(t-\tau) - c]. \end{cases} \tag{1.4}$$

The purpose of this paper is to discuss the stability and the properties of the Hopf bifurcation of model (1.4). This paper is organized as follows. In Section 2, the stability of the equilibrium and the existence of a Hopf bifurcation at the equilibrium are studied. In Section 3, the direction of the Hopf bifurcation and the stability and periodic of bifurcating periodic solutions on the center manifold are determined. In Section 4, numerical simulations are carried out to illustrate the validity of the main results. Some main conclusions are drawn in Section 5.

2 Stability of the equilibrium and local Hopf bifurcations

Let $E_*(x_1^*, x_2^*, x_3^*, p^*)$ be the non-zero equilibrium point of system (1.4), then we have

$$x_1^* = \frac{1}{\alpha}\sqrt{\frac{1-p^*}{\beta_1 p^*}}, \qquad x_2^* = \frac{1}{\alpha}\sqrt{\frac{1-p^*}{\beta_2 p^*}}, \qquad x_3^* = \frac{1}{\alpha}\sqrt{\frac{1-p^*}{\beta_3 p^*}}, \tag{2.1}$$

where p^* satisfies the following equation:

$$\sqrt{\frac{1-p^*}{\beta_1 p^*}} + \sqrt{\frac{1-p^*}{\beta_2 p^*}} + \sqrt{\frac{1-p^*}{\beta_3 p^*}} = c\alpha.$$

Let

$$\begin{aligned} &y_1(t) = x_1(t) - x_1^*, \qquad y_2(t) = x_2(t) - x_2^*, \\ &y_3(t) = x_3(t) - x_3^*, \qquad y_4(t) = p(t) - p^*. \end{aligned} \tag{2.2}$$

Substituting (2.2) into (1.4), we obtain the following linearized system:

$$\begin{cases} \dot{y}_1(t) = a_1 y_1(t) + a_2 y_4(t), \\ \dot{y}_2(t) = b_1 y_2(t) + b_2 y_4(t), \\ \dot{y}_3(t) = c_1 y_3(t) + c_2 y_4(t), \\ \dot{y}_4(t) = d_1 y_1(t-\tau) + d_1 y_2(t-\tau) + d_1 y_3(t-\tau), \end{cases} \tag{2.3}$$

where

$$\begin{aligned} &a_1 = -2k_1\beta_1 p^* x_1^*, \qquad b_1 = -2k_2\beta_2 p^* x_2^*, \qquad c_1 = -2k_3\beta_3 p^* x_3^*, \qquad d_1 = k_3 p^*, \\ &a_2 = -\frac{k_1\beta_1 (x_1^*)^2}{1-p^*}, \qquad b_2 = -\frac{k_2\beta_2 (x_2^*)^2}{1-p^*}, \qquad c_2 = -\frac{k_3\beta_3 (x_3^*)^2}{1-p^*}. \end{aligned}$$

Then the associated characteristic equation of (2.3) is

$$\det \begin{pmatrix} \lambda - a_1 & 0 & 0 & -a_2 \\ 0 & \lambda - b_1 & 0 & -b_2 \\ 0 & 0 & \lambda - c_1 & -c_2 \\ -d_1 e^{-\lambda\tau} & -d_1 e^{-\lambda\tau} & -d_1 e^{-\lambda\tau} & \lambda \end{pmatrix} = 0. \tag{2.4}$$

Then we obtain the following fourth degree exponential polynomial equation:

$$\lambda^4 + m_1\lambda^3 + m_2\lambda^2 + m_3\lambda + \left(n_1\lambda^2 + n_2\lambda + n_3\right)e^{-\lambda\tau} = 0, \tag{2.5}$$

where

$$\begin{aligned} &m_1 = -(a_1 + b_1 + c_1), \qquad m_2 = a_1 b_1 + a_1 c_1 + b_1 c_1, \qquad m_3 = -a_1 b_1 c_1, \\ &n_1 = -(b_2 d_2 + c_2 d_3 + a_2 d_1), \qquad n_2 = b_2 d_2 (a_1 + c_1) + c_2 d_3 (a_1 + b_1) + a_2 d_1 (b_1 + c_1), \\ &n_3 = -(a_1 b_2 c_1 d_2 + a_1 b_1 c_2 d_3 + a_2 b_1 c_1 d_1). \end{aligned}$$

Let $\lambda = i\omega_0$, $\tau = \tau_0$, and substitute this into (2.5), for the sake of simplicity, denote ω_0 and τ_0 by ω, τ, respectively, Separating the real and imaginary parts, we have

$$\left(n_3 - n_1\omega^2\right)\cos\omega\tau + n_2\omega\sin\omega\tau = m_2\omega^2 - \omega^4, \tag{2.6}$$

$$n_2\omega\cos\omega\tau - \left(n_3 - n_1\omega^2\right)\sin\omega\tau = m_1\omega^3 - m_3\omega. \tag{2.7}$$

Taking the square on both sides of (2.6) and (2.7) and summing them up, we obtain

$$\left(n_3 - n_1\omega^2\right)^2 + (n_2\omega)^2 = \left(m_2\omega^2 - \omega^4\right)^2 + \left(m_1\omega^3 - m_3\omega\right)^2,$$

i.e.,

$$\omega^8 + (m_1 - 2n_2)\omega^6 + \left(n_2 - 2m_1m_3 - n_1^2\right)\omega^4 + \left(m_3 + 2n_1n_3 - n_2^2\right)\omega^2 - n_3^2 = 0. \tag{2.8}$$

Let $p = m_1 - 2n_2$, $q = n_2 - 2m_1m_3 - n_1^2$, $u = m_3 + 2n_1n_3 - n_2^2$, $v = -n_3^2$, $z = \omega^2$. Then equation (2.8) becomes

$$z^4 + pz^3 + qz^2 + uz + v = 0. \tag{2.9}$$

Let

$$l(z) = z^4 + pz^3 + qz^2 + uz + v. \tag{2.10}$$

If the coefficients k_i $(i = 1,2,3,4)$, α, β_j $(j = 1,2,3)$ of system (1.4) are given, it is easy to use computer to calculate the roots of (2.9). Since $\lim_{z\to\infty} l(z) = +\infty$ and $v < 0$, we can conclude that (2.9) has at least one positive real root.

Without loss of generality, we assume that (2.9) has four positive roots, denoted by z_1, z_2, z_3, z_4, respectively. Then (2.8) has four positive roots

$$\omega_1 = \sqrt{z_1}, \qquad \omega_2 = \sqrt{z_2}, \qquad \omega_3 = \sqrt{z_3}, \qquad \omega_4 = \sqrt{z_4}.$$

By (2.6) and (2.7) we have

$$\tau_k^{(j)} = \frac{1}{\omega_k}\left[\arccos\frac{(m_2\omega_k^2 - m^4)(n_3 - n_1\omega_k^2) + (m_1\omega_k^3 - m_3\omega)n_2\omega}{(n_3 - n_1\omega_k^2)^2 + (n_2\omega)^2} + 2j\pi\right], \tag{2.11}$$

where $k = 1,2,3,4$ and $j = 0,1,2,\ldots$. Then $\pm i\omega_k$ are a pair of purely imaginary roots of equation (2.5) with $\tau = \tau_k^{(j)}$. Obviously, the sequence $\{\tau_k^{(j)}\}_{j=0}^{+\infty}$ is increasing, and

$$\lim_{j\to+\infty} \tau_k^{(j)} = +\infty, \quad k = 1,2,3,4.$$

Then we can define

$$\tau_0 = \tau_{k0}^{(0)} = \min_{1\le k\le 4}\left\{\tau_k^{(0)}\right\}, \qquad \omega_0 = \omega_{k0}. \tag{2.12}$$

Note that, when $\tau = 0$, (2.5) becomes

$$\lambda^4 + q_1\lambda^3 + q_2\lambda^2 + q_3\lambda + q_4 = 0, \tag{2.13}$$

where

$$q_1 = -(a_1 + b_1 + c_1),$$

$$q_2 = (a_1b_1 + a_1c_1 + b_1c_1) - (b_2d_2 + c_2d_3 + a_2d_1),$$

$$q_3 = b_2d_2(a_1 + c_1) + c_2d_3(a_1 + b_1) + a_2d_1(c_1 + d_1) - a_1b_1c_1,$$
$$q_4 = a_1b_2c_1d_2 + a_1b_1c_2d_3 + a_2b_1c_1d_1.$$

A set of necessary and sufficient conditions for all roots of (2.13) to have a negative real part is given by the well-known Routh-Hurwitz criteria in the following form:

$$q_1 > 0, \qquad q_1q_2 - q_3 > 0, \qquad q_1q_2q_3 - q_3^2 - q_1^2q_4 > 0, \qquad q_4 > 0. \tag{2.14}$$

Let $\lambda(\tau) = \alpha(\tau) + i\omega(\tau)$ be the root of equation (2.5) near $\tau = \tau_k^{(j)}$ satisfying $\alpha(\tau_k^{(j)}) = 0$, $\omega(\tau_k^{(j)}) = \omega_k$. Then the following lemma holds.

Lemma 2.1 *Suppose that* $l'(z_k) \neq 0$, *where* $l(z)$ *is defined by* (2.10). *If* $\tau = \tau_k^{(j)}$, *then* $\pm i\omega_k$ *are a pair of simple purely imaginary roots of equation* (2.5). *Moreover,*

$$\left[\frac{d(Re\lambda(\tau))}{d\tau}\right]_{\tau=\tau_k^{(j)}} \neq 0$$

and the sign of $[\frac{d(Re\lambda(\tau))}{d\tau}]_{\tau=\tau_k^{(j)}}$ *is consistent with that of* $l'(z_k)$.

Proof Let

$$F(\lambda) = \lambda^4 + m_1\lambda^3 + m_2\lambda^2 + m_3\lambda, \qquad H(\lambda) = n_1\lambda^2 + n_2\lambda + n_3. \tag{2.15}$$

Then (2.5) can be written as

$$F(\lambda) + H(\lambda)e^{-\lambda\tau} = 0, \tag{2.16}$$

which leads to

$$F(\lambda)\overline{F(\lambda)} - H(\lambda)\overline{H(\lambda)} = 0. \tag{2.17}$$

Thus, together with (2.9) and (2.10), we get

$$l(\omega^2) = F(i\omega)\overline{F(i\omega)} - H(i\omega)\overline{H(i\omega)}. \tag{2.18}$$

Differentiating both sides of equation (2.5) with respect to ω, we obtain

$$2\omega l'(\omega^2) = i\big[F'(i\omega)\overline{F(i\omega)} - \overline{F'(i\omega)}F(i\omega) - H'(i\omega)\overline{H(i\omega)} + \overline{H'(i\omega)}H(i\omega)\big]. \tag{2.19}$$

If $i\omega_k$ is not simple, then ω_k must satisfy

$$\frac{d}{d\lambda}\big[F(\lambda) + H(\lambda)e^{-\lambda\tau_k^{(j)}}\big]\bigg|_{\lambda=i\omega_k} = 0,$$

i.e.,

$$F'(i\omega_k) + H'(i\omega_k)e^{-\omega_k\tau_k^{(j)}} - \tau_k^{(j)}H(i\omega_k)e^{-\omega_k\tau_k^{(j)}} = 0.$$

By (2.16), we obtain

$$\tau_k^{(j)} = \frac{H'(i\omega_k)}{H(i\omega_k)} - \frac{F'(i\omega_k)}{F(i\omega_k)}.$$

Thus, by (2.17) and (2.19), we obtain

$$\begin{aligned}\operatorname{Im}\tau_k^{(j)} &= \operatorname{Im}\left\{\frac{H'(i\omega_k)}{H(i\omega_k)} - \frac{F'(i\omega_k)}{F(i\omega_k)}\right\} = \operatorname{Im}\left\{\frac{H'(i\omega_k)\overline{H(i\omega_k)}}{H(i\omega_k)\overline{H(i\omega_k)}} - \frac{F'(i\omega_k)\overline{F(i\omega_k)}}{F(i\omega_k)\overline{F(i\omega_k)}}\right\}\\ &= \operatorname{Im}\left\{\frac{H'(i\omega_k)\overline{H(i\omega_k)} - F'(i\omega_k)\overline{F(i\omega_k)}}{F(i\omega_k)\overline{F(i\omega_k)}}\right\} = \frac{\omega_k l'(\omega_k^2)}{|F(i\omega_k)|^2}.\end{aligned}$$

Since $\tau_k^{(j)}$ is real, we have $l'(z_k) = l'(\omega_k^2) = 0$, which is a contradiction to the condition $l'(z_k) \neq 0$. This proves the first part of Lemma 2.1.

