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Mathematical modeling of rhizosphere microbial degradation with impulsive diffusion on nutrient

Zhong Zhao[1*], Yanli Song[2] and Liuyong Pang[1]

*Correspondence:
zhaozhong8899@163.com
[1]Department of Mathematics, Huanghuai University, Zhumadian, Henan 463000, P.R. China

Abstract

In this paper, a new mathematical modeling of rhizosphere microbial degradation with impulsive diffusion is proposed. By using the Floquet theorem, we find the rhizosphere microbe-eradication periodic solution is globally asymptotically stable if some conditions are satisfied. At the same time we also obtain the permanent conditions of the nutrients and rhizosphere microbe. Finally, a system of equations is solved by using a numerical simulation to justify our results.

MSC: 34C05; 92D25

Keywords: rhizosphere microbial degradation; impulsive diffusion; rhizosphere microbe-eradication periodic solution; permanent

1 Introduction

The constructed wetland is usually used for the purpose of treating the oversupply of nutrients such as nitrogen and phosphorus in the lake. The rhizosphere microbe can play a key role in decomposing organic matter through releasing inorganic nutrient available to wetland plant. The degradation process is complex because it includes some biological and chemical reactions. Therefore, understanding of the degradation process of the rhizosphere microbe has been widely attractive to many authors [1–14]. Bunwong *et al.* [1] formulated a three-dimensional system of ordinary differential equations and investigated the existence of equilibria and local Hopf bifurcation. In [4], the authors compared the efficiency of a laboratory scale subsurface hybrid constructed wetland (SS-HCW) for domestic waste water treatment planted with different plants species at different hydraulic retention times. Strigul and Kravchenko [7] introduced beneficial microbes to the plant rhizosphere. They showed that the competition for limiting resources between the introduced population and the resident microorganisms was the most important factor determining PGPR survival. The authors [8] showed the exudation dynamics leading to the development of the emerging attractors and synchronized oscillations of microbial populations, carbon and oxygen concentrations. Zhao *et al.* [14] proposed a nonlinear mathematical model of the rhizosphere microbial degradation based on impulsive state feedback control. The sufficient conditions for existence of the positive order-1 or order-2 periodic solution were obtained by using the geometrical theory of the semi-continuous dynamical system.

In fact, the dispersal phenomenon is ubiquitous, which has attracted many interests of the researchers [15–18]. Jiao *et al.* [17] considered a five-dimensioned chemostat model with impulsive diffusion and pulse input and they obtained the stability of microorganism-extinction periodic solution and permanence of the model. In [18], the authors indicated population dispersal is beneficial to pest control for some ranges of dispersal rates.

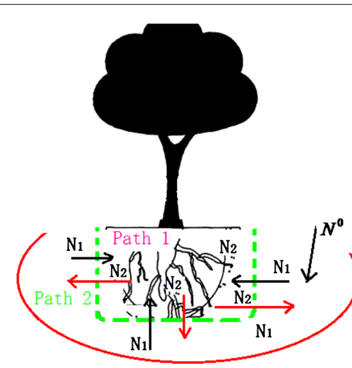
Since the rhizosphere microbial degradation is a complex process including some biological and chemical reactions, it is important to understand how this rhizosphere system operates. The mathematical model may be an important tool to understand the complex process and key parameters affecting the rhizosphere microbial degradation.

The paper is organized as follows: a mathematical model with impulsive diffusion is proposed in Section 2. In Sections 3 and 4, the stability of rhizosphere microbe eradication periodic solution and permanence are given, respectively. Finally, we give a brief discussion.

2 Model description and preliminaries

The rhizosphere is usually defined as a narrow zone of soil directly affected by the presence of plant root [19]. Considering the complexity of the degradation process, we suppose the rhizosphere system comprises two patches which is connected by impulsive diffusion (see Figure 1). The plant rhizosphere is directly considered as a chemostat, which is defined as patch 1 and the region outside the plant rhizosphere is called patch 2 (see Figure 1). Let $N_1(t)$ be the organic concentration of the region outside the rhizosphere (patch 2). $N_2(t)$ denotes the organic concentration of the rhizosphere (patch 1). N^0 denotes the input concentration of the organic matter. Q and D are the dilution rates. The growth of the rhizosphere microbe is supposed to follow the Monod equation involving the organic concentration $N_2(t)$ as well as the microbial concentration $x(t)$ (*i.e.* $\frac{\mu N_2(t)x(t)}{\delta(K+N_2(t))}$), where μ is the maximum specific growth rate and the constant δ is a yield term, K is a half-saturation constant. m is the mortality of the rhizosphere microbe. d $(0 < d < 1)$ is diffusive rate between patch 1 and patch 2, which shows the net exchange from patch j to patch i is proportional to the difference $N_j - N_i$.

Figure 1 Illustration of the impulsive diffusion.



