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The matrix representation of the delta shape operator on time scales

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Abstract
In this work, we consider the delta shape operator of a surface parameterized by the product of two arbitrary time scales. In particular, we present a matrix representation of the delta shape operators with respect to partial delta derivatives.

Keywords: time scales; shape operator; delta derivative

1 Introduction

The calculus of time scales, which has recently received a lot of attention, was introduced by Hilger in his PhD thesis in 1988 (supervised by Aulbach) in order to create a theory that can unify discrete and continuous analysis [1]. Then Hilger and Aulbach published [1] and [2] in 1990. A time scale is an arbitrary nonempty closed subset of the real numbers. Thus $\mathbb{R}, \mathbb{Z}, \mathbb{N}, \mathbb{N}_0$ are examples of time scales [1, 2]. Linear and nonlinear Hamiltonian systems were studied on time scales by Ahlbrandt *et al.* in [3]. The authors unify symplectic flow properties of discrete and continuous Hamiltonian systems. An introduction to the study of dynamic equations on time scales was developed in [4] in 2001. A generalization of the notion of regular curve, tangent vector, and natural parametrization were introduced by Guseinov and Ozyilmaz in [5]. The theory of time scales has proved to be useful in the mathematical modeling of several important dynamic processes [6–8]. In [9], the notion of pseudospherical surfaces in asymptotic coordinates on time scales was presented by Cieslinski. Thus the author extended the well-known notions of discrete and smooth pseudospherical surfaces. Also they presented the Gaussian curvature of the surface. Some applications of a vector field along a curve and a derivative mapping on a time scale were studied by Kusak and Caliskan in [10]. Some properties of directional nabla-derivative according to vector fields and curves on n-dimensional time scales were presented by Aktan *et al.* in [11]. The normal and osculating planes of the curves on time scales were developed by Pasali Atmaca in [12]. Also the authors defined the concept of vector-valued functions on time scales. A connection of a vector field in the direction of another vector field was accomplished on time scales in [13]. The forward curvature of a curve and some its properties were studied by Seyyidoglu *et al.* in [14]. Nabla 1-forms for multivariable functions on an n-dimensional time scale were presented Aktan *et al.* in [15]. In [16, 17], Lie brackets, the parameter map, and the velocity vector are introduced. A theoretical framework for surfaces parametrized by the product of two arbitrary time scales was developed by Pasali Atmaca and Akguller in [18]. The authors studied the metric properties of the surfaces. The surface normal vector, the first fundamental form, and the length measurement were presented in the paper. Also some fundamental properties as regards differential geometry can be obtained in [19, 20].

The general idea in this paper is to investigate the matrix representation of the delta shape operator (delta Weingarten transformation) which was not considered in the literature before. Hence we survey here the Weingarten delta shape operator which combines discrete space and continuous space on time scales.

2 Preliminaries

Let $n \in \mathbb{N}$ be fixed. Further, for each $i \in \{1, \dots, n\}$ let $\mathbb{T}_i$ denote a time scale, that is, $\mathbb{T}_i$ is a nonempty closed subset of the real numbers $\mathbb{R}$. Let us set $\Lambda^n = \mathbb{T}_1 \times \cdots \times \mathbb{T}_n = \{t = (t_1, \dots, t_n) : t_i \in \mathbb{T}_i \text{ for all } i \in \{1, \dots, n\}\}$. We call Λ^n an n-dimensional time scale. The set Λ^n is a complete metric space with the metric d defined by $d(t,s) = (\sum_{i=1}^{n} \|t_i - s_i\|^2)^{\frac{1}{2}}$ for $t, s \in \Lambda^n$ [7].

Lemma 2.1 *The delta derivation of the inner product for the two vector-valued functions $x(t)$ and $y(t)$ is defined with*

$$\begin{aligned} \frac{\partial}{\Delta t}\langle x(t), y(t)\rangle &= \frac{\partial x(t)}{\Delta t} \cdot y(t) + x^{\sigma}(t) \cdot \frac{\partial y(t)}{\Delta t} \\ &= \frac{\partial x(t)}{\Delta t} \cdot y^{\sigma}(t) + x(t) \cdot \frac{\partial y(t)}{\Delta t} \quad [12]. \end{aligned}$$

Lemma 2.2 *The delta derivation of the vector product for the two vector-valued functions $x(t)$ and $y(t)$ is* [12]

$$\begin{aligned} \frac{\partial}{\Delta t}\bigl(x(t) \times y(t)\bigr) &= \frac{\partial x(t)}{\Delta t} \times y(t) + x^{\sigma}(t) \times \frac{\partial y(t)}{\Delta t} \\ &= \frac{\partial x(t)}{\Delta t} \times y^{\sigma}(t) + x(t) \times \frac{\partial y(t)}{\Delta t}. \end{aligned}$$

Definition 2.1 Suppose that the function f is σ_1-completely delta differentiable at the point $(t^0, s^0) \in \Lambda^2$. Then the directional delta derivative of f at (t^0, s^0) in the direction of the vector $w(w_1, w_2)$ exists and is expressed by the formula [7]

$$\frac{\partial f(t^0, s^0)}{\Delta w} = \frac{\partial f(t^0, s^0)}{\Delta_1 t} w_1 + \frac{\partial f(\sigma_1(t^0), s^0)}{\Delta_2 s} w_2.$$

Definition 2.2 The metric of a surface S is determined by the partial Δ-derivatives of the surface patch φ. Let the cross product on time scales be a binary operation on the time scale spaces Λ^2 and be denoted by the symbol $\times$. Assuming that $\frac{\partial \varphi}{\Delta_1 t} \times \frac{\partial \varphi}{\Delta_2 s} \neq 0$, the tangent plane to S is spanned by the two tangent vectors $\frac{\partial \varphi}{\Delta_1 t}$ and $\frac{\partial \varphi}{\Delta_2 s}$. The surface normal vector is orthogonal to both tangent vectors and can be computed as [18]

$$N = \frac{\frac{\partial \varphi}{\Delta_1 t} \times \frac{\partial \varphi}{\Delta_2 s}}{\|\frac{\partial \varphi}{\Delta_1 t} \times \frac{\partial \varphi}{\Delta_2 s}\|}.$$

Definition 2.3 Let V and W be vector field on the space Λ^2. By considering the delta covariant derivative of W with respect to V implies the following equation:

$$(\Delta_V W)(p) = \Delta_{V(p)} W,$$

and the following mapping:

$$\begin{aligned} \Delta : T(\Lambda^2) \times T(\Lambda^2) &\to T(\Lambda^2) \\ (V, W) &\to \Delta_V W, \end{aligned}$$

is called '*the delta nature connection*' on the time scale [13].

$C_{\sigma_1}^{\Delta}$ is the set of continuous functions which are σ_1-completely delta differentiable [5].