Taking the derivative of λ with respect to τ in (2.5), it is easy to obtain

$$\left[F'(\lambda) + H'(\lambda)e^{-\lambda\tau} - \tau H(\lambda)e^{-\lambda\tau}\right]\frac{d\lambda}{d\tau} - \lambda H(\lambda)e^{-\lambda\tau} = 0,$$

which means

$$\begin{aligned}\frac{d\lambda(\tau)}{d\tau} &= \frac{\lambda H(\lambda)}{F'(\lambda)e^{\lambda\tau} + H'(\lambda) - \tau H(\lambda)}\\ &= \frac{\lambda H(\lambda)[\overline{F'(\lambda)}e^{\lambda\tau} + \overline{H'(\lambda)} - \tau\overline{H(\lambda)}]}{|F'(\lambda)e^{\lambda\tau} + H'(\lambda) - \tau H(\lambda)|^2}\\ &= \frac{\lambda[-\overline{F'(\lambda)}F(\lambda) + \overline{H'(\lambda)}H(\lambda) - \tau|H(\lambda)|^2]}{|F'(\lambda)e^{\lambda\tau} + H'(\lambda) - \tau H(\lambda)|^2}.\end{aligned}$$

It follows from this together with (2.19) that

$$\begin{aligned}&\left.\frac{d(\operatorname{Re}\lambda(\tau))}{d\tau}\right|_{\tau=\tau_k^{(j)}}\\ &\quad= \left.\frac{\operatorname{Re}\{\lambda[-\overline{F'(\lambda)}F(\lambda) + \overline{H'(\lambda)}H(\lambda) - \tau|H(\lambda)|^2]\}}{|F'(\lambda)e^{\lambda\tau} + H'(\lambda) - \tau H(\lambda)|^2}\right|_{\tau=\tau_k^{(j)}}\\ &\quad= \frac{i\omega_k[-\overline{F'(i\omega_k)}F(i\omega_k) + \overline{H'(i\omega_k)}H(i\omega_k) + F'(i\omega_k)\overline{F(i\omega_k)} - H'(i\omega_k)\overline{H(i\omega_k)}]}{2|F'(i\omega_k)e^{i\tau_k^{(j)}\omega_k} + H'(i\omega_k) - \tau_k^{(j)}H(i\omega_k)|^2}\\ &\quad= \frac{\omega_k^2 l'(\omega_k^2)}{|F'(i\omega_k)e^{i\tau_k^{(j)}\omega_k} + H'(i\omega_k) - \tau_k^{(j)}H(i\omega_k)|^2}\\ &\quad= \frac{\omega_k^2 l'(z_k)}{|F'(i\omega_k)e^{i\tau_k^{(j)}\omega_k} + H'(i\omega_k) - \tau_k^{(j)}H(i\omega_k)|^2} \neq 0.\end{aligned}$$

Obviously, the sign of $\frac{d(\operatorname{Re}\lambda(\tau))}{d\tau}|_{\tau=\tau_k^{(j)}}$ is determined by that of $l'(z_k)$. This completes the proof. □

In order to investigate the distribution of roots of the transcendental equation (2.5), the following lemma, which is stated in [14], is useful.

Lemma 2.2 [14] *For the transcendental equation*

$$\begin{aligned} P\left(\lambda, e^{-\lambda\tau_1}, \ldots, e^{-\lambda\tau_m}\right) &= \lambda^n + p_1^{(0)}\lambda^{n-1} + \cdots + p_{n-1}^{(0)}\lambda + p_n^{(0)} \\ &\quad + \left[p_1^{(1)}\lambda^{n-1} + \cdots + p_{n-1}^{(1)}\lambda + p_n^{(1)}\right]e^{-\lambda\tau_1} + \cdots \\ &\quad + \left[p_1^{(m)}\lambda^{n-1} + \cdots + p_{n-1}^{(m)}\lambda + p_n^{(m)}\right]e^{-\lambda\tau_m} = 0, \end{aligned}$$

as $(\tau_1, \tau_2, \tau_3, \ldots, \tau_m)$ *vary, the sum of orders of the zeros of* $P(\lambda, e^{-\lambda\tau_1}, \ldots, e^{-\lambda\tau_m})$ *in the open right half plane can change, and only a zero appears on or crosses the imaginary axis.*

From Lemma 2.2, it is easy to obtain the following results.

Theorem 2.3 *Let* $\tau_k^{(j)}$ *and* τ_0 *be defined by* (2.11) *and* (2.12), *respectively.*

(i) *If* (2.14) *holds, the equilibrium* $E_*(x_1^*, x_2^*, x_3^*, p^*)$ *of system* (1.4) *is asymptotically stable for* $\tau \in [0, \tau_0)$.

(ii) *If* (2.14) *and* $l'(z_k)$ *hold, then system* (1.4) *undergoes a Hopf bifurcation at the equilibrium* $E_*(x_1^*, x_2^*, x_3^*, p^*)$ *when* $\tau = \tau_k^{(j)}$, *i.e., system* (1.4) *has a branch of periodic solutions bifurcating from the equilibrium* $E_*(x_1^*, x_2^*, x_3^*, p^*)$ *near* $\tau = \tau_k^{(j)}$.

3 Direction and stability of the Hopf bifurcation

In the previous section, we have obtained conditions for Hopf bifurcation to occur when $\tau = \tau_k^{(j)}$. In this section, we shall derive the explicit formulas for determining the direction, stability, and period of these periodic solutions bifurcating from the equilibrium $E_*(x_1^*, x_2^*, x_3^*, p^*)$ at these critical value of τ, by using techniques from the normal form and center manifold theory [15], Throughout this section, we always assume that system (2.2) undergoes Hopf bifurcation at the equilibrium $E_*(x_1^*, x_2^*, x_3^*, p^*)$ for $\tau = \tau_k^{(j)}$, and then $\pm i\omega_k$ are corresponding purely imaginary roots of the characteristic equation at the equilibrium $E_*(x_1^*, x_2^*, x_3^*, p^*)$. The linear part of system (1.4) at $E_*(x_1^*, x_2^*, x_3^*, p^*)$ is given by

$$\begin{cases} \dot{y}_1(t) = a_1 y_1(t) + a_2 y_4(t), \\ \dot{y}_2(t) = b_1 y_2(t) + b_2 y_4(t), \\ \dot{y}_3(t) = c_1 y_3(t) + c_2 y_4(t), \\ \dot{y}_4(t) = d_1 y_1(t-\tau) + d_1 y_2(t-\tau) + d_1 y_3(t-\tau), \end{cases} \tag{3.1}$$

and the non-linear part is given by

$$f(\mu, u_t) = (f_1, f_2, f_3, f_4)^T, \tag{3.2}$$

where

$$\begin{aligned} f_1 &= a_3 y_1^2(t) + a_4 y_1(t) y_1(t-\tau) + a_5 y_1(t) y_4(t) + a_6 y_1(t-\tau) y_4(t) \\ &\quad + a_7 y_1^3(t) + a_8 y_1^2(t) y_1(t-\tau) + a_9 y_1^2(t) y_4(t) + a_{10} y_1(t) y_1(t-\tau) y_4(t) + \text{h.o.t.}, \\ f_2 &= b_3 y_2^2(t) + b_4 y_2(t) y_2(t-\tau) + b_5 y_2(t) y_4(t) + b_6 y_2(t-\tau) y_4(t) \\ &\quad + b_7 y_2^3(t) + b_8 y_2^2(t) y_2(t-\tau) + b_9 y_2^2(t) y_4(t) + b_{10} y_2(t) y_2(t-\tau) y_4(t) + \text{h.o.t.}, \\ f_3 &= c_3 y_3^2(t) + c_4 y_3(t) y_3(t-\tau) + c_5 y_3(t) y_4(t) + c_6 y_3(t-\tau) y_4(t) \end{aligned}$$

$$+ c_7 y_3^3(t) + c_8 y_3^2(t) y_3(t-\tau) + c_9 y_3^2(t) y_4(t) + c_{10} y_3(t) y_3(t-\tau) y_4(t) + \text{h.o.t.},$$
$$f_4 = d_1 y_1(t-\tau) y_4(t) + d_1 y_2(t-\tau) y_4(t) + d_1 y_3(t-\tau) y_4(t),$$

where

$$a_3 = k_1\beta_1 p^* x_1^*, \qquad a_4 = -2k_1\beta_1 p^*, \qquad a_5 = \frac{k_1\beta_1 x_1^*(2p^*-1)}{1-p^*}, \qquad a_6 = \frac{k_1\beta_1 x_1^*}{1-p^*},$$
$$a_7 = -\frac{k_1\beta_1 p^*}{x_1^*}, \qquad a_8 = \frac{k_1\beta_1 p^*}{x_1^*}, \qquad a_9 = -\frac{k_1\beta_1 p^*}{1-p^*}, \qquad a_{10} = \frac{k_1\beta_1(2p^*-1)}{1-p^*},$$
$$b_3 = k_2\beta_2 p^* x_2^*, \qquad b_4 = -2k_2\beta_2 p^*, \qquad b_5 = \frac{k_2\beta_2 x_2^*(2p^*-1)}{1-p^*},$$
$$b_6 = \frac{k_2\beta_2 x_2^*}{1-p^*}, \qquad b_7 = -\frac{k_2\beta_2 p^*}{x_2^*}, \qquad b_8 = \frac{k_2\beta_2 p^*}{x_2^*},$$
$$b_9 = -\frac{k_2\beta_2 p^*}{1-p^*}, \qquad b_{10} = \frac{k_2\beta_2(2p^*-1)}{1-p^*}, \qquad d_1 = k_4.$$

Set $\tau = \tau_k^{(j)} + \mu$ and denote

$$C^k[-\tau,0] = \bigl\{\varphi \mid \varphi : [-\tau,0] \to R^4, \text{each component of } \varphi$$
$$\text{has } k\text{th order continuous derivative}\bigr\}.$$

For convenience, denote $C[-\tau,0]$ by $C^0[-\tau,0]$.

For $\varphi(\theta) = (\varphi_1(\theta), \varphi_2(\theta), \varphi_3(\theta), \varphi_4(\theta))^T \in C([-\tau,0], R^4)$, define a family of operators

$$L_\mu \varphi = B\varphi(0) + B_1\varphi(-\tau) \tag{3.3}$$

and

$$G(\mu,\varphi) = (k_1, k_2, k_3, k_4)^T, \tag{3.4}$$

where

$$B = \begin{pmatrix} a_1 & 0 & 0 & a_2 \\ 0 & b_1 & 0 & b_2 \\ 0 & 0 & c_1 & c_2 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad B_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ d_1 & d_1 & d_1 & 0 \end{pmatrix},$$

$$k_1 = a_3\varphi_1^2(0) + a_4\varphi_1(0)\varphi_1(-\tau) + a_5\varphi_1(0)\varphi_4(0) + a_6\varphi_1(-\tau)\varphi_4(0)$$
$$+ a_7\varphi_1^3(0) + a_8\varphi_1^2(0)\varphi_1(-\tau) + a_9\varphi_1^2(t)\varphi_4(0) + a_{10}\varphi_1(t)\varphi_1(-\tau)\varphi_4(0) + o\bigl(\|\varphi\|^4\bigr),$$
$$k_2 = b_3\varphi_2^2(0) + b_4\varphi_2(0)\varphi_2(-\tau) + b_5\varphi_2(0)\varphi_4(0) + b_6\varphi_2(-\tau)y_4(0)$$
$$+ b_7\varphi_2^3(0) + b_8\varphi_2^2(0)\varphi_2(-\tau) + b_9\varphi_2^2(0)\varphi_4(0) + b_{10}\varphi_2(0)\varphi_2(-\tau)\varphi_4(0) + o\bigl(\|\varphi\|^4\bigr),$$
$$k_3 = c_3\varphi_3^2(0) + c_4\varphi_3(0)\varphi_3(-\tau) + c_5\varphi_3(0)\varphi_4(0) + c_6\varphi_3(t-\tau)\varphi_4(0)$$
$$+ c_7\varphi_3^3(0) + c_8\varphi_3^2(0)\varphi_3\bigl(0-\tau_k^{(j)}\bigr) + c_9\varphi_3^2(0)\varphi_4(0) + c_{10}\varphi_3(0)\varphi_3(-\tau)\varphi_4(0) + o\bigl(\|\varphi\|^4\bigr),$$
$$k_4 = d_1\varphi_1(-\tau)\varphi_4(0) + d_1\varphi_2(-\tau)\varphi_4(0) + d_1\varphi_3(-\tau)\varphi_4(0),$$

and L_μ is a one-parameter family of bounded linear operators in $C([-\tau,0],R^4) \to R^4$. By the Riesz representation theorem, there exists a matrix whose components are bounded variation functions $\eta(\theta,\mu)$ in $[-\tau,0] \to R^{4^2}$, such that

$$L_\mu \varphi = \int_{-\tau}^{0} d\eta(\theta,\mu)\varphi(\theta). \tag{3.5}$$

In fact, choosing

$$\eta(\theta,\mu) = B\delta(\theta) + B_1\delta(\theta+\tau), \tag{3.6}$$

where $\delta(\theta)$ is the Dirac delta function, then (3.5) is satisfied. For $(\varphi_1,\varphi_2,\varphi_3,\varphi_4) \in (C^1[-\tau,0], R^4)$, define

$$A(\mu)\varphi = \begin{cases} \frac{d\varphi(\theta)}{d\theta}, & -\tau \le \theta < 0, \\ \int_{-\tau}^{0} d\eta(s,\mu)\varphi(s), & \theta = 0 \end{cases} \tag{3.7}$$

and

$$R\varphi = \begin{cases} 0, & -\tau \le \theta < 0, \\ f(\mu,\varphi), & \theta = 0. \end{cases} \tag{3.8}$$