Based on the above description and [1–18], we formulate the following model:

$$\begin{cases} \left.\begin{array}{l} \frac{dN_1}{dt} = Q(N^0 - N_1), \\ \frac{dN_2}{dt} = -DN_2 - \frac{\mu N_2 x}{\delta(K+N_2)}, \\ \frac{dx}{dt} = \frac{\mu N_2 x}{K+N_2} - mx, \end{array}\right\} & t \neq nT, \\ \left.\begin{array}{l} \Delta N_1 = d(N_2 - N_1), \\ \Delta N_2 = d(N_1 - N_2), \\ \Delta x = 0, \end{array}\right\} & t = nT, \end{cases} \tag{2.1}$$

where T is the impulsive period, $n \in N = \{1, 2, 3, \ldots\}$, $\Delta N_i = N_i(t^+) - N_i(t)$ $(i = 1, 2)$, $\Delta x = x(t^+) - x(t)$.

For convenience, we first give the basic properties of the following system:

$$\begin{cases} \frac{dN_1}{dt} = Q(N^0 - N_1), & \multirow{2}{*}{} t \neq nT, \\ \frac{dN_2}{dt} = -DN_2, & \\ \Delta N_1 = d(N_2 - N_1), & t = nT. \\ \Delta N_2 = d(N_1 - N_2), & \end{cases} \tag{2.2}$$

For $(nT, (n+1)T]$, we solve the first two equations of system (2.2) and obtain

$$\begin{cases} N_1(t) = N^0 + (N_1(nT^+) - N^0)e^{-Q(t-nT)}, & nT \leq t < (n+1)T, \\ N_2(t) = N_2(nT^+)e^{-D(t-nT)}, & nT \leq t < (n+1)T, \end{cases} \tag{2.3}$$

system (2.3) describes the nutrient concentrations of patch 1 and patch 2 for $(nT, (n+1)T]$.

At the impulsive moment, system (2.2) becomes

$$\begin{cases} N_1(n+1)T^+ = (1-d)N^0(1 - e^{-QT}) + (1-d)N_1(nT^+)e^{-QT} + dN_2(nT^+)e^{-DT}, \\ N_2(n+1)T^+ = dN^0(1 - e^{-QT}) + (1-d)N_2(nT^+)e^{-DT} + dN_1(nT^+)e^{-QT}. \end{cases} \tag{2.4}$$

System (2.4) reflects the nutrient concentrations at the impulsive moment. The dynamical properties of systems (2.3) and (2.4) determine the dynamical behaviors of system (2.2). Obviously, system (2.4) has a fixed point

$$\begin{cases} N_1^* = \frac{N^0(1-e^{-QT})(1-d-(1-2d)e^{-DT})}{1-(1-d)(e^{-DT}+e^{-QT})+(1-2d)e^{-(D+Q)T}}, \\ N_2^* = \frac{dN^0(1-e^{-QT})}{1-(1-d)(e^{-QT}+e^{-DT})+(1-2d)e^{-(Q+D)T}}. \end{cases} \tag{2.5}$$

Similar to the method of [20], we have the following lemma.

Lemma 2.1 *System* (2.2) *has a unique positive T-periodic solution $(\bar{N}_1(t), \bar{N}_2(t))$, which is globally asymptotically stable, where*

$$\begin{cases} \bar{N}_1(t) = N^0 + (N_1^* - N^0)e^{-Q(t-nT)}, \\ \bar{N}_2(t) = N_2^* e^{-D(t-nT)}, \\ N_1^* = \frac{N^0(1-e^{-QT})(1-d-(1-2d)e^{-DT})}{1-(1-d)(e^{-DT}+e^{-QT})+(1-2d)e^{-(D+Q)T}}, \\ N_2^* = \frac{dN^0(1-e^{-QT})}{1-(1-d)(e^{-QT}+e^{-DT})+(1-2d)e^{-(Q+D)T}}, \end{cases} \quad nT \leq t < (n+1)T. \tag{2.6}$$

3 Stability of rhizosphere microbe-eradication periodic solution

Theorem 3.1 *The rhizosphere microbe-eradication periodic solution* $(\bar{N}_1^*(t), \bar{N}_2^*(t), 0)$ *is globally asymptotically stable if* $R < 1$, *where* $R = \frac{\mu}{mTD} \ln \frac{K+\bar{N}_2^*}{K+\bar{N}_2^* e^{-DT}}$ *and* N_2^* *is defined in* (2.5).