Theorem 2.1 *Let V, W, Y, Z be vector fields on Λ^2, and $f, g \in C_{\sigma_1}^{\Delta}$ be given for every $a, b \in \mathbb{R}$, then we have some properties of the delta nature connection as follows* [13]:

(i) $\Delta_V(fY + gZ) = f\Delta_V Y + g\Delta_V Z$;

(ii) $\Delta_{fV+gW} Y = f\Delta_V Y + g\Delta_W Y$;

(iii) $$\begin{aligned} \Delta_V(fY) = {} & \frac{\partial f}{\Delta V} Y(\sigma_1(t^0), s^0) + f(\sigma_1(t^0), \sigma_1(s^0))\Delta_V Y \\ & + \left\{\mu_1 V_1 \frac{\partial f}{\Delta_1 V_1} + \mu_2 V_1 \frac{\partial f(\sigma_1(t^0), s^0)}{\Delta_2 V_2}\right\} \sum_{i=1}^{2} \frac{\partial y_i}{\Delta_1 V_1} \frac{\partial}{\partial x_i}; \end{aligned}$$

(iv) $$\begin{aligned} \Delta_V\langle Y, Z\rangle = {} & \langle \Delta_V Y, Z\rangle + \langle \Delta_V Z, Y\rangle - \mu_1 V_1 \sum_{i=1}^{2} \frac{\partial y_i}{\Delta_1 V_1} \frac{\partial z_i}{\Delta_1 V_1} \frac{\partial}{\partial x_i} \\ & - \mu_2 V_1 \sum_{i=1}^{2} \frac{\partial z_i}{\Delta_1 V_1} \frac{\partial y_i(\sigma_1(t^0), s^0)}{\Delta_2 V_2} \frac{\partial}{\partial x_i}. \end{aligned}$$

Definition 2.4 The Lie parenthesis operator has the expression

$$[V_1, V_2]_\nabla(f) = \nabla_{V_1} V_2 - \nabla_{V_2} V_1 + \nu\left[\frac{\partial V_2}{\nabla x_1} V_1[f] - \frac{\partial V_1}{\nabla x_1} V_2[f]\right].$$

Similarly if the delta derivation of the Lie parenthesis operator is taken:

$$[V_1, V_2]_\Delta(f) = \Delta_{V_1} V_2 - \Delta_{V_2} V_1 + \mu\left[\frac{\partial V_2}{\Delta x_1} V_1[f] - \frac{\partial V_1}{\Delta x_1} V_2[f]\right].$$

Here $\nu = \rho(x) - x$ and $\mu = \sigma(x) - x$ [16].

3 Main result

Definition 3.1 Let M is a surface and $N = (N_1, N_2, \ldots, N_n)$ be the normal vector field of M; $\chi(M)$ be the vector field space of M; $T_M(P)$ be the tangent space of M at the point P; and Δ is the delta nature connection in Λ^n; the normal vector N is σ_1-completely delta differentiable; $V \in \chi(M)$ is a vector field;

$$S_\Delta(V) = -\Delta_V N = -\big(V[N_1], V[N_2], \ldots, V[N_n]\big)$$
$$= -\left(\frac{\partial N_1}{\Delta V}, \ldots, \frac{\partial N_n}{\Delta V}\right)$$

or we may find the delta shape operator at the point P as follows:

$$S_{\Delta p}(V_p) = -\Delta_{V_p} N_p = -\big(V_p[N_1], \ldots, V_p[N_n]\big)$$
$$= -\left(\frac{\partial N_1}{\Delta V_p}, \ldots, \frac{\partial N_n}{\Delta V_p}\right)$$

defined with the above value $S_{\Delta p}(V_p)$ transformation the surface M at the P point in V_p direction is called the '*Delta shape operator*' or the '*Delta Weingarten transformation*'. To facilitate the notation, we will denote the $S_{\Delta p}(V_p)$ transformation with $S_\Delta(V)$.

Theorem 3.1 *The delta shape operator $S_\Delta(V)$ is linear.*

Proof The proof is obvious. □

Theorem 3.2 *Let M be a surface defined with Λ^n time scales. The $\chi(M)$ shows vector fields space of the surface M, and the S_Δ delta shape operator is the conversion defined by*

$$S_\Delta : \chi(M) \to \chi(M)$$
$$V \to S_\Delta(V) = \Delta_V N.$$

Proof Since the N is the normal vector, the inner product $\langle N; N\rangle = 1$ is obtained. If we take the vector field derivation of both sides of this equation in the direction of V vector field, *i.e.* by making use of the $V[\langle N, N\rangle] = V[1]$ equation, and for the solution, by making use of the properties of the Δ_V connection, we obtain

$$0 = \langle \Delta_V N, N\rangle + \langle N, \Delta_V N\rangle - \mu_1 V_1 \sum_{i=1}^{n} \frac{\partial N_i}{\Delta_1 V_1} \frac{\partial N_i}{\Delta_1 V_1} \frac{\partial}{\partial x_i}$$
$$- \mu_2 V_1 \sum_{i=1}^{n} \frac{\partial N_i}{\Delta_1 V_1} \frac{\partial N_i(\sigma_1(t^0), s^0)}{\Delta_2 V_2} \frac{\partial}{\partial x_i}.$$

Here, to facilitate the equations, let us take the coefficient of μ_1 as δ_1 and the coefficient of μ_2 as δ_2. Then

$$2\langle \Delta_V N, N\rangle = \mu_1 \delta_1 - \mu_2 \delta_2 \tag{3.1}$$

is obtained. If we take $\mathbb{T}_1 \times \mathbb{T}_2 = \mathbb{R} \times \mathbb{R}$, the properties $\mu_1 = 0$, $\mu_2 = 0$ and $\langle \Delta_V N, N\rangle = 0$ are satisfied. From this, it is observed that $\Delta_V N \perp N$, *i.e.* $\Delta_V N \in T_p(\Lambda^2)$. In the case $\mathbb{T}_1 \times \mathbb{T}_2 \neq \mathbb{R} \times \mathbb{R}$ it becomes $\Delta_V N \in \chi(M)$. Thus the theorem is proven. □

Theorem 3.3 *The delta shape operator S_Δ is not symmetrical for $\Lambda^2 = \mathbb{T}_1 \times \mathbb{T}_2$; however, in the case $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{R}$, it is symmetrical, i.e. self-adjoint.*

Proof $\langle S_\Delta(X), Y\rangle = \langle X, S_\Delta(Y)\rangle$. For $\forall X, Y \in \chi(M)$, $\langle X, N\rangle = 0$ and $\langle Y, N\rangle = 0$. When the derivation of the inner product of $\langle X, N\rangle = 0$ is taken, and the derivation of the inner product for $\Delta_Y(\langle X, N\rangle) = 0$ and $\langle Y, N\rangle = 0$ is taken, subtract these two equations from each other by making use of the properties of the $\Delta_X\langle Y, N\rangle = 0$ connections;

$$0 = \langle \Delta_Y X, N\rangle + \langle \Delta_Y N, X\rangle - \mu_1 \overbrace{Y_1 \sum_{i=1}^{2} \frac{\partial X_i}{\Delta_1 Y_1} \frac{\partial N_i}{\Delta_1 Y_1} \frac{\partial}{\partial x_i}}$$

$$- \mu_2 \overbrace{Y_1 \sum_{i=1}^{2} \frac{\partial N_i}{\Delta_1 Y_1} \frac{\partial X_i(\sigma_1(t^0), s^0)}{\Delta_2 Y_2} \frac{\partial}{\partial x_i}},$$

$$0 = \langle \Delta_X Y, N\rangle + \langle \Delta_X N, Y\rangle - \mu_1 \overbrace{X_1 \sum_{i=1}^{2} \frac{\partial Y_i}{\Delta_1 X_1} \frac{\partial N_i}{\Delta_1 X_1} \frac{\partial}{\partial x_i}}$$

$$- \mu_2 \overbrace{X_1 \sum_{i=1}^{2} \frac{\partial N_i}{\Delta_1 X_1} \frac{\partial Y_i(\sigma_1(t^0), s^0)}{\Delta_2 X_2} \frac{\partial}{\partial x_i}}.$$

Here, let us use the letters a, b, c, d to the right side of μ_1, μ_2 to facilitate and abbreviate:

$$\langle \Delta_Y X - \Delta_X Y, N\rangle + \bigl[\bigl\langle S_\Delta(Y), X\bigr\rangle - \bigl\langle S_\Delta(X), Y\bigr\rangle\bigr] - \mu_1(a - c) - \mu_2(b - d) = 0,$$

$$\bigl[\bigl\langle S_\Delta(Y), X\bigr\rangle - \bigl\langle S_\Delta(X), Y\bigr\rangle\bigr] = -\langle \Delta_Y X - \Delta_X Y, N\rangle + \mu_1(a - c) + \mu_2(b - d).$$