Then (1.4) is equivalent to the abstract differential equation

$$\dot{u_t} = A(\mu)u_t + R(\mu)u_t, \tag{3.9}$$

where $u = (u_1,u_2,u_3,u_4)^T$, $u_t(\theta) = u(t+\theta)$, $\theta \in [-\tau,0]$.

For $\psi \in C([-\tau,0],(R^4)^*)$, define

$$A^*\psi(s) = \begin{cases} -\frac{d\psi(s)}{ds}, & s \in (0,\tau], \\ \int_{-\tau}^{0} d\eta^T(t,0)\psi(-t), & s = 0. \end{cases} \tag{3.10}$$

For $\phi \in C([-\tau,0],R^4)$ and $\psi \in C([0,\tau],(R^4)^*)$, define the bilinear form

$$\langle \psi,\phi\rangle = \bar{\psi}(0)\phi(0) - \int_{-\tau}^{0}\int_{\xi=0}^{\theta} \psi^T(\xi-\theta)\,d\eta(\theta)\phi(\xi)\,d\xi, \tag{3.11}$$

where $\eta(\theta) = \eta(\theta,0)$. We have the following result on the relation between the operators $A = A(0)$ and A^*.

Lemma 3.1 *$A = A(0)$ and A^* are adjoint operators.*

Proof Let $\phi \in C^1([-\tau,0],R^4)$ and $\psi \in C^1([0,\tau],R^4)^*$. It follows from (3.11) and the definitions of $A = A(0)$ and A^* that

$$\begin{aligned} \langle \psi(s), A(0)\phi(\theta)\rangle &= \bar{\psi}(0)A(0)\phi(0) - \int_{-\tau_k^{(j)}}^{0}\int_{\xi=0}^{\theta} \bar{\psi}(\xi-\theta)\,d\eta(\theta)A(0)\phi(\xi)\,d\xi \\ &= \bar{\psi}(0)\int_{-\tau_k^{(j)}}^{0} d\eta(\theta)\phi(\theta) - \int_{-\tau_k^{(j)}}^{0}\int_{\xi=0}^{\theta} \bar{\psi}(\xi-\theta)\,d\eta(\theta)A(0)\phi(\xi)\,d\xi \end{aligned}$$

$$
\begin{aligned}
&= \bar{\psi}(0)\int_{-\tau_k^{(j)}}^{0} d\eta(\theta)\phi(\theta) - \int_{-\tau_k^{(j)}}^{0}\big[\bar{\psi}(\xi-\theta)\,d\eta(\theta)\phi(\xi)\big]_{\xi=0}^{\theta} \\
&\quad + \int_{-\tau_k^{(j)}}^{0}\int_{\xi=0}^{\theta}\frac{d\bar{\psi}(\xi-\theta)}{d\xi}\,d\eta(\theta)\phi(\xi)\,d\xi \\
&= \int_{-\tau_k^{(j)}}^{0}\bar{\psi}(-\theta)\,d\eta(\theta)\phi(0) - \int_{-\tau_k^{(j)}}^{0}\int_{\xi=0}^{\theta}\left[-\frac{d\bar{\psi}(\xi-\theta)}{d\xi}\right]d\eta(\theta)\phi(\xi)\,d\xi \\
&= A*\bar{\psi}(0)\phi(0) - \int_{-\tau_k^{(j)}}^{0}\int_{\xi=0}^{\theta} A^*\bar{\psi}(\xi-\theta)\,d\eta(\theta)\phi(\xi)\,d\xi \\
&= \big\langle A^*\psi(s),\phi(\theta)\big\rangle.
\end{aligned}
$$

This shows that $A = A(0)$ and A^* are adjoint operators and the proof is complete. □

By the discussions in Section 2, we know that $\pm i\omega_k$ are eigenvalues of $A(0)$, and they are also eigenvalues of A^* corresponding to $i\omega_k$ and $-i\omega_k$, respectively. We have the following result.

Lemma 3.2 *The vector*

$$q(\theta) = (1, r_1, r_2, r_3)^T e^{i\omega_k\theta}, \quad \theta \in [-\tau, 0],$$

is the eigenvector of $A(0)$ *corresponding to the eigenvalue* $i\omega_k$, *and*

$$q^*(s) = D\big(1, r_1^*, r_2^*, r_3^*\big)e^{i\omega_k s}, \quad s \in [0, \tau],$$

is the eigenvector of A^* *corresponding to the eigenvalue* $-i\omega_k$, *moreover,* $\langle q^*(s), q(\theta)\rangle = 1$, *where*

$$D = \frac{1}{C}, \tag{3.12}$$

where

$$C = 1 + \sum_{i=1}^{3}\bar{r}_i r_i^* + (1 + \bar{r}_1 + \bar{r}_2 + \bar{r}_3)d_1 r_3^* e^{i\omega_k\tau_k^{(j)}}.$$

Proof Let $q(\theta)$ be the eigenvector of $A(0)$ corresponding to the eigenvalue $i\omega_k$ and $q^*(s)$ be the eigenvector of A^* corresponding to the eigenvalue $-i\omega_k$, namely, $A(0)q(\theta) = i\omega_k q(\theta)$ and $A^*q(s) = -i\omega_k q^*(s)$. From the definitions of $A(0)$ and A^*, we have $A(0)q(\theta) = dq(\theta)/d\theta$ and $A^*q(s) = -dq^*(s)/ds$. Thus, $q(\theta) = q(0)e^{i\omega_k\theta}$ and $q^*(s) = q(0)e^{i\omega_0 s}$. In addition,

$$
\begin{aligned}
\int_{-\tau_k^{(j)}}^{0} d\eta(\theta)q(\theta) &= \begin{pmatrix} a_1 & 0 & 0 & a_2 \\ 0 & b_1 & 0 & b_2 \\ 0 & 0 & c_1 & c_2 \\ 0 & 0 & 0 & 0 \end{pmatrix} q(0) + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ d_1 & d_1 & d_1 & 0 \end{pmatrix} q\big(-\tau_k^{(j)}\big) \\
&= A(0)q(0) = i\omega_k q(0).
\end{aligned} \tag{3.13}
$$

That is,

$$\begin{pmatrix} a_1 + a_2 r_3 \\ b_1 r_1 + b_2 r_3 \\ c_1 r_2 + c_2 r_3 \\ d_1(1 + r_1 + r_2)e^{-i\omega_k \tau_k^{(j)}} \end{pmatrix} = \begin{pmatrix} i\omega_k \\ i\omega_k r_1 \\ i\omega_k r_2 \\ i\omega_k r_3 \end{pmatrix}. \tag{3.14}$$

Therefore, we can easily obtain

$$r_1 = \frac{b_2(a_1 - i\omega_k)}{a_2(b_1 - i\omega_k)}, \qquad r_2 = \frac{c_2(a_1 - i\omega_k)}{a_2(c_1 - i\omega_k)}, \qquad r_3 = \frac{i\omega_k - a_1}{a_2}.$$

On the other hand,

$$\begin{aligned} \int_{-\tau_k^{(j)}}^{0} q^*(-t)\,d\eta(t) &= \begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & b_1 & 0 & 0 \\ 0 & 0 & c_1 & 0 \\ a_2 & b_2 & c_2 & 0 \end{pmatrix} q(0) + \begin{pmatrix} 0 & 0 & 0 & d_1 \\ 0 & 0 & 0 & d_1 \\ 0 & 0 & 0 & d_1 \\ 0 & 0 & 0 & 0 \end{pmatrix} q^*\bigl(-\tau_k^{(j)}\bigr) \\ &= A^* q^*(0) = -i\omega_0 q^*(0). \end{aligned} \tag{3.15}$$

Namely,

$$\begin{pmatrix} a_1 + d_1 r_3^* e^{-i\omega_k \tau_k^{(j)}} \\ b_1 r_1^* + d_1 r_3^* e^{-i\omega_k \tau_k^{(j)}} \\ c_1 r_2^* + d_1 r_3^* e^{-i\omega_k \tau_k^{(j)}} \\ a_2 + b_2 r_1^* + c_2 r_3^* \end{pmatrix} = \begin{pmatrix} -i\omega_k \\ -i\omega_k r_1^* \\ -i\omega_k r_2^* \\ -i\omega_k r_3^* \end{pmatrix}. \tag{3.16}$$

Therefore, we can easily obtain

$$r_1^* = \frac{a_1 + i\omega_k}{b_1 + i\omega_k}, \qquad r_2^* = \frac{a_1 + i\omega_k}{c_1 + i\omega_k}, \qquad r_3^* = -\frac{a_1 + i\omega_k}{d_1 e^{-i\omega_k \tau_k^{(j)}}}.$$

In the sequel, we shall verify that $\langle q^*(s), q(\theta)\rangle = 1$. In fact, from (3.11), we have

$$\begin{aligned} \bigl\langle q^*(s), q(\theta)\bigr\rangle &= \bar{D}\bigl(1, \bar{r}_1^*, \bar{r}_2^*, \bar{r}_3^*\bigr)(1, r_1, r_2, r_3)^T \\ &\quad - \int_{-\tau_k^{(j)}}^{0} \int_{\xi=0}^{\theta} \bar{D}\bigl(1, \bar{r}_1^*, \bar{r}_2^*, \bar{r}_3^*\bigr) e^{-i\omega_k(\xi-\theta)}\,d\eta(\theta)(1, r_1, r_2, r_3)^T e^{i\omega_k \xi}\,d\xi \\ &= \bar{D}\left[1 + \sum_{i=1}^{3} r_i \bar{r}_i^* - \int_{-\tau_k^{(j)}}^{0} \bigl(1, \bar{r}_1^*, \bar{r}_2^*, \bar{r}_3^*\bigr)\theta e^{i\omega_k \theta}\,d\eta(\theta)(1, r_1, r_2, r_3)^T\right] \\ &= \bar{D}\left\{1 + \sum_{i=1}^{3} r_i \bar{r}_i^* + \bigl(1, \bar{r}_1^*, \bar{r}_2^*, \bar{r}_3^*\bigr) B_1 e^{-i\omega_k \tau_k^{(j)}} (1, r_1, r_2, r_3)^T\right\} \\ &= \bar{D}\left[1 + \sum_{i=1}^{3} r_i \bar{r}_i^* + (1 + r_1 + r_2 + r_3) d_1 \bar{r}_3^* e^{-i\omega_k \tau_k^{(j)}}\right] = 1. \end{aligned}$$

□

Next, we use the same notations as those in Hassard *et al.* [15], and we first compute the coordinates to describe the center manifold C_0 at $\mu = 0$. Let y_t be the solution of equation (1.4) when $\mu = 0$.