Proof First of all, we prove the local stability of the rhizosphere microbe-eradication periodic solution $(\bar{N}_1^*(t), \bar{N}_2^*(t), 0)$. The local stability of the periodic solution $(\bar{N}_1^*(t), \bar{N}_2^*(t), 0)$ is determined by considering the small-amplitude perturbations of the solution. Define $N_1(t) = \bar{N}_1^*(t) + u(t)$, $N_2(t) = \bar{N}_2^*(t) + v(t)$, $x(t) = w(t)$, where $u(t)$, $v(t)$, and $w(t)$ are small enough. We have

$$\frac{d}{dt}\begin{pmatrix} u(t) \\ v(t) \\ w(t) \end{pmatrix} = \begin{pmatrix} -Q & 0 & 0 \\ 0 & -D & \frac{\mu \bar{N}_2^*(t)}{\delta(K+\bar{N}_2^*(t))} \\ 0 & 0 & \frac{\mu \bar{N}_2^*(t)}{K+\bar{N}_2^*(t)} - m \end{pmatrix} \begin{pmatrix} u(t) \\ v(t) \\ w(t) \end{pmatrix}. \tag{3.1}$$

It is easy to obtain the fundamental solution matrix:

$$\Phi(t) = \begin{pmatrix} e^{-Qt} & 0 & 0 \\ 0 & e^{-Dt} & \frac{\mu \bar{N}_2^*(t)}{\delta(K+\bar{N}_2^*(t))} \\ 0 & 0 & e^{\int_0^t (\frac{\mu \bar{N}_2^*(t)}{K+\bar{N}_2^*(t)} - m)\,dt} \end{pmatrix}. \tag{3.2}$$

The linearization of the equation from the fourth to the sixth is

$$\begin{pmatrix} u(nT^+) \\ v(nT^+) \\ w(nT^+) \end{pmatrix} = \begin{pmatrix} 1-d & d & 0 \\ d & 1-d & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u(nT) \\ v(nT) \\ w(nT) \end{pmatrix}.$$

Thus, the monodromy matrix of (3.1) is

$$M' = \begin{pmatrix} 1-d & d & 0 \\ d & 1-d & 0 \\ 0 & 0 & 1 \end{pmatrix} \Phi(T).$$

Let $\lambda_1, \lambda_2, \lambda_3$ be eigenvalues of M'. It is obvious for $\lambda_1 = (1-d)e^{-QT} < 1$, $\lambda_2 = (1-d)e^{-DT} < 1$. According to Floquet theory [21], we see that the rhizosphere microbe-eradication periodic solution $(\bar{N}_1^*(t), \bar{N}_2^*(t), 0)$ is locally asymptotically stable if $\lambda_3 = e^{\int_0^T (\frac{\mu \bar{N}_2^*(t)}{K+\bar{N}_2^*(t)} - m)\,dt} < 1$, that is, $\frac{\mu}{mTD} \ln \frac{K+\bar{N}_2^*}{K+\bar{N}_2^* e^{-DT}} < 1$.

In the following, we will prove the global attraction. Choose $\varepsilon > 0$ such that $\varrho = \frac{\mu}{D} \ln \frac{K+\varepsilon+\bar{N}_2^*}{K+\varepsilon+\bar{N}_2^* e^{-DT}} + \frac{\mu\varepsilon}{D(K+\varepsilon)} \ln \frac{(K+\varepsilon)e^{DT}+\bar{N}_2^*}{K+\varepsilon+\bar{N}_2^*} - mT < 0$.

Noticing that $\frac{dN_2}{dt} \leq -QN_2$, we consider the following comparison system:

$$\begin{cases} \left.\begin{aligned} \frac{du_1}{dt} &= QN^0 - Qu_1, \\ \frac{dv_1}{dt} &= -Dv_1, \end{aligned}\right\} & t \neq nT, \\ \left.\begin{aligned} \Delta u_1 &= d(v_1 - u_1), \\ \Delta v_1 &= d(u_1 - v_1), \end{aligned}\right\} & t = nT, \end{cases} \tag{3.3}$$

we have $N_1(t) \le u_1(t)$, $N_2(t) \le v_1(t)$, and $u_1(t) \to \bar{N}_1(t)$, $v_1(t) \to \bar{N}_2(t)$ as $t \to \infty$. Then we have

$$N_1(t) \le u_1(t) \le \bar{N}_1(t) + \varepsilon, \qquad N_2(t) \le v_1(t) \le \bar{N}_2(t) + \varepsilon \tag{3.4}$$

for t large enough. For simplicity, we suppose system (3.4) holds for all $t > 0$. From the third equation of system (2.1), we obtain

$$\frac{dx}{dt} \le x\left(\frac{\mu(\bar{N}_2(t) + \varepsilon)}{K + \bar{N}_2(t) + \varepsilon} - m\right). \tag{3.5}$$

Integrating the above inequality on the interval $(nT, (n+1)T]$, we get

$$x\bigl((n+1)T\bigr) \le x\bigl(nT^+\bigr)\exp\left(\int_{nT}^{(n+1)T}\left(\frac{\mu(\bar{N}_2(t) + \varepsilon)}{K + \bar{N}_2(t) + \varepsilon} - m\right)dt\right),$$

therefore, we have $x((n+1)T) \le x(nT^+)\exp(\varrho)$, thus $x(nT) \le x(0^+)\exp(n\varrho)$ and $x(nT) \to \infty$ as $t \to \infty$. Since $x(t) \le x(nT)$, we obtain $x(t) \to 0$ as $t \to \infty$.