As seen in the above equation, in the case $\mathbb{T}_1 \neq \mathbb{R}$ and $\mathbb{T}_2 \neq \mathbb{R}$, *i.e.* on any time scale, the delta shape operator is not symmetrical. However, in $\mathbb{T}_1 = \mathbb{R}$ and $\mathbb{T}_2 = \mathbb{R}$, $\mu_1 = \mu_2 = 0$ and $\langle \Delta_Y X - \Delta_X Y, N\rangle = 0$. It is observed that $\langle S_\Delta(Y), X\rangle = \langle S_\Delta(X), Y\rangle$. In other words, the $S_\Delta(V)$ delta shape operator is observed to be symmetrical when the $\Lambda^2 = \mathbb{R} \times \mathbb{R}$ time scale is taken. This property coincides with the property that is described by stating that the 'delta shape operator is symmetrical' in the surfaces theory in Euclidean space. □

Theorem 3.4

$$\varphi : \Lambda^2 \to \Lambda^3$$
$$(t, s) \to \varphi(t, s),$$

are differentiable; let the surface $\varphi(\Lambda^2) = M$ *be given. For the* M *surface's* N *unit vector field, the following equations are obtained*:

$$S_\Delta\left(\frac{\partial \varphi}{\Delta_1 t}\right) = \frac{\partial N}{\Delta_1 t} \quad \textit{and} \quad S_\Delta\left(\frac{\partial \varphi}{\Delta_2 s}\right) = \frac{\partial N}{\Delta_2 s}.$$

Proof t with a variable value, s with a constant value are taken for the $\varphi(t, s_0) = \alpha(t)$ curve,

$$S_\Delta\left(\frac{\partial \varphi}{\Delta_1 t}\right) = S\left(\frac{\partial \varphi}{\Delta_1 t}(t, s_0)\right) = S_\Delta\bigl(\alpha^{\Delta_1}(t)\bigr)$$
$$= D_{\alpha^{\Delta_1}(t)} N$$

$$= \big(\alpha(t)[N_1], \dots, \alpha(t)[N_2]\big)$$
$$= \frac{\partial N}{\Delta_1 t}(t);$$

t with constant value, s with variable value, and $\varphi(t_0, s) = B(t)$,

$$\begin{aligned} S_\Delta\left(\frac{\partial\varphi}{\Delta_2 s}\right) &= S_\Delta\left(\frac{\partial\varphi}{\Delta_2 s}(t_0, s)\right) = S_\Delta\big(\beta^{\Delta_2(t)}\big) \\ &= D_{\beta^{\Delta_2(t)}} N \\ &= \big(\beta(t)[N_1], \dots, \beta(t)[N_2]\big) \\ &= \frac{\partial N}{\Delta_2 s}(t). \end{aligned}$$

□

Theorem 3.5 *Since the principal normal N of the surface φ is vertical to $\frac{\partial\varphi}{\Delta_1 t}$ and $\frac{\partial\varphi}{\Delta_2 t}$, the following equations are obtained*:

(i) $\left\langle \frac{\partial N}{\Delta_1 t}, \frac{\partial\varphi}{\Delta_1 t}\right\rangle = -\left\langle N, \frac{\partial^2\varphi^{\sigma_1}}{\Delta_1 t^2}\right\rangle$;

(ii) $\left\langle \frac{\partial N}{\Delta_2 s}, \frac{\partial\varphi}{\Delta_2 s}\right\rangle = -\left\langle N, \frac{\partial^2\varphi^{\sigma_2}}{\Delta_2 s^2}\right\rangle$;

(iii) $\left\langle \frac{\partial N}{\Delta_1 t}, \frac{\partial\varphi}{\Delta_2 s}\right\rangle = -\left\langle N, \frac{\partial^2\varphi^{\sigma_1}}{\Delta_1 t\Delta_2 s}\right\rangle$;

(iv) $\left\langle \frac{\partial N}{\Delta_2 s}, \frac{\partial\varphi}{\Delta_1 t}\right\rangle = -\left\langle N, \frac{\partial^2\varphi^{\sigma_2}}{\Delta_2 s\Delta_1 t}\right\rangle$.

Proof (i) When the derivation is taken on both sides of $\langle N, \frac{\partial\varphi}{\Delta_1 t}\rangle = 0$, we have

$$\frac{\partial}{\Delta_1 t}\left\langle N, \frac{\partial\varphi}{\Delta_1 t}\right\rangle = 0,$$
$$\left\langle \frac{\partial N}{\Delta_1 t}, \frac{\partial\varphi}{\Delta_1 t}\right\rangle + \left\langle N, \frac{\partial^2\varphi^{\sigma_1}}{\Delta_1 t^2}\right\rangle = 0,$$
$$\left\langle \frac{\partial N}{\Delta_1 t}, \frac{\partial\varphi}{\Delta_1 t}\right\rangle = -\left\langle N, \frac{\partial^2\varphi^{\sigma_1}}{\Delta_1 t^2}\right\rangle.$$

(ii) $\langle N, \frac{\partial\varphi}{\Delta_1 t}\rangle = 0$; let us take the derivation of both sides of the above equation:

$$\frac{\partial}{\Delta_2 s}\left\langle N, \frac{\partial\varphi}{\Delta_2 s}\right\rangle = 0,$$
$$\left\langle \frac{\partial N}{\Delta_2 s}, \frac{\partial\varphi}{\Delta_2 s}\right\rangle + \left\langle N, \frac{\partial^2\varphi^{\sigma_2}}{\Delta_2 s^2}\right\rangle = 0,$$
$$\left\langle \frac{\partial N}{\Delta_2 s}, \frac{\partial\varphi}{\Delta_2 s}\right\rangle = -\left\langle N, \frac{\partial^2\varphi^{\sigma_2}}{\Delta_2 s^2}\right\rangle.$$

(iii) $\langle N, \frac{\partial\varphi}{\Delta_2 s}\rangle = 0$; let us take the derivation of both sides of the above equation:

$$\frac{\partial}{\Delta_1 t}\left\langle N, \frac{\partial\varphi}{\Delta_2 s}\right\rangle = 0,$$

$$\left\langle \frac{\partial N}{\Delta_1 t}, \frac{\partial \varphi}{\Delta_2 s} \right\rangle + \left\langle N, \frac{\partial^2 \varphi^{\sigma_1}}{\Delta_1 t \Delta_2 s} \right\rangle = 0,$$
$$\left\langle \frac{\partial N}{\Delta_1 t}, \frac{\partial \varphi}{\Delta_2 s} \right\rangle = -\left\langle N, \frac{\partial^2 \varphi^{\sigma_1}}{\Delta_1 t \Delta_2 s} \right\rangle = 0.$$

(iv) $\frac{\partial}{\Delta_2 s}\langle N, \frac{\partial \varphi}{\Delta_1 t}\rangle = 0$; let us take the derivation of both sides of the above equation:

$$\frac{\partial}{\Delta_2 s}\left\langle N, \frac{\partial \varphi}{\Delta_1 t} \right\rangle = 0,$$
$$\left\langle \frac{\partial N}{\Delta_2 s}, \frac{\partial \varphi}{\Delta_1 t} \right\rangle + \left\langle N, \frac{\partial^2 \varphi^{\sigma_2}}{\Delta_2 s \Delta_1 t} \right\rangle = 0,$$
$$\left\langle \frac{\partial N}{\Delta_2 s}, \frac{\partial \varphi}{\Delta_1 t} \right\rangle = -\left\langle N, \frac{\partial^2 \varphi^{\sigma_1}}{\Delta_2 s \Delta_1 t} \right\rangle = 0.$$
□

Theorem 3.6 *Let the $\varphi : \mathbb{T}_1 \times \mathbb{T}_2 \subset \Lambda^2 \to \Lambda^3$ surface be given for the completely differentiable function that is defined as $\varphi(\mathbb{T}_1 \times \mathbb{T}_2) = M$.*