Define

$$z(t) = \langle q^*, y_t \rangle, \qquad W(t,\theta) = y_t(\theta) - 2\operatorname{Re}\{z(t)q(\theta)\} \tag{3.17}$$

on the center manifold C_0, and we have

$$W(t,\theta) = W\big(z(t), \bar{z}(t), \theta\big), \tag{3.18}$$

where

$$W\big(z(t), \bar{z}(t), \theta\big) = W(z, \bar{z}) = W_{20}\frac{z^2}{2} + W_{11}z\bar{z} + W_{02}\frac{\bar{z}^2}{2} + \cdots \tag{3.19}$$

and z and $\bar{z}$ are local coordinates for center manifold C_0 in the direction of q^* and $\bar{q}^*$. Noting that W is also real if y_t is real, we consider only real solutions. For solutions $y_t \in C_0$ of (1.4),

$$\begin{aligned}
\dot{z}(t) &= \langle q^*(s), \dot{x}_t \rangle = \langle q^*(s), A(0)u_t + R(0)u_t \rangle \\
&= \langle q^*(s), A(0)y_t \rangle + \langle q^*(s), R(0)y_t \rangle \\
&= \langle A^*q^*(s), y_t \rangle + \bar{q}^*(0)R(0)y_t \\
&\quad - \int_{-\tau_k^{(j)}}^{0} \int_{\xi=0}^{\theta} \bar{q}^*(\xi - \theta)\, d\eta(\theta)A(0)R(0)y_t(\xi)\, d\xi \\
&= \langle i\omega_k q^*(s), y_t \rangle + \bar{q}^*(0)f\big(0, y_t(\theta)\big) \\
&\stackrel{\text{def}}{=} i\omega_k z(t) + \bar{q}^*(0)f_0\big(z(t), \bar{z}(t)\big).
\end{aligned} \tag{3.20}$$

That is,

$$\dot{z}(t) = i\omega_k z + g(z, \bar{z}), \tag{3.21}$$

where

$$g(z, \bar{z}) = g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + g_{21}\frac{z^2\bar{z}}{2} + \cdots. \tag{3.22}$$

Hence, we have

$$\begin{aligned}
g(z, \bar{z}) &= \bar{q}^*(0)f_0(z, \bar{z}) = f(0, y_t) \\
&= \bar{D}\big(1, \bar{r}_1^*, \bar{r}_2^*, \bar{r}_3^*\big)\big(f_1(0, y_t), f_2(0, y_t), f_3(0, y_t), f_4(0, y_t)\big)^T,
\end{aligned} \tag{3.23}$$

where

$$\begin{aligned}
f_1(0, y_t) &= a_3 y_{1t}^2(0) + a_4 y_{1t}(0)y_{1t}\big(-\tau_k^{(j)}\big) + a_5 y_{1t}(0)y_{4t}(0) + a_6 y_{1t}\big(-\tau_k^{(j)}\big)y_{4t}(0) \\
&\quad + a_7 y_{1t}^3(0) + a_8 y_{1t}^2(0)y_{1t}\big(-\tau_k^{(j)}\big) + a_9 y_{1t}^2(0)y_{4t}(0)
\end{aligned}$$

$$
\begin{aligned}
&\quad + a_{10}y_{1t}(0)y_{1t}\bigl(-\tau_k^{(j)}\bigr)y_{4t}(0) + \text{h.o.t.},\\
f_2(0,y_t) &= b_3y_{2t}^2(0) + b_4y_{2t}(0)y_{2t}\bigl(-\tau_k^{(j)}\bigr) + b_5y_{2t}(0)y_{4t}(0) + b_6y_{2t}\bigl(-\tau_k^{(j)}\bigr)y_{4t}(0)\\
&\quad + b_7y_{2t}^3(0) + b_8y_{2t}^2(0)y_{2t}\bigl(-\tau_k^{(j)}\bigr) + b_9y_{2t}^2(0)y_{4t}(0)\\
&\quad + b_{10}y_{2t}(0)y_{2t}\bigl(-\tau_k^{(j)}\bigr)y_{4t}(0) + \text{h.o.t.},\\
f_3(0,y_t) &= c_3y_{3t}^2(0) + c_4y_{3t}(0)y_{3t}\bigl(-\tau_k^{(j)}\bigr) + c_5y_{3t}(0)y_{4t}(0) + c_6y_{3t}\bigl(-\tau_k^{(j)}\bigr)y_{4t}(0)\\
&\quad + c_7y_{3t}^3(0) + c_8y_{3t}^2(0)y_{3t}\bigl(-\tau_k^{(j)}\bigr) + c_9y_{3t}^2(0)y_{4t}(0)\\
&\quad + c_{10}y_{3t}(0)y_{3t}\bigl(-\tau_k^{(j)}\bigr)y_{4t}(0) + \text{h.o.t.},\\
f_4(0,y_t) &= d_1y_{1t}\bigl(-\tau_k^{(j)}\bigr)y_{4t}(0) + d_1y_{2t}\bigl(-\tau_k^{(j)}\bigr)y_{4t}(0) + d_1y_{3t}\bigl(-\tau_k^{(j)}\bigr)y_{4t}(0).
\end{aligned}
$$

Noticing that

$$
y_t(\theta) = \bigl(y_{1t}(\theta), y_{2t}(\theta), y_{3t}(\theta), y_{4t}(\theta)\bigr)^T = W(t,\theta) + zq(\theta) + \bar{z}\bar{q}
$$

and

$$
q(\theta) = (1, r_1, r_2, r_3)^T e^{i\omega_k\theta},
$$

we have

$$
\begin{aligned}
y_{1t}(0) &= z + \bar{z} + W_{20}^{(1)}(0)\frac{z^2}{2} + W_{11}^{(1)}(0)z\bar{z} + W_{02}^{(1)}(0)\frac{\bar{z}^2}{2} + \cdots,\\
y_{2t}(0) &= r_1z + \bar{r}_1\bar{z} + W_{20}^{(2)}(0)\frac{z^2}{2} + W_{11}^{(2)}(0)z\bar{z} + W_{02}^{(2)}(0)\frac{\bar{z}^2}{2} + \cdots,\\
y_{3t}(0) &= r_2z + \bar{r}_2\bar{z} + W_{20}^{(3)}(0)\frac{z^2}{2} + W_{11}^{(3)}(0)z\bar{z} + W_{02}^{(3)}(0)\frac{\bar{z}^2}{2} + \cdots,\\
y_{4t}(0) &= r_3z + \bar{r}_3\bar{z} + W_{20}^{(4)}(0)\frac{z^2}{2} + W_{11}^{(4)}(0)z\bar{z} + W_{02}^{(4)}(0)\frac{\bar{z}^2}{2} + \cdots,\\
y_{1t}\bigl(-\tau_k^{(j)}\bigr) &= e^{-i\omega_k\tau_k^{(j)}}z + e^{i\omega_k\tau_k^{(j)}}\bar{z} + W_{20}^{(1)}\bigl(-\tau_k^{(j)}\bigr)\frac{z^2}{2}\\
&\quad + W_{11}^{(1)}\bigl(-\tau_k^{(j)}\bigr)z\bar{z} + W_{02}^{(1)}\bigl(-\tau_k^{(j)}\bigr)\frac{\bar{z}^2}{2} + \cdots,\\
y_{2t}\bigl(-\tau_k^{(j)}\bigr) &= r_1e^{-i\omega_k\tau_k^{(j)}}z + \bar{r}_1e^{i\omega_k\tau_k^{(j)}}\bar{z} + W_{20}^{(2)}\bigl(-\tau_k^{(j)}\bigr)\frac{z^2}{2}\\
&\quad + W_{11}^{(2)}\bigl(-\tau_k^{(j)}\bigr)z\bar{z} + W_{02}^{(2)}\bigl(-\tau_k^{(j)}\bigr)\frac{\bar{z}^2}{2} + \cdots,\\
y_{3t}\bigl(-\tau_k^{(j)}\bigr) &= r_2e^{-i\omega_k\tau_k^{(j)}}z + \bar{r}_2e^{i\omega_k\tau_k^{(j)}}\bar{z} + W_{20}^{(3)}\bigl(-\tau_k^{(j)}\bigr)\frac{z^2}{2}\\
&\quad + W_{11}^{(3)}\bigl(-\tau_k^{(j)}\bigr)z\bar{z} + W_{02}^{(3)}\bigl(-\tau_k^{(j)}\bigr)\frac{\bar{z}^2}{2} + \cdots,\\
y_{4t}\bigl(-\tau_k^{(j)}\bigr) &= r_3e^{-i\omega_k\tau_k^{(j)}}z + \bar{r}_3e^{i\omega_k\tau_k^{(j)}}\bar{z} + W_{20}^{(4)}\bigl(-\tau_k^{(j)}\bigr)\frac{z^2}{2}\\
&\quad + W_{11}^{(4)}\bigl(-\tau_k^{(j)}\bigr)z\bar{z} + W_{02}^{(4)}\bigl(-\tau_k^{(j)}\bigr)\frac{\bar{z}^2}{2} + \cdots.
\end{aligned}
$$