Next, we prove $N_1(t) \to \bar{N}_1^*(t)$ and $N_2(t) \to \bar{N}_2^*(t)$ as $t \to \infty$. There exists a $t_0 > 0$ such that $0 < x(t) \le \varepsilon$ for $t \ge t_0$. Therefore, we have $-DN_2(t) - \frac{\mu\varepsilon}{\delta} \le \frac{dN_2}{dt} \le -DN_2$. Hence, we get $u_2(t) \le N_1(t) \le u_3(t)$ and $v_2(t) \le N_2(t) \le v_3(t)$, where $(u_2(t), v_2(t))$ and $(u_3(t), v_3(t))$ are the solutions of the following two comparison systems, respectively:

$$\begin{cases} \left.\begin{array}{l} \frac{du_2}{dt} = QN^0 - Qu_2, \\ \frac{dv_2}{dt} = -Dv_2 - \frac{\mu\varepsilon}{\delta}, \end{array}\right\} & t \neq nT, \\ \left.\begin{array}{l} \Delta u_2 = d(v_2 - u_2), \\ \Delta v_2 = d(u_2 - v_2), \end{array}\right\} & t = nT, \end{cases} \tag{3.6}$$

and

$$\begin{cases} \left.\begin{array}{l} \frac{du_3}{dt} = QN^0 - Qu_3, \\ \frac{dv_3}{dt} = -Dv_3, \end{array}\right\} & t \neq nT, \\ \left.\begin{array}{l} \Delta u_3 = d(v_3 - u_3), \\ \Delta v_3 = d(u_3 - v_3), \end{array}\right\} & t = nT. \end{cases} \tag{3.7}$$

The periodic solution of system (3.6) is

$$\begin{cases} u_2^* = \frac{D\delta N^0(1-e^{-QT})-d\mu\varepsilon(1-e^{-DT})}{D\delta(1-(1-d)e^{-QT})} + \frac{de^{-DT}v_2^*}{1-()e^{-QT}}, \\ v_2^* = \frac{(1-(1-d)e^{-QT})(D\delta dN^0(1-e^{-QT})-(1-d)\mu\varepsilon(1-e^{-DT}))}{D\delta(1-(1-d)(e^{-QT}+e^{-DT})+(1-d)e^{-(D+Q)T})}, \\ \bar{u}_2(t) = N_0(1 - e^{-Q(t-nT)}) + u_2^* e^{-Q(t-nT)}, \\ \bar{v}_2(t) = v_2^* e^{-D(t-nT)} - \frac{\mu\varepsilon}{D\delta}(1 - e^{-D(t-nT)}), \end{cases} \quad t \in \bigl(nT, (n+1)T\bigr]. \tag{3.8}$$

The periodic solution of system (3.7) is

$$\begin{cases} u_3^* = \frac{N^0(1-e^{-QT})(1-d-(1-2d)e^{-DT})}{1-(1-d)(e^{-DT}+e^{-QT})+(1-2d)e^{-(D+Q)T}}, \\ v_3^* = \frac{dN^0(1-e^{-QT})}{1-(1-d)(e^{-QT}+e^{-DT})+(1-2d)e^{-(Q+D)T}}, \\ \bar{u}_3(t) = N_0(1 - e^{-Q(t-nT)}) + u_3^* e^{-Q(t-nT)}, \\ \bar{v}_3(t) = v_3^* e^{-D(t-nT)}, \end{cases} \quad t \in \bigl(nT, (n+1)T\bigr]. \tag{3.9}$$

Therefore, for small enough ε_1, there exists a $t_1 > 0$ such that

$$\bar{u}_2(t) - \varepsilon_1 \le N_1(t) \le \bar{u}_3(t) + \varepsilon_1, \qquad \bar{v}_2(t) - \varepsilon_1 \le N_2(t) \le \bar{v}_3(t) + \varepsilon_1.$$

Let $\varepsilon \to 0$, we get

$$\bar{N}_1^*(t) - \varepsilon_1 \le N_1(t) \le \bar{N}_1^*(t) + \varepsilon_1, \qquad \bar{N}_2^*(t) - \varepsilon_1 \le N_2(t) \le \bar{N}_2^*(t) + \varepsilon_1$$

for t large enough, which shows $N_1(t) \to \bar{N}_1^*(t)$ and $N_2(t) \to \bar{N}_2^*(t)$ as $t \to \infty$. The proof is completed. □

4 Permanent

First of all, we show that all solutions of (2.1) are ultimately bounded.