If $T_M(P) = sp\{\frac{\partial \varphi}{\Delta_1 t}|_p, \frac{\partial \varphi}{\Delta_2 s}|_p\}$ and $\chi(M) = sp\{\frac{\partial \varphi}{\Delta_1 t}, \frac{\partial \varphi}{\Delta_2 s}\}$, $S_\Delta = \chi(M) \to \chi(M)$ the matrix is as follows according to the figure operator basis $\{\frac{\partial \varphi}{\Delta_1 t}, \frac{\partial \varphi}{\Delta_2 s}\}$:

$$S_\Delta = \begin{bmatrix} \frac{-1}{\|\frac{\partial \varphi}{\Delta_1 t}\|^3 \|\frac{\partial \varphi}{\Delta_2 s}\|} \det(\frac{\partial^2 \varphi^{\sigma_1}}{\Delta_1 t^2}, \frac{\partial \varphi}{\Delta_1 t}, \frac{\partial \varphi}{\Delta_2 s}) & \frac{-1}{\|\frac{\partial \varphi}{\Delta_1 t}\|^2 \|\frac{\partial \varphi}{\Delta_2 s}\|^2} \det(\frac{\partial^2 \varphi^{\sigma_2}}{\Delta_2 s \Delta_1 t}, \frac{\partial \varphi}{\Delta_1 t}, \frac{\partial \varphi}{\Delta_2 s}) \\ \frac{-1}{\|\frac{\partial \varphi}{\Delta_1 t}\|^2 \|\frac{\partial \varphi}{\Delta_2 s}\|} \det(\frac{\partial^2 \varphi^{\sigma_1}}{\Delta_1 t \Delta_2 s}, \frac{\partial \varphi}{\Delta_1 t}, \frac{\partial \varphi}{\Delta_2 s}) & \frac{-1}{\|\frac{\partial \varphi}{\Delta_1 t}\| \|\frac{\partial \varphi}{\Delta_2 s}\|^3} \det(\frac{\partial^2 \varphi^{\sigma_2}}{\Delta_2 s^2}, \frac{\partial \varphi}{\Delta_1 t}, \frac{\partial \varphi}{\Delta_2 s}) \end{bmatrix}.$$

Proof The matrix of the delta shape operators $S_\Delta(\frac{\partial \varphi}{\Delta_1 t}) = a\frac{\partial \varphi}{\Delta_1 t} + b\frac{\partial \varphi}{\Delta_2 s}$ and $S_\Delta(\frac{\partial \varphi}{\Delta_2 s}) = c\frac{\partial \varphi}{\Delta_1 t} + d\frac{\partial \varphi}{\Delta_2 s}$ may be written as

$$\left[S_\Delta\left(\frac{\partial \varphi}{\Delta_1 t}\right), S_\Delta\left(\frac{\partial \varphi}{\Delta_2 s}\right) \right] = \left[\frac{\partial \varphi}{\Delta_1 t} \; \frac{\partial \varphi}{\Delta_2 s} \right] \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Then we get

$$S_\Delta = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

If we take multiplication of $S_\Delta(\frac{\partial \varphi}{\Delta_1 t})$ and $S_\Delta(\frac{\partial \varphi}{\Delta_2 s})$ by $\frac{\partial \varphi}{\Delta_1 t}$ and $\frac{\partial \varphi}{\Delta_2 s}$, respectively, we find the a, b, c, d components of the S matrix. We have

$$\left\langle S_\Delta\left(\frac{\partial \varphi}{\Delta_1 t}\right), \frac{\partial \varphi}{\Delta_1 t} \right\rangle = \left\langle a\frac{\partial \varphi}{\Delta_1 t} + b\frac{\partial \varphi}{\Delta_2 s}, \frac{\partial \varphi}{\Delta_1 t} \right\rangle$$
$$= a\left\langle \frac{\partial \varphi}{\Delta_1 t}, \frac{\partial \varphi}{\Delta_1 t} \right\rangle + b\left\langle \frac{\partial \varphi}{\Delta_2 s}, \frac{\partial \varphi}{\Delta_1 t} \right\rangle,$$
$$\left\langle S_\Delta\left(\frac{\partial \varphi}{\Delta_1 t}\right), \frac{\partial \varphi}{\Delta_2 s} \right\rangle = \left\langle a\frac{\partial \varphi}{\Delta_1 t} + b\frac{\partial \varphi}{\Delta_2 s}, \frac{\partial \varphi}{\Delta_2 s} \right\rangle$$
$$= a\left\langle \frac{\partial \varphi}{\Delta_1 t}, \frac{\partial \varphi}{\Delta_2 s} \right\rangle + b\left\langle \frac{\partial \varphi}{\Delta_2 s}, \frac{\partial \varphi}{\Delta_2 s} \right\rangle,$$

$$\begin{aligned}\left\langle S_\Delta\left(\frac{\partial\varphi}{\Delta_2 s}\right), \frac{\partial\varphi}{\Delta_1 t}\right\rangle &= \left\langle a\frac{\partial\varphi}{\Delta_1 t} + b\frac{\partial\varphi}{\Delta_2 s}, \frac{\partial\varphi}{\Delta_1 t}\right\rangle \\ &= a\left\langle \frac{\partial\varphi}{\Delta_1 t}, \frac{\partial\varphi}{\Delta_1 t}\right\rangle + b\left\langle \frac{\partial\varphi}{\Delta_2 s}, \frac{\partial\varphi}{\Delta_1 t}\right\rangle, \\ \left\langle S_\Delta\left(\frac{\partial\varphi}{\Delta_2 s}\right), \frac{\partial\varphi}{\Delta_2 s}\right\rangle &= \left\langle a\frac{\partial\varphi}{\Delta_1 t} + b\frac{\partial\varphi}{\Delta_2 s}, \frac{\partial\varphi}{\Delta_2 s}\right\rangle \\ &= a\left\langle \frac{\partial\varphi}{\Delta_1 t}, \frac{\partial\varphi}{\Delta_2 s}\right\rangle + b\left\langle \frac{\partial\varphi}{\Delta_2 s}, \frac{\partial\varphi}{\Delta_2 s}\right\rangle.\end{aligned}$$

Here, from [18], since $E = \langle \frac{\partial\varphi}{\Delta_1 t}, \frac{\partial\varphi}{\Delta_1 t}\rangle$, $F = \langle \frac{\partial\varphi}{\Delta_1 t}, \frac{\partial\varphi}{\Delta_2 s}\rangle$, $G = \langle \frac{\partial\varphi}{\Delta_2 s}, \frac{\partial\varphi}{\Delta_2 s}\rangle$ and if the above equations are made use of we have the following results:

$$\begin{aligned}&\left\langle S_\Delta\left(\frac{\partial\varphi}{\Delta_1 t}\right), \frac{\partial\varphi}{\Delta_1 t}\right\rangle = aE + bF, \\ &\left\langle S_\Delta\left(\frac{\partial\varphi}{\Delta_1 t}\right), \frac{\partial\varphi}{\Delta_2 s}\right\rangle = aF + bG, \\ &\left\langle S_\Delta\left(\frac{\partial\varphi}{\Delta_2 s}\right), \frac{\partial\varphi}{\Delta_1 t}\right\rangle = cE + dF, \\ &\left\langle S_\Delta\left(\frac{\partial\varphi}{\Delta_2 s}\right), \frac{\partial\varphi}{\Delta_2 s}\right\rangle = cF + dG,\end{aligned}$$

are obtained. Then for this linear equation system to have a solution, one needs

$$\begin{vmatrix} E & F \\ F & G \end{vmatrix} = EG - F^2.$$

Here, the coefficient matrix determinant must be zero. In addition, since the vector product of the parameter curves of the M surface gives the following result:

$$\begin{aligned}\left\| \frac{\partial\varphi}{\Delta_1 t} \times \frac{\partial\varphi}{\Delta_2 s}\right\|^2 &= \left\langle \frac{\partial\varphi}{\Delta_1 t} \times \frac{\partial\varphi}{\Delta_2 s}, \frac{\partial\varphi}{\Delta_1 t} \times \frac{\partial\varphi}{\Delta_2 s}\right\rangle \\ &= \begin{vmatrix} \langle \frac{\partial\varphi}{\Delta_1 t}, \frac{\partial\varphi}{\Delta_1 t}\rangle & \langle \frac{\partial\varphi}{\Delta_1 t}, \frac{\partial\varphi}{\Delta_2 s}\rangle \\ \langle \frac{\partial\varphi}{\Delta_1 t}, \frac{\partial\varphi}{\Delta_2 s}\rangle & \langle \frac{\partial\varphi}{\Delta_2 s}, \frac{\partial\varphi}{\Delta_2 s}\rangle \end{vmatrix} \\ &= \begin{vmatrix} E & F \\ F & G \end{vmatrix} = EG - F^2 \neq 0,\end{aligned}$$

there is only one solution for the a, b, c, d unknowns of the above-given linear equation system. When we write the vectors $S_\Delta(\frac{\partial\varphi}{\Delta_1 t})$ and $S_\Delta(\frac{\partial\varphi}{\Delta_2 s})$, which are on the left side on this equation system as the connection of the vertical vector area, the following equations are obtained:

$$\begin{aligned}&S_\Delta\left(\frac{\partial\varphi}{\Delta_1 t}\right) = \Delta_{\frac{\partial\varphi}{\Delta_1 t}} N = \frac{\partial N}{\Delta_1 t}, \\ &S_\Delta\left(\frac{\partial\varphi}{\Delta_2 s}\right) = \Delta_{\frac{\partial\varphi}{\Delta_2 s}} N = \frac{\partial N}{\Delta_2 s}.\end{aligned}$$

For the $\{V_1, V_2\}$ orthonormal basis system, the basis vectors are defined with

$$V_1 = \frac{\frac{\partial\varphi}{\Delta_1 t}}{\|\frac{\partial\varphi}{\Delta_1 t}\|} \quad \text{and} \quad V_2 = \frac{\frac{\partial\varphi}{\Delta_2 s}}{\|\frac{\partial\varphi}{\Delta_2 s}\|},$$

the delta shape operators are

$$S_\Delta(V_1) = c_1 V_1 + c_2 V_2,$$
$$S_\Delta(V_2) = c_3 V_1 + c_4 V_2.$$

If the matrix notation is

$$S_\Delta = \begin{bmatrix} c_1 & c_3 \\ c_2 & c_4 \end{bmatrix},$$

the elements of this matrix are calculated with the following formulas:

$$c_1 = \langle S_\Delta(V_1), V_1\rangle, \qquad c_3 = \langle S_\Delta(V_1), V_1\rangle,$$
$$c_2 = \langle S_\Delta(V_1), V_2\rangle, \qquad c_4 = \langle S_\Delta(V_1), V_2\rangle.$$

Thus, the c_1 coefficient is

$$\begin{aligned}
c_1 &= \langle S_\Delta(V_1), V_1\rangle = \left\langle S_\Delta\left(\frac{\frac{\partial\varphi}{\Delta_1 t}}{\|\frac{\partial\varphi}{\Delta_1 t}\|}\right), \frac{\frac{\partial\varphi}{\Delta_1 t}}{\|\frac{\partial\varphi}{\Delta_1 t}\|}\right\rangle \\
&= \frac{1}{\|\frac{\partial\varphi}{\Delta_1 t}\|^2}\left\langle S_\Delta\left(\frac{\partial\varphi}{\Delta_1 t}\right), \frac{\partial\varphi}{\Delta_1 t}\right\rangle \\
&= \frac{1}{\|\frac{\partial\varphi}{\Delta_1 t}\|^2}\left\langle \frac{\partial N}{\Delta_1 t}, \frac{\partial\varphi}{\Delta_1 t}\right\rangle \\
&= \frac{1}{\|\frac{\partial\varphi}{\Delta_1 t}\|^2}\left[-\left\langle N, \frac{\partial\varphi^{\sigma_1}}{\Delta_1 t^2}\right\rangle\right] = -\frac{1}{\|\frac{\partial\varphi}{\Delta_1 t}\|^2}\left\langle N, \frac{\partial\varphi^{\sigma_1}}{\Delta_1 t^2}\right\rangle \\
&= -\frac{1}{\|\frac{\partial\varphi}{\Delta_1 t}\|^2}\left\langle \frac{\frac{\partial\varphi}{\Delta_1 t}\times\frac{\partial\varphi}{\Delta_2 s}}{\|\frac{\partial\varphi}{\Delta_1 t}\times\frac{\partial\varphi}{\Delta_2 s}\|}, \frac{\partial\varphi^{\sigma_1}}{\Delta_1 t^2}\right\rangle \\
&= -\frac{1}{\|\frac{\partial\varphi}{\Delta_1 t}\|^2}\frac{1}{\|\frac{\partial\varphi}{\Delta_1 t}\times\frac{\partial\varphi}{\Delta_2 s}\|}\left\langle \frac{\partial\varphi}{\Delta_1 t}\times\frac{\partial\varphi}{\Delta_2 s}, \frac{\partial\varphi^{\sigma_1}}{\Delta_1 t^2}\right\rangle \\
&= -\frac{1}{\|\frac{\partial\varphi}{\Delta_1 t}\|^2}\frac{1}{\|\frac{\partial\varphi}{\Delta_1 t}\|\cdot\|\frac{\partial\varphi}{\Delta_2 s}\|\cdot\sin 90^\circ}\left(\frac{\partial\varphi}{\Delta_1 t}, \frac{\partial\varphi}{\Delta_2 s}, \frac{\partial\varphi^{\sigma_1}}{\Delta_1 t^2}\right) \\
&= -\frac{1}{\|\frac{\partial\varphi}{\Delta_1 t}\|^3\cdot\|\frac{\partial\varphi}{\Delta_2 s}\|}\left(\frac{\partial\varphi}{\Delta_1 t}, \frac{\partial\varphi}{\Delta_2 s}, \frac{\partial\varphi^{\sigma_1}}{\Delta_1 t^2}\right),
\end{aligned}$$

the c_2 coefficient is

$$\begin{aligned}
c_2 &= \langle S_\Delta(V_1), V_2\rangle \\
&= \left\langle S_\Delta\left(\frac{\frac{\partial\varphi}{\Delta_1 t}}{\|\frac{\partial\varphi}{\Delta_1 t}\|}\right), \frac{\frac{\partial\varphi}{\Delta_2 s}}{\|\frac{\partial\varphi}{\Delta_2 s}\|}\right\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\|\frac{\partial\varphi}{\Delta_1 t}\|} \cdot \frac{1}{\|\frac{\partial\varphi}{\Delta_2 s}\|} \left\langle S_\Delta\left(\frac{\partial\varphi}{\Delta_1 t}\right), \frac{\partial\varphi}{\Delta_2 s}\right\rangle \\
&= \frac{1}{\|\frac{\partial\varphi}{\Delta_1 t}\| \cdot \|\frac{\partial\varphi}{\Delta_2 s}\|} \cdot \left\langle \frac{\partial N}{\Delta_1 t}, \frac{\partial\varphi}{\Delta_2 s}\right\rangle \\
&= \frac{1}{\|\frac{\partial\varphi}{\Delta_1 t}\| \cdot \|\frac{\partial\varphi}{\Delta_2 s}\|} \left[-\left\langle N, \frac{\partial^2\varphi^{\sigma_1}}{\Delta_1 t \cdot \Delta_2 s}\right\rangle\right] \\
&= -\frac{1}{\|\frac{\partial\varphi}{\Delta_1 t}\| \cdot \|\frac{\partial\varphi}{\Delta_2 s}\|} \left[\left\langle \frac{\frac{\partial\varphi}{\Delta_1 t} \times \frac{\partial\varphi}{\Delta_2 s}}{\|\frac{\partial\varphi}{\Delta_1 t} \times \frac{\partial\varphi}{\Delta_2 s}\|}, \frac{\partial^2\varphi^{\sigma_1}}{\Delta_1 t \Delta_2 s}\right\rangle\right] \\
&= -\frac{1}{\|\frac{\partial\varphi}{\Delta_1 t}\| \cdot \|\frac{\partial\varphi}{\Delta_2 s}\| \|\frac{\partial\varphi}{\Delta_1 t} \times \frac{\partial\varphi}{\Delta_2 s}\|} \left\langle \frac{\partial\varphi}{\Delta_1 t} \times \frac{\partial\varphi}{\Delta_2 s}, \frac{\partial^2\varphi^{\sigma_1}}{\Delta_1 t \Delta_2 s}\right\rangle \\
&= -\frac{1}{\|\frac{\partial\varphi}{\Delta_1 t}\|^2 \cdot \|\frac{\partial\varphi}{\Delta_2 s}\|^2 \sin 90^\circ} \left(\frac{\partial\varphi}{\Delta_1 t} \frac{\partial\varphi}{\Delta_2 s} \frac{\partial^2\varphi^{\sigma_1}}{\Delta_1 t \Delta_2 s}\right) \\
&= -\frac{1}{\|\frac{\partial\varphi}{\Delta_1 t}\|^2 \cdot \|\frac{\partial\varphi}{\Delta_2 s}\|^2} \left(\frac{\partial\varphi}{\Delta_1 t} \frac{\partial\varphi}{\Delta_2 s} \frac{\partial^2\varphi^{\sigma_1}}{\Delta_1 t \Delta_2 s}\right),
\end{aligned}
$$