From (3.22) and (3.23), we have

$$\begin{aligned}
g(z,\bar{z}) &= \bar{q}^*(0)f_0(z,\bar{z}) \\
&= \bar{D}\big[f_1(0,y_t) + \bar{r}_1^* f_2(0,x_t) + \bar{r}_2^* f_3(0,y_t) + \bar{r}_3^* f_4(0,y_t)\big] \\
&= \bar{D}\big[a_3 + a_4 e^{-i\omega_k\tau_k^{(j)}} + a_5 r_3 + a_6 r_3 e^{-i\omega_k\tau_k^{(j)}} \\
&\quad + \bar{r}_1^*\big(b_3 r_1^2 + b_4 r_1^2 e^{-i\omega_k\tau_k^{(j)}} + b_5 r_1 r_3 + b_6 r_1 r_3 e^{-i\omega_k\tau_k^{(j)}}\big) \\
&\quad + \bar{r}_2^*\big(c_3 r_2^2 + c_4 r_2^2 e^{-i\omega_k\tau_k^{(j)}} + c_5 r_2 r_3 + c_6 r_2 r_3 e^{-i\omega_k\tau_k^{(j)}}\big) \\
&\quad + \bar{r}_3^*\big(d_1 r_3 e^{-i\omega_k\tau_k^{(j)}} + d_1 r_1 r_3 e^{-i\omega_k\tau_k^{(j)}} + d_1 r_2 r_3 e^{-i\omega_k\tau_k^{(j)}}\big)\big]z^2 \\
&\quad + \bar{D}\big[2a_3 + 2a_4 \operatorname{Re}\big\{r_3 e^{i\omega_k\tau_k^{(j)}}\big\} + 2a_5 \operatorname{Re}\{r_3\} + 2a_6 \operatorname{Re}\big\{r_3 e^{i\omega_k\tau_k^{(j)}}\big\} \\
&\quad + r_1^*\big(2b_3|r_1|^2 + 2b_4 \operatorname{Re}\big\{r_1\bar{r}_1 e^{-i\omega_k\tau_k^{(j)}}\big\} + 2b_5 \operatorname{Re}\{r_1\bar{r}_3\} + 2b_6 \operatorname{Re}\big\{r_1\bar{r}_3 e^{-i\omega_k\tau_k^{(j)}}\big\}\big) \\
&\quad + r_2^*\big(2c_3|r_2|^2 + 2c_4 \operatorname{Re}\big\{|r_2|^2 e^{i\omega_k\tau_k^{(j)}}\big\} + 2c_5 \operatorname{Re}\{r_2\bar{r}_3\} + 2c_6 \operatorname{Re}\big\{r_2\bar{r}_3 e^{-i\omega_k\tau_k^{(j)}}\big\}\big) \\
&\quad + r_3^*\big(2d_1 \operatorname{Re}\big\{r_3 e^{i\omega_k\tau_k^{(j)}}\big\} + 2d_1 \operatorname{Re}\big\{r_1 e^{-i\omega_k\tau_k^{(j)}}\big\} + 2d_1 \operatorname{Re}\big\{r_2 e^{-i\omega_k\tau_k^{(j)}}\big\}\big)\big]z\bar{z} \\
&\quad + \bar{D}\big[a_3 + a_4 e^{i\omega_k\tau_k^{(j)}} + a_5\bar{r}_3 + a_6\bar{r}_3 e^{i\omega_k\tau_k^{(j)}} \\
&\quad + r_1^*\big(b_3\bar{r}_1^2 + b_4\bar{r}_1^2 e^{i\omega_k\tau_k^{(j)}} + b_5\bar{r}_1\bar{r}_3 + b_6\bar{r}_1\bar{r}_3 e^{i\omega_k\tau_k^{(j)}}\big) \\
&\quad + r_2^*\big(c_3\bar{r}_2^2 + c_4\bar{r}_2^2 e^{i\omega_k\tau_k^{(j)}} + c_5\bar{r}_2 r_3 + c_6\bar{r}_2\bar{r}_3 e^{i\omega_k\tau_k^{(j)}}\big) \\
&\quad + r_3\big(d_1\bar{r}_3 e^{i\omega_k\tau_k^{(j)}} + d_1\bar{r}_1\bar{r}_3 e^{i\omega_k\tau_k^{(j)}} + d_1\bar{r}_2\bar{r}_3 e^{i\omega_k\tau_k^{(j)}}\big)\big]\bar{z}^2 \\
&\quad + \bar{D}\bigg\{r_1^*\bigg[b_3\big(2r_1 W_{11}^{(2)}(0) + W_{201}^{(2)}(0)\bar{r}_1\big) \\
&\quad + b_4\bigg(r_1 W_{11}^{(2)}\big(-\tau_k^{(j)}\big) + \frac{1}{2}\bar{r}_1 W_{20}^{(2)}\big(-\tau_k^{(j)}\big) \\
&\quad + \frac{1}{2}\bar{r}_1 W_{20}^{(2)}(0)e^{-i\omega_k\tau_k^{(j)}} + r_1 W_{11}^{(2)}(0)e^{-i\omega_k\tau_k^{(j)}}\bigg) + b_5\bigg(r_1 W_{11}^{(4)}(0) + \frac{1}{2}\bar{r}_1 W_{20}^{(4)}(0) \\
&\quad + \frac{1}{2}\bar{r}_3 W_{20}^{(2)}(0) + r_3 W_{11}^{(2)}(0)\bigg) + b_6\bigg(r_1 W_{11}^{(4)}(0) + \frac{1}{2}\bar{r}_1 W_{20}^{(4)}(0)e^{i\omega_k\tau_k^{(j)}} \\
&\quad + \frac{1}{2}\bar{r}_3 W_{20}^{(2)}(-\tau)e^{i\omega_k\tau_k^{(j)}} + r_3 W_{11}^{(2)}\big(-\tau_k^{(j)}\big)\bigg) \\
&\quad + 3b_7 r_1^2\bar{r}_1 + b_8\big(r_1^2\bar{r}_1 e^{i\omega_k\tau_k^{(j)}} + 2r_1^2\bar{r}_1 e^{-i\omega_k\tau_k^{(j)}}\big) \\
&\quad + b_9\big(r_1^2\bar{r}_3 + 2|r_1|^2 r_3\big) + b_{10}\big(r_1^2 e^{-i\omega_k\tau_k^{(j)}} + |r_1|^2 r_3 e^{i\omega_k\tau_k^{(j)}} + |r_1|^2 r_3 e^{-i\omega_k\tau_k^{(j)}}\big)\bigg] \\
&\quad + r_2^*\bigg[3c_3 r_2^2\bar{r}_2 + c_4\bigg(r_2 W_{11}^{(3)}\big(-\tau_k^{(j)}\big) + \frac{1}{2}\bar{r}_2 W_{20}^{(3)}\big(-\tau_k^{(j)}\big) \\
&\quad + \frac{1}{2}\bar{r}_2 W_{20}^{(3)}\big(-\tau_k^{(j)}\big)e^{i\omega_k\tau_k^{(j)}} + r_2 W_{11}^{(3)}(0)e^{-i\omega_k\tau_k^{(j)}}\bigg) \\
&\quad + c_5\bigg(r_2 W_{11}^{(4)}(0) + \frac{1}{2}\bar{r}_2 W_{20}^{(4)}(0) + \frac{1}{2}\bar{r}_3 W_{20}^{(3)}(0) + r_3 W_{11}^{(3)}(0)\bigg) \\
&\quad + c_6\bigg(r_2 W_{11}^{(4)}(0)e^{-i\omega_k\tau_k^{(j)}} + \frac{1}{2}\bar{r}_2 W_{20}^{(4)}(0)e^{i\omega_k\tau_k^{(j)}}
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{1}{2}\bar{r}_3 W_{20}^{(3)}\left(-\tau_k^{(j)}\right) + r_3 W_{11}^{(3)}\left(-\tau_k^{(j)}\right)\bigg) \\
&+ 3c_7 r_2^2\bar{r}_2 + c_8\left(r_2^2\bar{r}_2 e^{i\omega_k\tau_k^{(j)}} + 2r_2^2\bar{r}_2 e^{-i\omega_k\tau_k^{(j)}}\right) \\
&+ c_9\left(r_2^2\bar{r}_3 + 2|r_2|^2 r_3\right) + c_{10}\left(r_2^2\bar{r}_3 e^{-i\omega_k\tau_k^{(j)}}\right. \\
&\left.+ |r_2|^2 r_3 e^{-i\omega_k\tau_k^{(j)}} + |r_2|^2 r_3 e^{i\omega_k\tau_k^{(j)}}\right)\bigg] + \bar{r}_3^* d_1\bigg[W_{11}^{(4)}(0) + \frac{1}{2}W_{20}^{(4)}(0) + \frac{1}{2}\bar{r}_3 W_{20}^{(1)}(0) \\
&+ r_3 W_{11}^{(1)}(0) + r_1 W_{11}^{(4)}(0)e^{-i\omega_k\tau_k^{(j)}} + \frac{1}{2}\bar{r}_1 W_{20}^{(4)}(0)e^{i\omega_k\tau_k^{(j)}} \\
&+ \frac{1}{2}\bar{r}_3 W_{20}^{(4)}\left(-\tau_k^{(j)}\right) + r_3 W_{11}^{(2)}\left(-\tau_k^{(j)}\right) + r_2 W_{11}^{(4)}(0)e^{-i\omega_k\tau_k^{(j)}} \\
&+ \frac{1}{2}\bar{r}_2 W_{20}^{(4)}(0)e^{i\omega_k\tau_k^{(j)}} + \frac{1}{2}\bar{r}_3 W_{20}^{(3)}\left(-\tau_k^{(j)}\right)e^{i\omega_k\tau_k^{(j)}} + r_3 W_{11}^{(3)}\left(-\tau_k^{(j)}\right)\bigg] z^2\bar{z} + \cdots\bigg\}
\end{aligned}
$$

and we obtain

$$
\begin{aligned}
g_{20} = {}& 2\bar{D}\big[a_3 + a_4 e^{-i\omega_k\tau_k^{(j)}} + a_5 r_3 + a_6 r_3 e^{-i\omega_k\tau_k^{(j)}} \\
&+ \bar{r}_1^*\left(b_3 r_1^2 + b_4 r_1^2 e^{-i\omega_k\tau_k^{(j)}} + b_5 r_1 r_3 + b_6 r_1 r_3 e^{-i\omega_k\tau_k^{(j)}}\right) \\
&+ \bar{r}_2^*\left(c_3 r_2^2 + c_4 r_2^2 e^{-i\omega_k\tau_k^{(j)}} + c_5 r_2 r_3 + c_6 r_2 r_3 e^{-i\omega_k\tau_k^{(j)}}\right) \\
&+ \bar{r}_3^*\left(d_1 r_3 e^{-i\omega_k\tau_k^{(j)}} + d_1 r_1 r_3 e^{-i\omega_k\tau_k^{(j)}} + d_1 r_2 r_3 e^{-i\omega_k\tau_k^{(j)}}\right)\big], \\
g_{11} = {}& \bar{D}\big[2a_3 + 2a_4 \operatorname{Re}\big\{r_3 e^{i\omega_k\tau_k^{(j)}}\big\} + 2a_5 \operatorname{Re}\{r_3\} + 2a_6 \operatorname{Re}\big\{r_3 e^{i\omega_k\tau_k^{(j)}}\big\} \\
&+ r_1^*\left(2b_3|r_1|^2 + 2b_4 \operatorname{Re}\big\{r_1\bar{r}_1 e^{-i\omega_k\tau_k^{(j)}}\big\} + 2b_5 \operatorname{Re}\{r_1\bar{r}_3\} + 2b_6 \operatorname{Re}\big\{r_1\bar{r}_3 e^{-i\omega_k\tau_k^{(j)}}\big\}\right) \\
&+ r_2^*\left(2c_3|r_2|^2 + 2c_4 \operatorname{Re}\big\{|r_2|^2 e^{i\omega_0\tau_k^{(j)}}\big\} + 2c_5 \operatorname{Re}\{r_2\bar{r}_3\} + 2c_6 \operatorname{Re}\big\{r_2\bar{r}_3 e^{-i\omega_k\tau_k^{(j)}}\big\}\right) \\
&+ r_3^*\left(2d_1 \operatorname{Re}\big\{r_3 e^{i\omega_k\tau_k^{(j)}}\big\} + 2d_1 \operatorname{Re}\big\{r_1 e^{-i\omega_k\tau_k^{(j)}}\big\} + 2d_1 \operatorname{Re}\big\{r_2 e^{-i\omega_k\tau_k^{(j)}}\big\}\right)\big], \\
g_{02} = {}& 2\bar{D}\big[a_3 + a_4 e^{i\omega_k\tau_k^{(j)}} + a_5\bar{r}_3 + a_6\bar{r}_3 e^{i\omega_k\tau_k^{(j)}} \\
&+ r_1^*\left(b_3\bar{r}_1^2 + b_4\bar{r}_1^2 e^{i\omega_k\tau_k^{(j)}} + b_5\bar{r}_1\bar{r}_3 + b_6\bar{r}_1\bar{r}_3 e^{i\omega_k\tau_k^{(j)}}\right) \\
&+ r_2^*\left(c_3\bar{r}_2^2 + c_4\bar{r}_2^2 e^{i\omega_k\tau_k^{(j)}} + c_5\bar{r}_2 r_3 + c_6\bar{r}_2\bar{r}_3 e^{i\omega_k\tau_k^{(j)}}\right) \\
&+ r_3\left(d_1\bar{r}_3 e^{i\omega_k\tau_k^{(j)}} + d_1\bar{r}_1\bar{r}_3 e^{i\omega_k\tau_k^{(j)}} + d_1\bar{r}_2\bar{r}_3 e^{i\omega_k\tau_k^{(j)}}\right)\big], \\
g_{21} = {}& 2\bar{D}\bigg\{r_1^*\bigg[b_3\left(2r_1 W_{11}^{(2)}(0) + W_{201}^{(2)}(0)\bar{r}_1\right) + b_4\bigg(r_1 W_{11}^{(2)}\left(-\tau_k^{(j)}\right) + \frac{1}{2}\bar{r}_1 W_{20}^{(2)}\left(-\tau_k^{(j)}\right) \\
&+ \frac{1}{2}\bar{r}_1 W_{20}^{(2)}(0)e^{-i\omega_k\tau_k^{(j)}} + r_1 W_{11}^{(2)}(0)e^{-i\omega_k\tau_k^{(j)}}\bigg) + b_5\bigg(r_1 W_{11}^{(4)}(0) + \frac{1}{2}\bar{r}_1 W_{20}^{(4)}(0) \\
&+ \frac{1}{2}\bar{r}_3 W_{20}^{(2)}(0) + r_3 W_{11}^{(2)}(0)\bigg) + b_6\bigg(r_1 W_{11}^{(4)}(0) + \frac{1}{2}\bar{r}_1 W_{20}^{(4)}(0)e^{i\omega_k\tau_k^{(j)}} \\
&+ \frac{1}{2}\bar{r}_3 W_{20}^{(2)}\left(-\tau_k^{(j)}\right)e^{i\omega_k\tau_k^{(j)}} + r_3 W_{11}^{(2)}\left(-\tau_k^{(j)}\right)\bigg) \\
&+ 3b_7 r_1^2\bar{r}_1 + b_8\left(r_1^2\bar{r}_1 e^{i\omega_k\tau_k^{(j)}} + 2r_1^2\bar{r}_1 e^{-i\omega_k\tau_k^{(j)}}\right) \\
&+ b_9\left(r_1^2\bar{r}_3 + 2|r_1|^2 r_3\right) + b_{10}\left(r_1^2 e^{-i\omega_k\tau_k^{(j)}} + |r_1|^2 r_3 e^{i\omega_k\tau_k^{(j)}} + |r_1|^2 r_3 e^{-i\omega_k\tau_k^{(j)}}\right)\bigg]
\end{aligned}
$$