Lemma 4.1 *The system* (2.1) *is ultimately bounded.*

Proof Define a function $V(t) = N_1(t) + N_2(t) + \frac{x(t)}{\delta}$. When $t \neq nT$, we have $\frac{dV}{dt} = QN^0 - QN_1 - DN_2 - \frac{mx(t)}{\delta} \le QN^0 - \rho V(t)$, where $\rho = \min\{Q, D, m\}$. When $t = nT$, we also obtain $V(nT^+) = V(nT)$. For $(nT, (n+1)T]$, we have $V(t) \le V(0)e^{-\rho t} + \frac{QN^0}{\rho}(1 - e^{-\rho t}) \to \frac{QN^0}{\rho}$, as $t \to \infty$. So we have $N_1(t) \le M, N_2(t) \le M, x(t) \le M$, where $\frac{QN^0}{\rho} \triangleq M$. □

Theorem 4.2 *System* (2.1) *is permanent if* $R > 1$, *where* R *is defined in Theorem* 3.1.

Proof Suppose $(N_1(t), N_2(t), x(t))$ is a solution of (2.1) with positive initial value. From Lemma 4.1 $N_1(t) \le \frac{QN^0}{\rho} \triangleq M$, $N_2(t) \le \frac{QN^0}{\rho} \triangleq M$, $x(t) \le \frac{QN^0}{\rho} \triangleq M$, $t \ge 0$, we get $\frac{dN_2}{dt} \ge -DN_2 - \frac{\mu M}{\delta}$.

Considering the comparison system:

$$\begin{cases} \frac{du_4}{dt} = QN^0 - Qu_4, & \\ \frac{dv_4}{dt} = -Dv_4 - \frac{\mu M}{\delta}, & t \neq nT, \\ \Delta u_4 = d(v_4 - u_4), & \\ \Delta v_4 = d(u_4 - v_4), & t = nT. \end{cases} \tag{4.1}$$

Similar to system (2.2), the periodic solution $(\bar{u}_4(t), \bar{v}_4(t))$ can be given:

$$\begin{cases} u_4^* = \frac{(1-d)N^0 - d\frac{\mu M}{D\delta}(1-e^{-DT})}{1-(1-d)e^{-QT}} + \frac{de^{-DT}v_4^*}{1-(1-d)e^{-Q}}, & \\ v_4^* = \frac{[-(1-d)\frac{\mu M}{D\delta}(1-e^{-DT}) + dN^0(1-e^{-QT})](1-(1-d)e^{-QT})}{1-(1-d)(e^{-QT}+e^{-DT})+(1-2d)e^{-(Q+D)T}}, & t \in (nT, (n+1)T], \\ \bar{u}_4(t) = N_0(1 - e^{-Q(t-nT)}) + u_4^* e^{-Q(t-nT)}, & \\ \bar{v}_4(t) = -\frac{\mu M}{D\delta} + (v_4^* + \frac{\mu M}{D\delta})e^{-D(t-nT)}, & \end{cases} \tag{4.2}$$

which is globally asymptotically stable. Hence, there exists a $\varepsilon_2 > 0$ such that $N_1(t) \ge u_4(t) \ge u_4^*(t) - \varepsilon_2 \ge u_4^* - \varepsilon_2 \triangleq m_1, N_2(t) \ge v_4(t) \ge v_4^*(t) - \varepsilon_2 \ge v_4^* - \varepsilon_2 \triangleq m_2$ for t large enough.

In the following, we want to find m_3 such that $x(t) \ge m_3$ for t large enough. We shall do it in the following two steps.

Step I: Let $m_3 > 0$ and ε_3 be small enough such that

$$\rho = \frac{\mu\varepsilon_3 + \frac{\mu^2 m_3}{D\delta}}{D(K - \varepsilon_3 - \frac{\mu m_3}{D\delta})} \ln \frac{K - \varepsilon_3 + v_5^*}{(K - \varepsilon_3 - \frac{\mu m_3}{D\delta})e^{DT} + v_5^* + \frac{\mu m_3}{D\delta}} + \frac{1}{D} \ln \frac{K - \varepsilon_3 - \frac{\mu m_3}{D\delta} + (v_5^* + \frac{\mu m_3}{D\delta})e^{-DT}}{K - \varepsilon_3 + v_5^*} - mT > 0.$$