the c_3 coefficient is

$$
\begin{aligned}
c_3 &= \langle S_\Delta(V_2), V_1\rangle = \left\langle S_\Delta\left(\frac{\frac{\partial\varphi}{\Delta_2 s}}{\|\frac{\partial\varphi}{\Delta_2 s}\|}\right), \frac{\frac{\partial\varphi}{\Delta_1 t}}{\|\frac{\partial\varphi}{\Delta_1 t}\|}\right\rangle \\
&= \frac{1}{\|\frac{\partial\varphi}{\Delta_1 t}\| \|\frac{\partial\varphi}{\Delta_2 s}\|} \left\langle S_\Delta\left(\frac{\partial\varphi}{\Delta_2 s}\right), \frac{\partial\varphi}{\Delta_1 t}\right\rangle \\
&= \frac{1}{\|\frac{\partial\varphi}{\Delta_1 t}\| \|\frac{\partial\varphi}{\Delta_2 s}\|} \left\langle \frac{\partial N}{\Delta_2 s}, \frac{\partial\varphi}{\Delta_1 t}\right\rangle \\
&= \frac{1}{\|\frac{\partial\varphi}{\Delta_1 t}\| \|\frac{\partial\varphi}{\Delta_2 s}\|} \left[-\left\langle N, \frac{\partial^2\varphi^{\sigma_2}}{\Delta_2 s \cdot \Delta_1 t}\right\rangle\right] \\
&= -\frac{1}{\|\frac{\partial\varphi}{\Delta_1 t}\| \|\frac{\partial\varphi}{\Delta_2 s}\|} \left[\left\langle \frac{\frac{\partial\varphi}{\Delta_1 t} \times \frac{\partial\varphi}{\Delta_2 s}}{\|\frac{\partial\varphi}{\Delta_1 t} \times \frac{\partial\varphi}{\Delta_2 s}\|}, \frac{\partial^2\varphi^{\sigma_2}}{\Delta_2 s \cdot \Delta_1 t}\right\rangle\right] \\
&= -\frac{1}{\|\frac{\partial\varphi}{\Delta_1 t}\| \|\frac{\partial\varphi}{\Delta_2 s}\| \|\frac{\partial\varphi}{\Delta_1 t} \times \frac{\partial\varphi}{\Delta_2 s}\|} \left\langle \frac{\partial\varphi}{\Delta_1 t} \times \frac{\partial\varphi}{\Delta_2 s}, \frac{\partial^2\varphi^{\sigma_2}}{\Delta_2 s \Delta_1 t}\right\rangle \\
&= -\frac{1}{\|\frac{\partial\varphi}{\Delta_1 t}\|^2 \cdot \|\frac{\partial\varphi}{\Delta_2 s}\|^2 \sin 90^\circ} \left(\frac{\partial\varphi}{\Delta_1 t} \frac{\partial\varphi}{\Delta_2 s} \frac{\partial^2\varphi^{\sigma_2}}{\Delta_2 s \Delta_1 t}\right) \\
&= -\frac{1}{\|\frac{\partial\varphi}{\Delta_2 s}\|^2 \cdot \|\frac{\partial\varphi}{\Delta_1 t}\|^2} \left(\frac{\partial\varphi}{\Delta_1 t} \frac{\partial\varphi}{\Delta_2 s} \frac{\partial^2\varphi^{\sigma_2}}{\Delta_2 s \Delta_1 t}\right),
\end{aligned}
$$

and the c_4 coefficient is

$$
\begin{aligned}
c_4 &= \langle S_\Delta(V_2), V_2\rangle = \left\langle S_\Delta\left(\frac{\frac{\partial\varphi}{\Delta_2 s}}{\|\frac{\partial\varphi}{\Delta_2 s}\|}\right), \frac{\frac{\partial\varphi}{\Delta_2 s}}{\|\frac{\partial\varphi}{\Delta_2 s}\|}\right\rangle \\
&= \frac{1}{\|\frac{\partial\varphi}{\Delta_2 s}\| \cdot \|\frac{\partial\varphi}{\Delta_2 s}\|} \left\langle S_\Delta\left(\frac{\partial\varphi}{\Delta_2 s}\right), \frac{\partial\varphi}{\Delta_2 s}\right\rangle
\end{aligned}
$$

$$\begin{aligned}
&= \frac{1}{\|\frac{\partial\varphi}{\Delta_2 s}\|\cdot\|\frac{\partial\varphi}{\Delta_2 s}\|}\cdot\left\langle\frac{\partial N}{\Delta_2 s},\frac{\partial\varphi}{\Delta_2 s}\right\rangle \\
&= \frac{1}{\|\frac{\partial\varphi}{\Delta_2 s}\|\cdot\|\frac{\partial\varphi}{\Delta_2 s}\|}\left[-\left\langle N,\frac{\partial\varphi^{\sigma_2}}{\Delta_2 s}\right\rangle\right] \\
&= -\frac{1}{\|\frac{\partial\varphi}{\Delta_2 s}\|\cdot\|\frac{\partial\varphi}{\Delta_2 s}\|}\left[\left\langle\frac{\frac{\partial\varphi}{\Delta_1 t}\times\frac{\partial\varphi}{\Delta_2 s}}{\|\frac{\partial\varphi}{\Delta_1 t}\times\frac{\partial\varphi}{\Delta_2 s}\|},\frac{\partial\varphi^{\sigma_2}}{\Delta_2 s}\right\rangle\right] \\
&= -\frac{1}{\|\frac{\partial\varphi}{\Delta_1 t}\|\cdot\|\frac{\partial\varphi}{\Delta_2 s}\|^3\cdot\sin 90^\circ}\left(\frac{\partial\varphi}{\Delta_1 t}\frac{\partial\varphi}{\Delta_2 s}\frac{\partial\varphi^{\sigma_2}}{\Delta_2 s}\right) \\
&= -\frac{1}{\|\frac{\partial\varphi}{\Delta_1 t}\|\cdot\|\frac{\partial\varphi}{\Delta_2 s}\|^3}\left(\frac{\partial\varphi}{\Delta_1 t}\frac{\partial\varphi}{\Delta_2 s}\frac{\partial^2\varphi^{\sigma_2}}{\Delta_2 s}\right).
\end{aligned}$$