$$
\begin{aligned}
&+ r_2^*\bigg[3c_3 r_2^2\bar{r}_2 + c_4\bigg(r_2 W_{11}^{(3)}\big(-\tau_k^{(j)}\big) + \frac{1}{2}\bar{r}_2 W_{20}^{(3)}\big(-\tau_k^{(j)}\big) + \frac{1}{2}\bar{r}_2 W_{20}^{(3)}(-\tau)e^{i\omega_k\tau_k^{(j)}} \\
&+ r_2 W_{11}^{(3)}(0)e^{-i\omega_k\tau_k^{(j)}}\bigg) + c_5\bigg(r_2 W_{11}^{(4)}(0) + \frac{1}{2}\bar{r}_2 W_{20}^{(4)}(0) + \frac{1}{2}\bar{r}_3 W_{20}^{(3)}(0) + r_3 W_{11}^{(3)}(0)\bigg) \\
&+ c_6\bigg(r_2 W_{11}^{(4)}(0)e^{-i\omega_k\tau_k^{(j)}} + \frac{1}{2}\bar{r}_2 W_{20}^{(4)}(0)e^{i\omega_k\tau_k^{(j)}} + \frac{1}{2}\bar{r}_3 W_{20}^{(3)}\big(-\tau_k^{(j)}\big) + r_3 W_{11}^{(3)}\big(-\tau_k^{(j)}\big)\bigg) \\
&+ 3c_7 r_2^2\bar{r}_2 + c_8\big(r_2^2\bar{r}_2 e^{i\omega_k\tau_k^{(j)}} + 2r_2^2\bar{r}_2 e^{-i\omega_k\tau_k^{(j)}}\big) + c_9\big(r_2^2\bar{r}_3 + 2|r_2|^2 r_3\big) + c_{10}\big(r_2^2\bar{r}_3 e^{-i\omega_k\tau_k^{(j)}} \\
&+ |r_2|^2 r_3 e^{-i\omega_k\tau_k^{(j)}} + |r_2|^2 r_3 e^{i\omega_k\tau_k^{(j)}}\big)\bigg] + \bar{r}_3^* d_1\bigg[W_{11}^{(4)}(0) + \frac{1}{2}W_{20}^{(4)}(0) + \frac{1}{2}\bar{r}_3 W_{20}^{(1)}(0) \\
&+ r_3 W_{11}^{(1)}(0) + r_1 W_{11}^{(4)}(0)e^{-i\omega_k\tau_k^{(j)}} + \frac{1}{2}\bar{r}_1 W_{20}^{(4)}(0)e^{i\omega_k\tau_k^{(j)}} \\
&+ \frac{1}{2}\bar{r}_3 W_{20}^{(4)}\big(-\tau_k^{(j)}\big) + r_3 W_{11}^{(2)}\big(-\tau_k^{(j)}\big) + r_2 W_{11}^{(4)}(0)e^{-i\omega_k\tau_k^{(j)}} \\
&+ \frac{1}{2}\bar{r}_2 W_{20}^{(4)}(0)e^{i\omega_k\tau_k^{(j)}} + \frac{1}{2}\bar{r}_3 W_{20}^{(3)}\big(-\tau_k^{(j)}\big)e^{i\omega_k\tau_k^{(j)}} + r_3 W_{11}^{(3)}\big(-\tau_k^{(j)}\big)\bigg]\bigg\}.
\end{aligned}
$$

For

$$
\begin{array}{llllll}
W_{20}^{(2)}(0), & W_{11}^{(2)}(0), & W_{11}^{(2)}\big(-\tau_k^{(j)}\big), & W_{20}^{(2)}\big(-\tau_k^{(j)}\big), & W_{11}^{(4)}(0), & W_{20}^{(4)}(0), \\
W_{11}^{(4)}(0), & W_{20}^{(4)}(0), & W_{11}^{(2)}\big(-\tau_k^{(j)}\big), & W_{11}^{(3)}\big(-\tau_k^{(j)}\big), & W_{11}^{(2)}(0), & W_{20}^{(3)}(0), \\
W_{11}^{(3)}(0), & W_{11}^{(4)}(0), & W_{20}^{(3)}\big(-\tau_k^{(j)}\big), & W_{11}^{(3)}\big(-\tau_k^{(j)}\big), & W_{20}^{(3)}\big(-\tau_k^{(j)}\big) &
\end{array}
$$

unknown in g_{21}, we still need to compute them.

From (3.9), (3.23), we have

$$
\begin{aligned}
W' &= \begin{cases} AW - 2\operatorname{Re}\{\bar{q}^*(0)f_0 q(\theta)\}, & -\tau_k^{(j)} \le \theta < 0, \\ AW - 2\operatorname{Re}\{\bar{q}^*(0)f_0 q(\theta)\} + f_0, & \theta = 0 \end{cases} \\
&\stackrel{\text{def}}{=} AW + H(z,\bar{z},\theta),
\end{aligned}
\tag{3.24}
$$

where

$$
H(z,\bar{z},\theta) = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta)z\bar{z} + H_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots. \tag{3.25}
$$

Comparing the coefficients, we obtain

$$
(A - 2i\omega_0)W_{20}(\theta) = -H_{20}(\theta), \tag{3.26}
$$

$$
AW_{11}(\theta) = -H_{11}(\theta), \tag{3.27}
$$

$$
\ldots.
$$

And we know that for $\theta \in [-\tau_k^{(j)}, 0)$,

$$
H(z,\bar{z},\theta) = -\bar{q}^*(0)f_0 q(\theta) - q^*(0)\bar{f}_0\bar{q}(\theta) = -g(z,\bar{z})q(\theta) - \bar{g}(z,\bar{z})\bar{q}(\theta). \tag{3.28}
$$

Comparing the coefficients of (3.25) with (3.28) gives

$$H_{20}(\theta) = -g_{20}q(\theta) - \bar{g}_{02}\bar{q}(\theta), \tag{3.29}$$

$$H_{11}(\theta) = -g_{11}q(\theta) - \bar{g}_{11}\bar{q}(\theta). \tag{3.30}$$

From (3.26), (3.29), and the definition of A, we get

$$\dot{W}_{20}(\theta) = 2i\omega_k W_{20}(\theta) + g_{20}q(\theta) + \bar{g}_{02}\bar{q}(\theta). \tag{3.31}$$

Noting that $q(\theta) = q(0)e^{i\omega_0\theta}$, we have

$$W_{20}(\theta) = \frac{ig_{20}}{\omega_k}q(0)e^{i\omega_k\theta} + \frac{i\bar{g}_{02}}{3\omega_k}\bar{q}(0)e^{-i\omega_k\theta} + E_1e^{2i\omega_k\theta}, \tag{3.32}$$

where E_1 is a constant vector. Similarly, from (3.27), (3.30), and the definition of A, we have

$$\dot{W}_{11}(\theta) = g_{11}q(\theta) + \bar{g}_{11}\bar{q}(\theta), \tag{3.33}$$

$$W_{11}(\theta) = -\frac{ig_{11}}{\omega_k}q(0)e^{i\omega_k\theta} + \frac{i\bar{g}_{11}}{\omega_k}\bar{q}(0)e^{-i\omega_k\theta} + E_2, \tag{3.34}$$

where E_2 is a constant vector.

In the following, we shall seek appropriate E_1, E_2 in (3.32), (3.34), respectively. It follows from the definition of A and (3.29), (3.30) that

$$\int_{-\tau_k^{(j)}}^{0} d\eta(\theta)W_{20}(\theta) = 2i\omega_k W_{20}(0) - H_{20}(0) \tag{3.35}$$

and

$$\int_{-\tau_k^{(j)}}^{0} d\eta(\theta)W_{11}(\theta) = -H_{11}(0), \tag{3.36}$$

where $\eta(\theta) = \eta(0,\theta)$.

From (3.30), we have

$$H_{20}(0) = -g_{20}q(0) - \bar{g}_{02}\bar{q}(0) + (H_1, H_2, H_3, H_4)^T, \tag{3.37}$$

where

$$\begin{aligned}
H_1 &= 2\bigl(a_3 + a_4e^{-i\omega_k\tau_k^{(j)}} + a_5r_3 + a_6r_3e^{-i\omega_k\tau_k^{(j)}}\bigr), \\
H_2 &= 2\bigl(b_3r_1^2 + b_4r_1^2e^{-i\omega_k\tau_k^{(j)}} + b_5r_1r_3 + b_6r_1r_3e^{-i\omega_k\tau_k^{(j)}}\bigr), \\
H_3 &= 2\bigl(c_3r_2^2 + c_4r_2^2e^{-i\omega_k\tau_k^{(j)}} + c_5r_2r_3 + c_6r_2r_3e^{-i\omega_k\tau_k^{(j)}}\bigr), \\
H_4 &= 2\bigl(d_1r_3e^{-i\omega_k\tau_k^{(j)}} + d_1r_1r_3e^{-i\omega_k\tau_k^{(j)}} + d_1r_2r_3e^{-i\omega_k\tau_k^{(j)}}\bigr).
\end{aligned}$$

From (3.31), we have

$$H_{11}(0) = -g_{11}q(0) - \bar{g}_{11}(0)\bar{q}(0) + (P_1, P_2, P_3, P_4)^T, \tag{3.38}$$

where

$$\begin{aligned}
P_1 &= 2a_3 + 2a_4 \operatorname{Re}\left\{r_3 e^{i\omega_k \tau_k^{(j)}}\right\} + 2a_5 \operatorname{Re}\{r_3\} + 2a_6 \operatorname{Re}\left\{r_3 e^{i\omega_k \tau_k^{(j)}}\right\}, \\
P_2 &= 2b_3|r_1|^2 + 2b_4 \operatorname{Re}\left\{r_1\bar{r}_1 e^{-i\omega_k \tau_k^{(j)}}\right\} + 2b_5 \operatorname{Re}\{r_1\bar{r}_3\} + 2b_6 \operatorname{Re}\left\{r_1\bar{r}_3 e^{-i\omega_k \tau_k^{(j)}}\right\}, \\
P_3 &= 2c_3|r_2|^2 + 2c_4 \operatorname{Re}\left\{|r_2|^2 e^{i\omega_k \tau_k^{(j)}}\right\} + 2c_5 \operatorname{Re}\{r_2\bar{r}_3\} + 2c_6 \operatorname{Re}\left\{r_2\bar{r}_3 e^{-i\omega_k \tau_k^{(j)}}\right\}, \\
P_4 &= 2d_1 \operatorname{Re}\left\{r_3 e^{i\omega_k \tau_k^{(j)}}\right\} + 2d_1 \operatorname{Re}\left\{r_1 e^{-i\omega_k \tau_k^{(j)}}\right\} + 2d_1 \operatorname{Re}\left\{r_2 e^{-i\omega_k \tau_k^{(j)}}\right\}.
\end{aligned}$$

From (3.26), (3.27), and the definition of A, we have

$$\begin{cases} BW_{20}(0) + B_1 W_{20}(-\tau_k^{(j)}) = 2i\omega_k W_{20}(0) - H_{20}(0), \\ BW_{11}(0) + B_1 W_{11}(-\tau_k^{(j)}) = -H_{11}(0). \end{cases} \tag{3.39}$$

Noting that

$$\left(i\omega_k I - \int_{-\tau_k^{(j)}}^{0} e^{i\omega_k\theta}\, d\eta(\theta)\right) q(0) = 0, \tag{3.40}$$

$$\left(-i\omega_k I - \int_{-\tau_k^{(j)}}^{0} e^{-i\omega_k\theta}\, d\eta(\theta)\right) \bar{q}(0) = 0, \tag{3.41}$$

and substituting (3.36) and (3.41) into (3.39), we have

$$\left(2i\omega_k I - \int_{-\tau_k^{(j)}}^{0} e^{2i\omega_k\theta}\, d\eta(\theta)\right) E_1 = (H_1, H_2, H_3, H_4)^T. \tag{3.42}$$