We will prove $x_2(t) < m_3$ cannot hold for all $t \geq 0$. Otherwise, $\frac{dN_2}{dt} \geq -DN_2 - \frac{\mu m_3}{\delta}$, we consider the following comparison system:

$$\begin{cases} \frac{du_5}{dt} = QN^0 - Qu_5 & \\ \frac{dv_5}{dt} = -Dv_5 - \frac{\mu m_3}{\delta}, & t \neq nT, \\ \Delta u_5 = d(v_5 - u_5), & \\ \Delta v_5 = d(u_5 - v_5), & t = nT, \end{cases} \tag{4.3}$$

we get $N_1(t) \geq u_5(t)$, $N_2(t) \geq v_5(t)$, and $u_5(t) \to \bar{u}_5^*(t)$, $v_5(t) \to \bar{v}_5^*(t)$ as $t \to \infty$, where $(\bar{u}_5^*(t), \bar{v}_5^*(t))$ is the periodic solution of system (4.3) and $(\bar{u}_5^*(t), \bar{v}_5^*(t))$ is given as

$$\begin{cases} u_5^* = \frac{(1-d)N^0 - d\frac{\mu m_3}{D\delta}(1-e^{-DT})}{1-(1-d)e^{-QT}} + \frac{de^{-DT}v_5^*}{1-(1-d)e^{-Q}}, \\ v_5^* = \frac{[-(1-d)\frac{\mu m_3}{D\delta}(1-e^{-DT}) + dN^0(1-e^{-QT})](1-(1-d)e^{-QT})}{1-(1-d)(e^{-QT}+e^{-DT})+(1-2d)e^{-(Q+D)T}}, & t \in (nT, (n+1)T]. \\ \bar{u}_5^*(t) = N_0(1 - e^{-Q(t-nT)}) + u_5^* e^{-Q(t-nT)}, \\ \bar{v}_5^*(t) = -\frac{\mu m_3}{D\delta} + (v_5^* + \frac{\mu m_3}{D\delta})e^{-D(t-nT)}, \end{cases} \tag{4.4}$$

Hence for ε_3 small enough, there exists a $T_1 > 0$ such that $N_1(t) \geq u_5^*(t) - \varepsilon_3$, $N_2(t) \geq v_5^*(t) - \varepsilon_3$, and

$$\frac{dx}{dt} \geq \left(\frac{\mu(v_5^*(t) - \varepsilon_3)}{K + v_5^*(t) - \varepsilon_3} - m \right) x, \quad t \geq T_1. \tag{4.5}$$

Let $n_1 \in N$ and $n_1 T > T_1$, integrating (4.5) on the interval $(nT, (n+1)T]$, $n > n_1$, we obtain

$$x\big((n+1)T\big) \geq x\big(nT^+\big) \exp\left(\int_{nT}^{(n+1)T} \left(\frac{\mu(v_5^*(t) - \varepsilon_3)}{K + v_5^*(t) - \varepsilon_3} - m \right) dt \right) = x\big(nT^+\big) \exp(\rho).$$

Then $x((n_1 + k)T) \geq x(n_1 T) \exp(k\rho) \to \infty$ as $k \to \infty$, which is in contradiction to the boundedness of $x(t)$. Therefore, there is a $t_1 > 0$ such that $x(t_1) > m_3$. If $x(t) \geq m_3$ for all $t > t_1$, then our aim is obtained. Otherwise, there exists a $\bar{t}_1 > t_1$ such that $x(\bar{t}_1) < m_3$. Setting $t^* = \inf_{t > t^*}\{x_2(t) < m_3\}$, then we have $x_2(t) \geq m_3$ for $t \in [t, t^*)$, and $x_2(t^*) = m_3$.

Steps II: Since $x(t)$ is continuous, suppose $t^* \in (n_1 T, (n_1 + 1)T]$, $n_1 \in N$, select $n_2 \in N$, $n_3 \in N$, such that

$$n_2 T > \min\left\{ \frac{1}{Q} \frac{M + u_5^*}{\varepsilon_3}, \frac{1}{D} \frac{M + v_5^*}{\varepsilon_3} \right\},$$

$$\exp\big(\eta(n_2 + 1)T\big) \exp(n_3 \rho) > 1,$$

where $\eta = \frac{\mu\Phi}{K+\Phi} - m < 0$, where $\Phi = -\frac{\mu m_3}{D\delta} + (v_5^* + \frac{\mu m_3}{D\delta})e^{-DT}$ and v_5^* is defined in (4.4).