Finally, if we make use of the matrix components of the delta shape operator:

$$S_\Delta = \begin{bmatrix} \frac{-1}{\|\frac{\partial\varphi}{\Delta_1 t}\|^3\|\frac{\partial\varphi}{\Delta_2 s}\|}\det(\frac{\partial^2\varphi^{\sigma_1}}{\Delta_1 t^2},\frac{\partial\varphi}{\Delta_1 t},\frac{\partial\varphi}{\Delta_2 s}) & \frac{-1}{\|\frac{\partial\varphi}{\Delta_1 t}\|^2\|\frac{\partial\varphi}{\Delta_2 s}\|^2}\det(\frac{\partial^2\varphi^{\sigma_2}}{\Delta_2 s\Delta_1 t},\frac{\partial\varphi}{\Delta_1 t},\frac{\partial\varphi}{\Delta_2 s}) \\ \frac{-1}{\|\frac{\partial\varphi}{\Delta_1 t}\|^2\|\frac{\partial\varphi}{\Delta_2 s}\|}\det(\frac{\partial^2\varphi^{\sigma_1}}{\Delta_1 t\Delta_2 s},\frac{\partial\varphi}{\Delta_1 t},\frac{\partial\varphi}{\Delta_2 s}) & \frac{-1}{\|\frac{\partial\varphi}{\Delta_1 t}\|\|\frac{\partial\varphi}{\Delta_2 s}\|^3}\det(\frac{\partial^2\varphi^{\sigma_2}}{\Delta_2 s^2},\frac{\partial\varphi}{\Delta_1 t},\frac{\partial\varphi}{\Delta_2 s}) \end{bmatrix}.$$ □

Theorem 3.7 *On the surface $\varphi:\pi_1\times\pi_2\to\Lambda^3$, when $\pi_1=\pi_2=\pi$, the result will be $\sigma_1=\sigma_2=\sigma$, and then*

$$\frac{\partial^2\varphi^{\sigma_1}}{\Delta_1 t\Delta_2 s}=\frac{\partial^2\varphi^{\sigma_2}}{\Delta_2 s\Delta_1 t}=\frac{\partial^2\varphi^{\sigma}}{\Delta_1 t\Delta_2 s}.$$

Thus, $c_2=c_3$ is found in the delta shape operator.

Theorem 3.8 *For the $\pi_1\neq\pi_2$ situation, the matrix to the S_Δ delta shape operator corresponds is not symmetrical*; *however, for the $\pi_1=\pi_2$ situation, the matrix is symmetrical.*

4 Numeric examples

Example 1 Let us calculate the matrix representation of the delta shape operator of the surface $\varphi(t,s)=(t,s^2,t^2)$. Here, the partial derivations of the $\varphi(t,s)$ surface $\varphi_t=(1,0,2t)$, $\varphi_s=(0,2s,0)$, $\varphi_{ts}=(0,0,0)$, $\varphi_{st}=(0,0,0)$, $\varphi_{tt}=(0,0,2)$, $\varphi_{ss}=(0,2,0)$ are obtained. Also, we need the equations $\|\varphi_t\|=\sqrt{1+4t^2}$, $\|\varphi_s\|=2s$, $\det(\varphi_{tt},\varphi_t,\varphi_s)=4s$, and $\det(\varphi_{ts},\varphi_t\varphi_s)=0$, $\det(\varphi_{ss},\varphi_t,\varphi_s)=0$ for the matrix components. From these equations, we obtain the matrix of the delta shape operator on Euclidean space without time scales as in the following equation:

$$S=\begin{bmatrix}\frac{2}{(\sqrt{1+4t^2})^3} & 0\\ 0 & 0\end{bmatrix}. \tag{4.1}$$

Now, we will try to calculate the delta shape operator's matrix representation of $\varphi(t,s)$, the surface with using time scales $\mathbb{T}_1\times\mathbb{T}_2=\mathbb{R}\times\mathbb{R}$. The partial derivations of the $\varphi(t,s)$ surface on time scales

$$\frac{\partial\varphi}{\Delta_1 t}=\bigl(1,0,\sigma_1(t)+t\bigr),\qquad \frac{\partial\varphi}{\Delta_2 s}=\bigl(0,\sigma_2(s)+s,0\bigr),$$

$$\frac{\partial^2\varphi}{\Delta_1 t\cdot\Delta_2 s}=(0,0,0),\qquad \frac{\partial^2\varphi}{\Delta_2 s\cdot\Delta_1 t}=(0,0,0),$$
$$\frac{\partial^2\varphi^{\sigma_1}}{\Delta_1 t^2}=\left(0,0,\frac{\partial\sigma_1(t)}{\Delta_1 t}+1\right),\qquad \frac{\partial^2\varphi^{\sigma_2}}{\Delta_2 s^2}=\left(0,\frac{\partial\sigma_2(s)}{\Delta_2 s}+1,0\right),$$

are obtained, and here let σ_1 and σ_2 have delta differentiability. Also, we get the norms

$$\left\|\frac{\partial\varphi}{\Delta_1 t}\right\|=\sqrt{1+\bigl(\sigma_1(t)+t\bigr)^2},\qquad \left\|\frac{\partial\varphi}{\Delta_2 s}\right\|=\sqrt{\bigl(\sigma_2(s)+s\bigr)^2}=\sigma_2(s)+s,$$

and the determinants

$$\det\left(\frac{\partial\varphi}{\Delta_1 t},\frac{\partial\varphi}{\Delta_2 s},\frac{\partial^2\varphi^{\sigma_1}}{\Delta_1 t^2}\right)=\bigl[\sigma_2(s)+s\bigr]\cdot\left[\frac{\partial\sigma_1(t)}{\Delta_1 t}+1\right],$$
$$\det\left(\frac{\partial\varphi}{\Delta_1 t},\frac{\partial\varphi}{\Delta_2 s},\frac{\partial^2\varphi^{\sigma_2}}{\Delta_2 s\cdot\Delta_1 t}\right)=\det\left(\frac{\partial\varphi}{\Delta_1 t},\frac{\partial\varphi}{\Delta_2 s},\frac{\partial^2\varphi^{\sigma 1}}{\Delta_1 t\cdot\Delta_2 s}\right)$$
$$=\det\left(\frac{\partial\varphi}{\Delta_1 t},\frac{\partial\varphi}{\Delta_2 s},\frac{\partial^2\varphi^{\sigma 2}}{\Delta_2 s^2}\right)=0$$

for the matrix components. From these equations we get

$$S_\Delta=\begin{bmatrix}\frac{[s+s][\frac{\partial(t)}{\Delta_1 t}+1]}{(\sqrt{1+(t+t)^2})^3\cdot(s+s)} & 0\\ 0 & 0\end{bmatrix},\tag{4.2}$$

which is the matrix of the delta shape operator on the time scales. Note that if we take the time scale as $\mathbb{T}_1\times\mathbb{T}_2=\mathbb{R}\times\mathbb{R}$, the delta shape operator matrix on time scales S_Δ in equation (4.2) will be equal to the shape operator matrix S on Euclidean space without time scales in equation (4.1) as follows:

$$S_\Delta=\begin{bmatrix}\frac{2}{(\sqrt{1+4t^2})^3} & 0\\ 0 & 0\end{bmatrix}.$$

Here, for $\mathbb{T}_1\times\mathbb{T}_2=\mathbb{R}\times\mathbb{R}$ we have $\sigma_1(t)=t$, $\sigma_2(s)=s$. Thus we may find the indiscrete shape operator from the same matrix of the delta shape operator. In this easier and smoother method we have only one delta shape operator matrix included in both the discrete and the indiscrete cases.

If we take the time scale as $\mathbb{T}_1\times\mathbb{T}_2=\mathbb{Z}\times\mathbb{Z}$ for the discrete case, the forward jump operator of the set $\mathbb{Z}$ is $\sigma_1(t)=t+1$ and $\sigma_2(s)=s+1$. Hence, the delta shape operator matrix

$$S_\Delta=\begin{bmatrix}\frac{-2}{(\sqrt{1+(2t+1)^2})^3} & 0\\ 0 & 0\end{bmatrix}$$

is obtained for $\mathbb{T}_1\times\mathbb{T}_2=\mathbb{Z}\times\mathbb{Z}$.