That is,

$$\det \begin{pmatrix} 2i\omega_k - a_1 & 0 & 0 & -a_2 \\ 0 & 2i\omega_k - b_1 & 0 & -b_2 \\ 0 & 0 & 2i\omega_k - c_1 & -c_2 \\ -d_1 e^{-2i\omega_k\tau_k^{(j)}} & -d_1 e^{-2i\omega_k\tau_k^{(j)}} & -d_1 e^{-2i\omega_k\tau_k^{(j)}} & 2i\omega_k \end{pmatrix} \begin{pmatrix} E_1^{(1)} \\ E_1^{(2)} \\ E_1^{(3)} \\ E_1^{(4)} \end{pmatrix} = \begin{pmatrix} H_1 \\ H_2 \\ H_3 \\ H_4 \end{pmatrix}. \tag{3.43}$$

Hence,

$$E_1^{(1)} = \frac{\Delta_{11}}{\Delta_1}, \qquad E_1^{(2)} = \frac{\Delta_{12}}{\Delta_1}, \qquad E_1^{(3)} = \frac{\Delta_{13}}{\Delta_1}, \qquad E_1^{(4)} = \frac{\Delta_{14}}{\Delta_1}, \tag{3.44}$$

where

$$\Delta_1 = \det \begin{pmatrix} 2i\omega_k - a_1 & 0 & 0 & -a_2 \\ 0 & 2i\omega_k - b_1 & 0 & -b_2 \\ 0 & 0 & 2i\omega_k - c_1 & -c_2 \\ -d_1 e^{-2i\omega_k\tau_k^{(j)}} & -d_1 e^{-2i\omega_k\tau_k^{(j)}} & -d_1 e^{-2i\omega_k\tau_k^{(j)}} & 2i\omega_k \end{pmatrix},$$

$$\Delta_{11} = \det\begin{pmatrix} H_1 & 0 & 0 & -a_2 \\ H_2 & 2i\omega_k - b_1 & 0 & -b_2 \\ H_3 1 & 0 & 2i\omega_k - c_1 & -c_2 \\ H_4 & -d_1 e^{-2i\omega_k \tau_k^{(j)}} & -d_1 e^{-2i\omega_k \tau_k^{(j)}} & 2i\omega_k \end{pmatrix},$$

$$\Delta_{12} = \det\begin{pmatrix} 2i\omega_k - a_1 & H_1 & 0 & -a_2 \\ 0 & H_2 & 0 & -b_2 \\ 0 & H_3 & 2i\omega_k - c_1 & -c_2 \\ -d_1 e^{-2i\omega_k \tau_k^{(j)}} & H_4 & -d_1 e^{-2i\omega_k \tau_k^{(j)}} & 2i\omega_k \end{pmatrix},$$

$$\Delta_{13} = \det\begin{pmatrix} 2i\omega_k - a_1 & 0 & H_1 & -a_2 \\ 0 & 2i\omega_k - b_1 & H_2 & -b_2 \\ 0 & 0 & H_3 & -c_2 \\ -d_1 e^{-2i\omega_k \tau_k^{(j)}} & -d_1 e^{-2i\omega_k \tau_k^{(j)}} & H_4 & 2i\omega_k \end{pmatrix},$$

$$\Delta_{14} = \det\begin{pmatrix} 2i\omega_k - a_1 & 0 & 0 & H_1 \\ 0 & 2i\omega_k - b_1 & 0 & H_2 \\ 0 & 0 & 2i\omega_k - c_1 & H_3 \\ -d_1 e^{-2i\omega_k \tau_k^{(j)}} & -d_1 e^{-2i\omega_k \tau_k^{(j)}} & -d_1 e^{-2i\omega_k \tau_k^{(j)}} & H_4 \end{pmatrix}.$$

Similarly, substituting (3.37) and (3.42) into (3.40), we have

$$\left(\int_{-\tau_k^{(j)}}^{0} d\eta(\theta)\right) E_2 = (P_1, P_2, P_3, P_4)^T. \tag{3.45}$$

That is,

$$\begin{pmatrix} a_1 & 0 & 0 & a_2 \\ 0 & b_1 & 0 & b_2 \\ 0 & 0 & c_1 & c_2 \\ d_1 & d_1 & d_1 & 0 \end{pmatrix} \begin{pmatrix} E_2^{(1)} \\ E_2^{(2)} \\ E_2^{(3)} \\ E_2^{(4)} \end{pmatrix} = \begin{pmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{pmatrix}. \tag{3.46}$$

Hence,

$$E_2^{(1)} = \frac{\Delta_{21}}{\Delta_2}, \qquad E_2^{(2)} = \frac{\Delta_{22}}{\Delta_2}, \qquad E_2^{(3)} = \frac{\Delta_{23}}{\Delta_2}, \qquad E_2^{(4)} = \frac{\Delta_{24}}{\Delta_2}, \tag{3.47}$$

where

$$\Delta_2 = \det\begin{pmatrix} a_1 & 0 & 0 & a_2 \\ 0 & b_1 & 0 & b_2 \\ 0 & 0 & c_1 & c_2 \\ d_1 & d_1 & d_1 & 0 \end{pmatrix},$$

$$\Delta_{21} = \det\begin{pmatrix} -P_1 & 0 & 0 & a_2 \\ -P_2 & b_1 & 0 & b_2 \\ -P_3 & 0 & c_1 & c_2 \\ -P_4 & d_1 & d_1 & 0 \end{pmatrix},$$

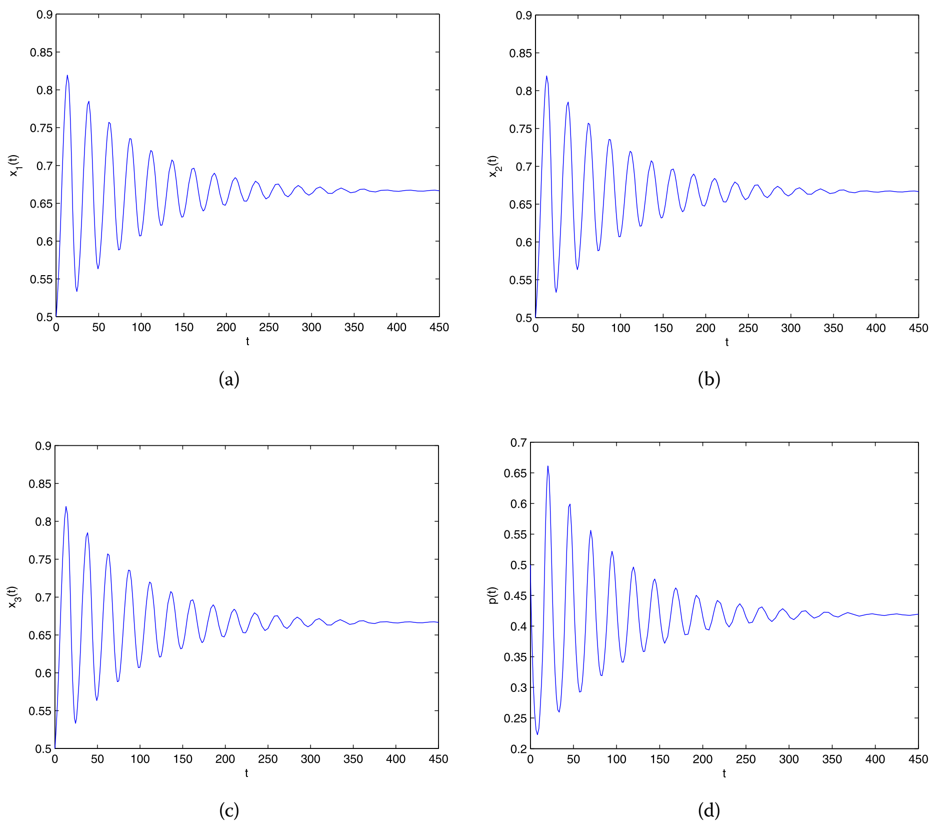

Figure 1 Dynamic behavior of system (4.1): times series of x_i ($i = 1, 2, 3$) and $p(t)$. A Matlab simulation of the asymptotically stable origin to system (4.1) with $\tau = 1.65 < \tau_0 \approx 1.85$. The initial value is $(0.5, 0.5, 0.5, 0.5)$.

$$\Delta_{22} = \det \begin{pmatrix} a_1 & -P_1 & 0 & a_2 \\ 0 & -P_2 & 0 & b_2 \\ 0 & -P_3 & c_1 & c_2 \\ d_1 & -P_4 & d_1 & 0 \end{pmatrix},$$

$$\Delta_{23} = \det \begin{pmatrix} a_1 & 0 & -P_1 & a_2 \\ 0 & b_1 & -P_2 & b_2 \\ 0 & 0 & -P_3 & c_2 \\ d_1 & d_1 & -P_4 & 0 \end{pmatrix},$$

$$\Delta_{24} = \det \begin{pmatrix} a_1 & 0 & 0 & -P_1 \\ 0 & b_1 & 0 & -P_2 \\ 0 & 0 & c_1 & -P_3 \\ d_1 & d_1 & d_1 & -P_4 \end{pmatrix}.$$

From (3.32), (3.34), (3.44), (3.47), we can calculate g_{21} and derive the following values:

$$c_1(0) = \frac{i}{2\omega_k}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right) + \frac{g_{21}}{2},$$

$$\mu_2 = -\frac{\operatorname{Re}\{c_1(0)\}}{\operatorname{Re}\{\lambda'(\tau_k^{(j)})\}},$$

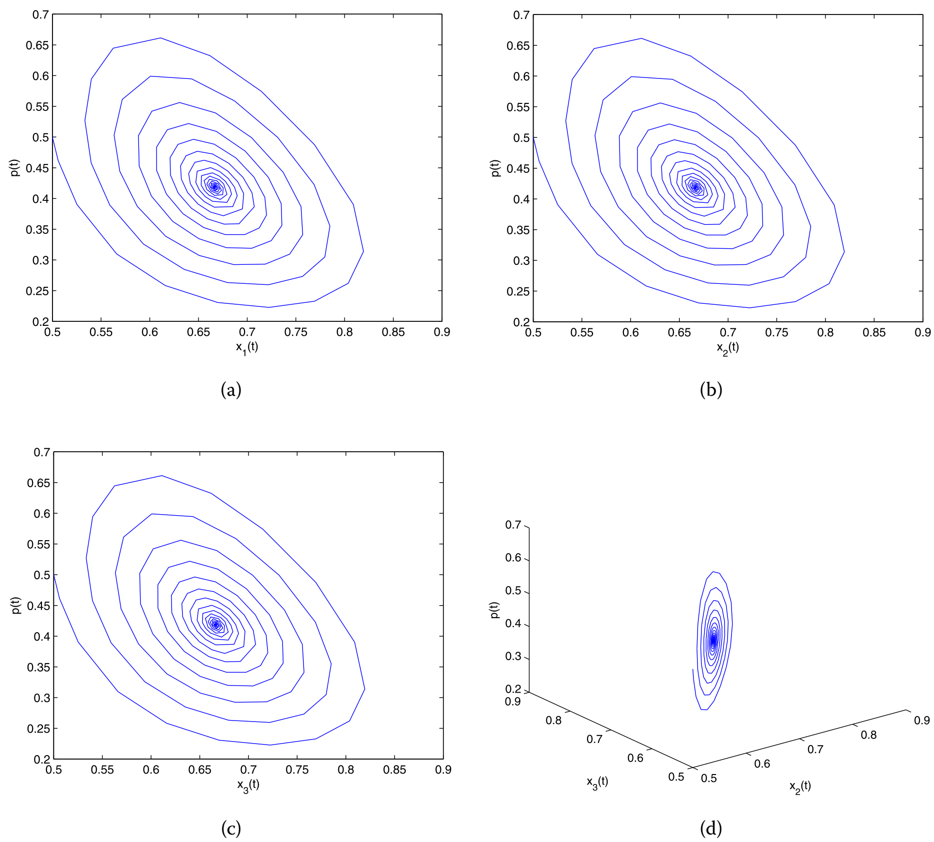

Figure 2 Dynamic behavior of system (4.1): projection on x_1-p, x_2-p, x_3-p plane and projection on x_2-x_3-p space. A Matlab simulation of the asymptotically stable origin to system (4.1) with $\tau = 1.65 < \tau_0 \approx 1.85$. The initial value is $(0.5, 0.5, 0.5, 0.5)$.

$$\beta_2 = 2\,\mathrm{Re}\left(c_1(0)\right),$$

$$T_2 = -\frac{\mathrm{Im}\{c_1(0)\} + \mu_2\,\mathrm{Im}\{\lambda'(\tau_k^{(j)})\}}{\omega_k}.$$

These formulas give a description of the Hopf bifurcation periodic solutions of (1.4) at $\tau = \tau_k^{(j)}$ on the center manifold. From the discussion above, we have the following result.