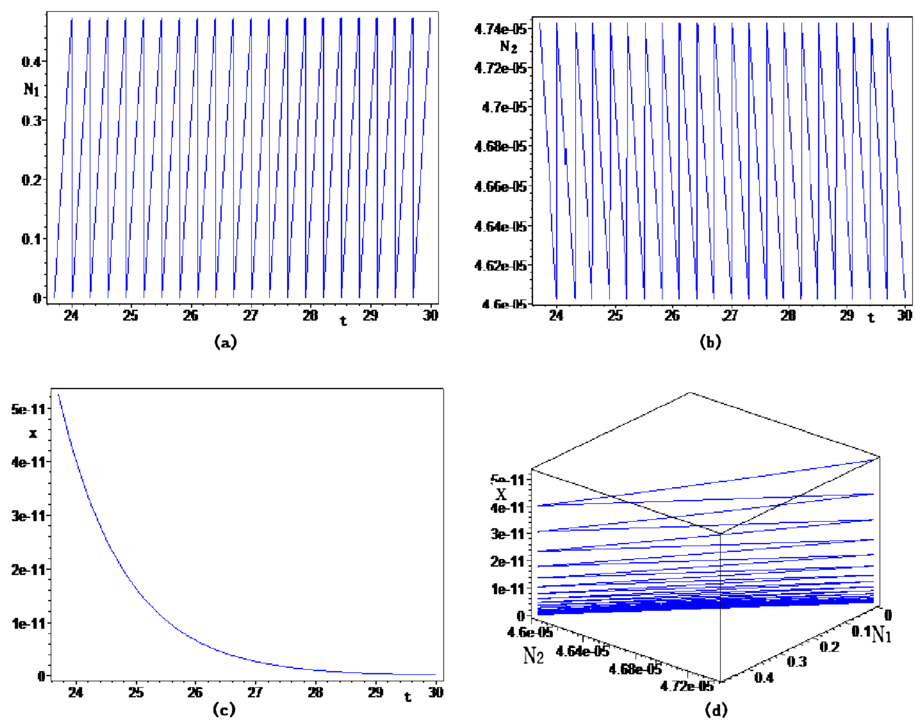


Figure 2 Dynamical behavior of the system (2.1) with $Q = 0.08$, $N^0 = 20$, $D = 0.1$, $\mu = 0.1$, $\delta = 0.5$, $K = 1.8$, $m = 0.9$, $d = 0.0001$, $N_1(0) = 18$, $N_2(0) = 0.2$, $x(0) = 0.1$, $R = 0.39 < 1$. (a) Time-series of the nutrient concentration outside the rhizosphere (patch 2). **(b)** Time-series of the nutrient concentration inside the rhizosphere (patch 1). **(c)** Phase portrait denoting the rhizosphere microbe-eradication periodic solution $(\bar{N}_1^*(t), \bar{N}_2^*(t), 0)$ is globally asymptotically stable for $R < 1$.

Let $T' = n_2T + n_3T$, we claim that there must exist a $t' \in ((n_1+1)T, (n_1+1)T + T']$ such that $x(t) \geq m_3$, otherwise $x(t) < m_3$ for $t \in ((n_1+1)T, (n_1+1)T + T']$.

Considering (4.3) with

$$u_5\big((n_1+1)T^+\big) = N_1\big((n_1+1)T^+\big), \qquad v_5\big((n_1+1)T^+\big) = N_2\big((n_1+1)T^+\big),$$

we have

$$u_5(t) = \big(u_5\big((n_1+1)T^+\big) - u_5^*\big)e^{-Q(t-(n_1+1)T)} + \bar{u}_5^*(t),$$
$$v_5(t) = \big(v_5\big((n_1+1)T^+\big) - v_5^*\big)e^{-D(t-(n_1+1)T)} + \bar{v}_5^*(t)$$

for $t \in (nT, (n+1)T]$, $n_1 + 1 < n \leq n_1 + n_2 + n_3 + 1$. Then

$$\big|u_5(t) - \bar{u}_5^*(t)\big| \leq \big(M + u_5^*\big)e^{-Q(t-(n_1+1)T)} < \varepsilon_3,$$
$$\big|v_5(t) - \bar{v}_5^*(t)\big| \leq \big(M + v_5^*\big)e^{-D(t-(n_1+1)T)} < \varepsilon_3,$$

we have $u(t) \leq \bar{u}^*(t) + \varepsilon_3$, $v(t) \leq \bar{v}^*(t) + \varepsilon_3$ for $(n_1+1+n_2)T \leq t \leq (n_1+1)T + T'$, which shows system (4.5) holds. As in step I, we have $x((n_1+n_2+n_3+1)T) \geq x((n_1+n_2+1)T)\exp(n_3\rho)$.