Theorem 3.3 *For system* (1.4), *if* (H1)-(H4) *hold, the periodic solution is supercritical* (*subcritical*) *if* $\mu_2 > 0$ ($\mu_2 < 0$); *the bifurcating periodic solutions are orbitally asymptotically stable with asymptotical phase* (*unstable*) *if* $\beta_2 < 0$ ($\beta_2 > 0$); *the periods of the bifurcating periodic solutions increase* (*decrease*) *if* $T_2 > 0$ ($T_2 < 0$).

4 Numerical examples

In this section, we present some numerical results of system (1.4) to verify the analytical predictions obtained in the previous section. From Section 3, we may determine the direction of a Hopf bifurcation and the stability of the bifurcation periodic solutions. Let us

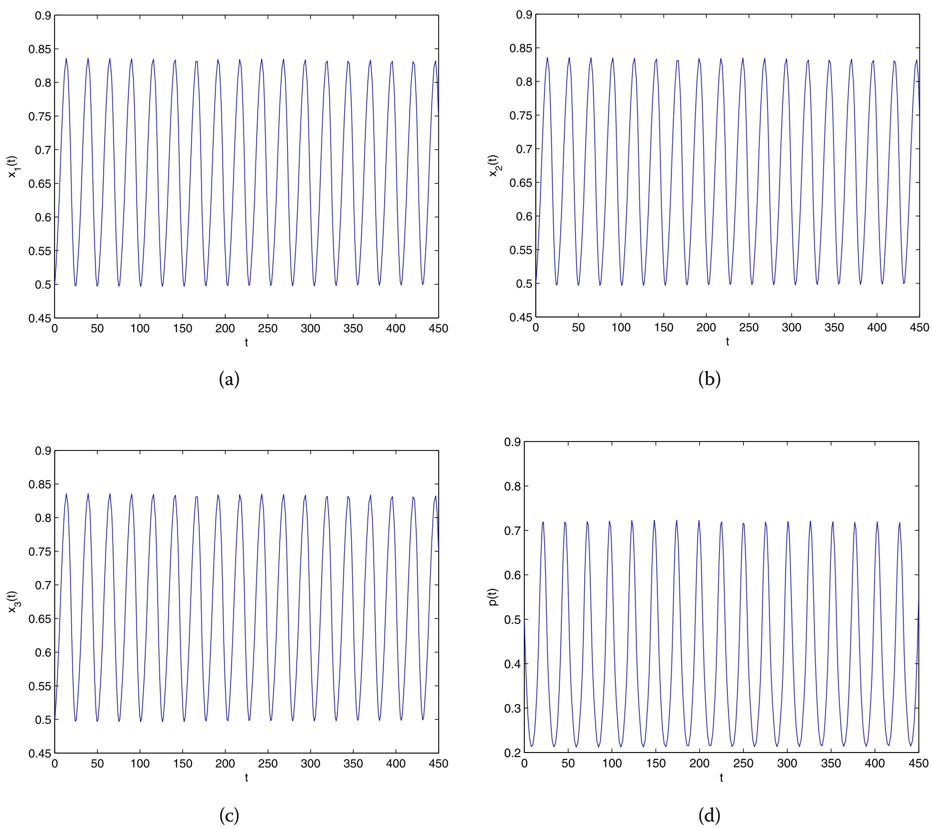

Figure 3 Dynamic behavior of system (4.1): times series of x_i ($i = 1, 2, 3$) and p. A Matlab simulation of a periodic solution to system (4.1) with $\tau = 1.97 > \tau_0 \approx 1.85$. The initial value is $(0.5, 0.5, 0.5, 0.5)$.

consider the following special case of system (1.4):

$$\begin{cases} \dot{x}_1(t) = 0.5x_1(t-\tau)[\frac{1-p(t)}{6.25x_1(t)} - 0.5x_1(t)p(t)], \\ \dot{x}_2(t) = 0.5x_2(t-\tau)[\frac{1-p(t)}{6.25x_2(t)} - 0.5x_2(t)p(t)], \\ \dot{x}_3(t) = 0.5x_3(t-\tau)[\frac{1-p(t)}{6.25x_3(t)} - 0.5x_3(t)p(t)], \\ \dot{p}(t) = 0.3p(t)[x_1(t-\tau) + x_2(t-\tau) + x_3(t-\tau) - 2]. \end{cases} \tag{4.1}$$

By some complicated computation by means of Matlab 7.0, we get $\omega_0 \approx 0.9824$, $\tau_0 \approx 1.85$, $\lambda'(\tau_0) \approx 2.1022 - 3.1513i$. Thus we can calculate the following values: $c_1(0) \approx -2.9542 - 22.2355i$, $\mu_2 \approx 0.5642$, $\beta_2 \approx -4.4636$, $T_2 \approx 22.1327$. We see that the conditions indicated in Theorem 2.3 are satisfied. Furthermore, it follows that $\mu_2 > 0$ and $\beta_2 < 0$. Choose $\tau = 1.65 < \tau_0 \approx 1.85$. Thus, the equilibrium $(x_1^*, x_2^*, x_3^*, p^*)$ is stable when $\tau < \tau_0$, which is illustrated by the computer simulations (see Figure 1 and Figure 2). When τ passes through the critical value $\tau_0 \approx 6.2$, the equilibrium $(x_1^*, x_2^*, x_3^*, p^*)$ loses its stability and a Hopf bifurcation occurs, *i.e.*, a family of periodic solutions bifurcate from the equilibrium $(x_1^*, x_2^*, x_3^*, p^*)$. Choose $\tau = 1.97 > \tau_0 \approx 1.85$. Since $\mu_2 > 0$ and $\beta_2 < 0$, the direction of the Hopf bifurcation is $\tau > \tau_0$, and these bifurcating periodic solutions from $(x_1^*, x_2^*, x_3^*, p^*)$ at τ_0 are stable; they are depicted in Figure 3 and Figure 4.

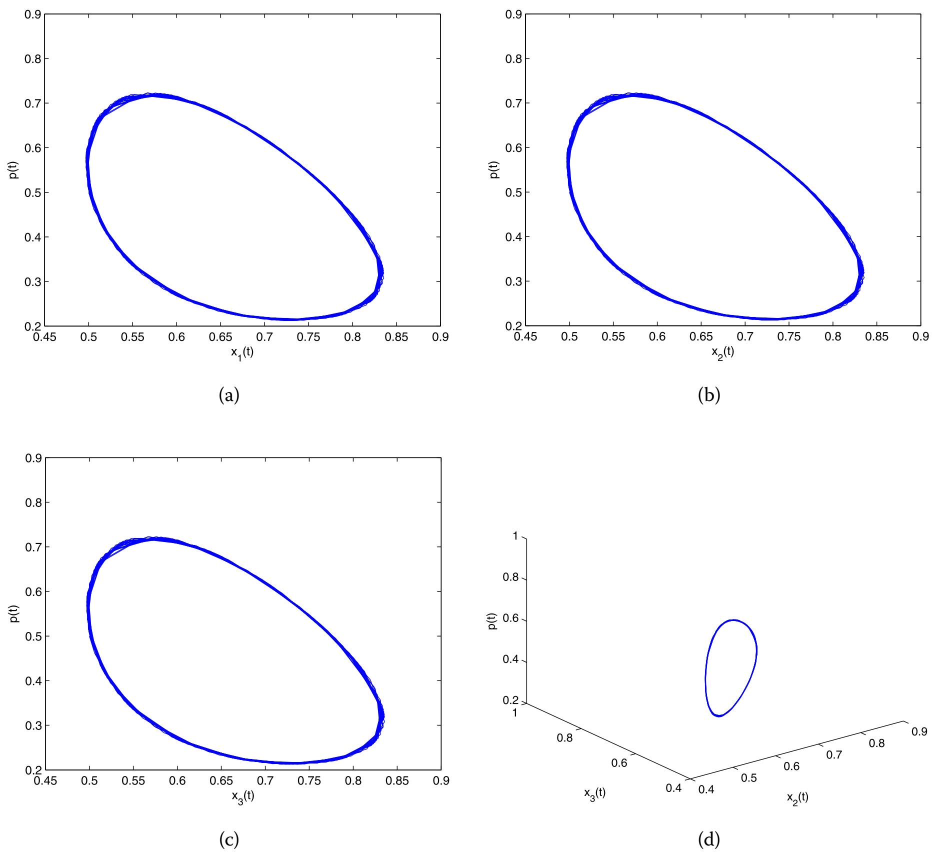

Figure 4 Dynamic behavior of system (4.1): projection on the x_1-p, x_2-p, x_3-p plane and projection on the x_2-x_3-p space, respectively. A Matlab simulation of a periodic solution to system (4.1) with $\tau = 1.97 > \tau_0 \approx 1.85$. The initial value is $(0.5, 0.5, 0.5, 0.5)$.

5 Conclusions

In this paper, we have investigated the properties of Hopf bifurcation in an exponential RED algorithm with communication delay. It is shown that under certain conditions, the Hopf bifurcation occurs as the delay τ passes through some critical values $\tau = \tau_k^{(j)}$, $k, j = 0, 1, 2, \ldots$. Moreover, the direction of the Hopf bifurcation and the stability of the bifurcating periodic orbits are derived by applying the normal form theory and the center manifold theorem.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors have made contributions of the same significance. All authors read and approved the final manuscript.

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References

1. Liu, F, Guan, ZH, Wang, HO: Controlling bifurcations and chaos in TCP-UDP-RED. Nonlinear Anal., Real World Appl. **11**(3), 1491-1501 (2010)
2. Raina, G, Heckmann, O: TCP: local stability and Hopf bifurcation. Perform. Eval. **64**, 266-275 (2007)
3. Guo, ST, Liao, XF, Li, CD, Yang, DG: Stability analysis of a novel exponential-RED model with heterogeneous delays. Comput. Commun. **30**, 1058-1074 (2007)
4. Xu, CJ, Tang, XH, Liao, MX: Local Hopf bifurcation and global existence of periodic solutions in TCP system. Appl. Math. Mech. **31**(6), 775-786 (2010)
5. Guo, ST, Feng, G, Liao, XF, Liu, Q: Novel delay-range-dependent stability analysis of the second-order congestion control algorithm with heterogenous communication delays. J. Netw. Comput. Appl. **32**, 568-577 (2009)
6. Tian, YP: A general stability criterion for congestion control with diverse communication delays. Automatica **41**, 1255-1262 (2005)
7. Abdallah, CT, Chiasson, J, Tarbouriech, S (eds.): Advances in Communication Control Networks. Lecture Notes in Control and Information Sciences. Springer, Berlin (2004)
8. Srikant, R: The Mathematics of Internet Congestion Control. Birkhäuser, Boston (2004)
9. Liu, S, Basar, T, Srikant, R: Exponential-RED: a stabilizing AQM scheme for low-and high-speed TCP protocols. IEEE/ACM Trans. Netw. **13**(5), 1068-1081 (2005)
10. Alpcan, T, Basar, T: A globally stable adaptive congestion control scheme for Internet-style networks with delays. IEEE/ACM Trans. Netw. **13**(6), 1261-1274 (2005)
11. Raina, G: Local bifurcation analysis of some dual congestion control algorithms. IEEE Trans. Autom. Control **50**(8), 1135-1146 (2005)
12. Sichitiu, ML, Bauter, PH: Asymptotic stability of congestion control systems with multiple sources. IEEE Trans. Autom. Control **51**(2), 292-298 (2006)
13. Guo, ST, Liao, XF, Liu, Q, Li, CD: Necessary and sufficient conditions for Hopf bifurcation in exponential RED algorithm with communication delay. Nonlinear Anal., Real World Appl. **9**(4), 1768-1793 (2008)
14. Ruan, S, Wei, J: On the zero of some transcendental functions with applications to stability of delay differential equations with two delays. Dyn. Contin. Discrete Impuls. Syst., Ser. A Math. Anal. **10**, 863-874 (2003)
15. Hassard, B, Kazatina, D, Wan, Y: Theory and Applications of Hopf Bifurcation. Cambridge University Press, Cambridge (1981)

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