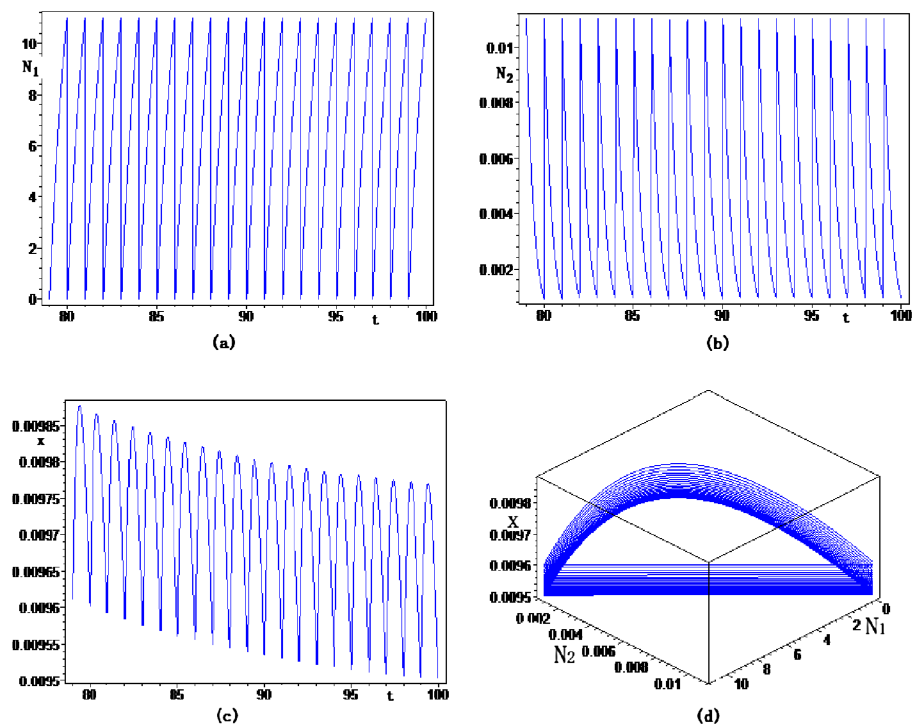


Figure 3 Dynamical behavior of the system (2.1) with $Q = 0.8$, $N^0 = 20$, $D = 0.1$, $\mu = 0.3$, $\delta = 0.1$, $K = 1.8$, $m = 0.1$, $d = 0.0001$, $N_1(0) = 18$, $N_2(0) = 0.2$, $x(0) = 0.1$, $R = 15.99 > 1$. (a) Time-series of the nutrient concentration outside the rhizosphere (patch 2). **(b)** Time-series of the nutrient concentration inside the rhizosphere (patch 1). **(c)** Phase portrait implying system (2.1) is permanent for $R > 1$.

From system (2.1), we have

$$\frac{dx}{dt} \geq \left(\frac{\mu\Phi}{K + \Phi} - m\right)x(t) = \eta x(t).$$

Integrating the above equation on $(t^*, (n_1 + 1 + n_2)T]$, we obtain $x((n_1 + 1 + n_2)T) \geq m_3 e^{\eta(n_2+1)T}$, then

$$x\big((n_1 + 1 + n_2 + N_3)T\big) \geq m_3 e^{\eta(n_2+1)T} e^{n_3 T} > m_3,$$

which is a contradiction. Let $\bar{t} = \inf_{t \geq t^*}\{x(t) \geq m_3\}$, thus $x(\bar{t}) \geq m_3$ for $t \in [t^*, \bar{t}]$, we get $x(t) \geq x(t^*)e^{\eta(t-t^*)} \geq m_3 e^{\eta(n_1+1+n_2)T} \triangleq \bar{m}_3$ for $t \geq \bar{t}$. The same arguments can be continued since $x(\bar{t}) \geq m_3$. Hence $x(t) \geq \bar{m}_3$ for $t \geq t_1$. □

5 Discussion

Since the rhizosphere microbial degradation undergoes a series of complex biochemical reactions, the degradation process of rhizosphere microbe may be affected by many factors. It is unrealistic to expect the existing model predicts all possible results of microbial degradation, therefore, each of the models has some validity and application limits [7]. In this paper, we have formulated a mathematical modeling of rhizosphere microbial degradation with impulsive diffusion and obtained the rhizosphere microbe-eradication periodic solution $(\bar{N}_1^*(t), \bar{N}_2^*(t), 0)$ is globally asymptotically stable for $R < 1$, which is showed in

Figure 2 with the parameters $Q = 0.08$, $N^0 = 20$, $D = 0.1$, $\mu = 0.1$, $\delta = 0.5$, $K = 1.8$, $m = 0.9$, $d = 0.0001$, $N_1(0) = 18$, $N_2(0) = 0.2$, $x(0) = 0.1$, $R = 0.39 < 1$. We can see the variables $N_1(t)$, $N_2(t)$ oscillate in a stable periodical cycle. On the contrary, $x(t)$ rapidly decreases to zero. From Theorem 4.2, we also have proved system (2.1) is permanent for $R > 1$, which is simulated in Figure 3 with the parameters $Q = 0.8$, $N^0 = 20$, $D = 0.1$, $\mu = 0.3$, $\delta = 0.1$, $K = 1.8$, $m = 0.1$, $d = 0.0001$, $N_1(0) = 18$, $N_2(0) = 0.2$, $x(0) = 0.1$, $R = 15.99 > 1$. The variables $N_1(t)$, $N_2(t)$ and $x(t)$ oscillate in a stable periodical cycle, respectively.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
ZZ formulated the mathematical modeling of rhizosphere microbial degradation and carried out the analysis. YS gave some constructive comments. LP corrected the manuscript. All authors have read and approved the final manuscript.

Author details
[1]Department of Mathematics, Huanghuai University, Zhumadian, Henan 463000, P.R. China. [2]Department of Economic Management, Henan College of Quality Engineering Vocation, Pingdingshan, Henan 467001, P.R. China.

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