Example 2 The parametric equation of the plane which is passing through the point $A(2,-1,3)$ and parallel to the vectors $\vec{u}=(1,1,1)$ and $\vec{v}=(3,2,4)$ is obtained: $\varphi(t,s)=$

$(2+t+3s,-1+t+4s,3+t+4s)$. We can calculate the delta shape operator for the time scales $\mathbb{R}\times\mathbb{R}$ and $\mathbb{Z}\times\mathbb{Z}$. The partial derivatives and their norms are $\frac{\partial\varphi}{\Delta_1 t}=(1,1,1)$, $\frac{\partial\varphi}{\Delta_2 s}=(3,4,4)$, $\|\frac{\partial\varphi}{\Delta_1 t}\|=\sqrt{3}$, $\|\frac{\partial\varphi}{\Delta_2 s}\|=\sqrt{41}$. The partial derivatives are independent from σ_1 and σ_2. Hence $\frac{\partial^2\varphi}{\Delta_1 t\Delta_2 s}=\frac{\partial^2\varphi}{\Delta_2 s\Delta_1 t}=\frac{\partial^2\varphi^{\sigma_1}}{\Delta_1 t^2}=\frac{\partial^2\varphi^{\sigma_2}}{\Delta_2 s^2}=(0,0,0)$. Thus for both time scales $\mathbb{T}_1\times\mathbb{T}_2=\mathbb{R}\times\mathbb{R}$ and $\mathbb{T}_1\times\mathbb{T}_2=\mathbb{Z}\times\mathbb{Z}$, the delta shape operator can be found to be the zero matrix, *i.e.* $S_\Delta=[0]$. The geometric interpretation shows that the shape operator of a plane which has a constant normal vector should be zero for both the discrete and the indiscrete cases.

Now, in the following example we will examine the delta shape operator of any discrete asymptotic weak Chebyshev net which is an example in [9].

Example 3 The discrete surfaces are defined as maps $\mathbf{r}:\varepsilon_1\mathbb{Z}\times\varepsilon_2\mathbb{Z}\to\mathbb{R}^3$ such that $\Delta_1\mathbf{r}$ and $\Delta_2\mathbf{r}$ are linearly independent. A discrete asymptotic weak Chebyshev net (discrete K-surface) is an immersion $\mathbf{r}$ such that

$$\Delta_1\mathbf{r}\cdot\Delta_1\mathbf{n}=\Delta_2\mathbf{r}\cdot\Delta_2\mathbf{n}=0, \tag{4.3}$$

$$\Delta_1\mathbf{r}\cdot\Delta_1\mathbf{r}=E,\qquad \Delta_2\mathbf{r}\cdot\Delta_2\mathbf{r}=G,\qquad \Delta_1\mathbf{r}\cdot\Delta_2\mathbf{r}=F \tag{4.4}$$

correspond to $E=G=1$. Also, for any discrete asymptotic weak Chebyshev net, the Gaussian curvature

$$K=-\frac{(\Delta_1\mathbf{n}\cdot\Delta_2\mathbf{r})(\Delta_2\mathbf{n}\cdot\Delta_1\mathbf{r})}{(\Delta_1\mathbf{r})^2\cdot(\Delta_2\mathbf{r})^2-(\Delta_1\mathbf{r}\Delta_2\mathbf{r})^2}$$

is constant [9].

In this example, we will try to calculate the delta shape operator of the discrete asymptotic weak Chebyshev net. If we take the partial derivatives as $\frac{\partial\varphi}{\Delta_1 t}=\Delta_1\mathbf{r}$ and $\frac{\partial\varphi}{\Delta_2 s}=\Delta_2\mathbf{r}$ in the delta shape operator matrix, the delta shape operator matrix of discrete surface will be

$$S_\Delta=\begin{bmatrix}\frac{-\det(\Delta_{11}\mathbf{r}^{\sigma_1},\Delta_1\mathbf{r},\Delta_2\mathbf{r})}{\|\Delta_1\mathbf{r}\|^3\|\Delta_2\mathbf{r}\|} & \frac{-\det(\Delta_{21}\mathbf{r}^{\sigma_2},\Delta_1\mathbf{r},\Delta_2\mathbf{r})}{\|\Delta_1\mathbf{r}\|^2\|\Delta_2\mathbf{r}\|^2}\\ \frac{-\det(\Delta_{12}\mathbf{r}^{\sigma_1},\Delta_1\mathbf{r},\Delta_2\mathbf{r})}{\|\Delta_1\mathbf{r}\|^2\|\Delta_2\mathbf{r}\|^2} & \frac{-\det(\Delta_{22}\mathbf{r}^{\sigma_2},\Delta_1\mathbf{r},\Delta_2\mathbf{r})}{\|\Delta_1\mathbf{r}\|\|\Delta_2\mathbf{r}\|^3}\end{bmatrix}.$$

The delta shape operator of the discrete asymptotic weak Chebyshev net

$$S_\Delta=\begin{bmatrix}0 & \frac{-\det(\Delta_{21}\mathbf{r}^{\sigma_2},\Delta_1\mathbf{r},\Delta_2\mathbf{r})}{\|\Delta_1\mathbf{r}\|^2\|\Delta_2\mathbf{r}\|^2}\\ \frac{-\det(\Delta_{12}\mathbf{r}^{\sigma_1},\Delta_1\mathbf{r},\Delta_2\mathbf{r})}{\|\Delta_1\mathbf{r}\|^2\|\Delta_2\mathbf{r}\|^2} & 0\end{bmatrix}$$

is obtained using equations (4.3) and (4.4). In other words the delta shape operator of the discrete asymptotic weak Chebyshev net can be written as

$$S_\Delta=\begin{bmatrix}\langle S(V_1),V_1\rangle & \langle S(V_2),V_1\rangle\\ \langle S(V_1),V_2\rangle & \langle S(V_2),V_2\rangle\end{bmatrix}=\frac{1}{\|\Delta_1\mathbf{r}\|\|\Delta_2\mathbf{r}\|}\begin{bmatrix}0 & \Delta_2\mathbf{n}\cdot\Delta_1\mathbf{r}\\ \Delta_1\mathbf{n}\cdot\Delta_2\mathbf{r} & 0\end{bmatrix}$$

from the definition of the matrix coefficients c_1, c_2, c_3, c_4. Here, we know that

$$\|\Delta_1\mathbf{r}\|\|\Delta_2\mathbf{r}\|=\|\Delta_1\mathbf{r}\times\Delta_2\mathbf{r}\|^2=EG-F^2=(\Delta_1\mathbf{r})^2(\Delta_2\mathbf{r})^2-\Delta_1\mathbf{r}\Delta_2\mathbf{r}.$$

Thus we can obtain another representation of the delta shape operator for the discrete asymptotic weak Chebyshev net:

$$\frac{1}{(\Delta_1\mathbf{r})^2(\Delta_2\mathbf{r})^2 - \Delta_1\mathbf{r}\Delta_2\mathbf{r}}\begin{bmatrix} 0 & \Delta_2\mathbf{n}\cdot\Delta_1\mathbf{r} \\ \Delta_1\mathbf{n}\cdot\Delta_2\mathbf{r} & 0 \end{bmatrix}.$$

The Gaussian curvature is equal to the determinant of the shape operator, *i.e.*

$$K = \det S_\Delta = \frac{-(\Delta_1\mathbf{n}\cdot\Delta_2\mathbf{r})(\Delta_2\mathbf{n}\cdot\Delta_1\mathbf{r})}{(\Delta_1\mathbf{r})^2(\Delta_2\mathbf{r})^2 - \Delta_1\mathbf{r}\Delta_2\mathbf{r}},$$

as can be found similarly in [9].

5 Conclusion

In this paper we obtained the matrix representation of the shape operator on time scales. The advantage is the fact that it is an easier and smoother procedure to use the shape operator in discrete differential geometry and the time scale analysis. Therefore it is possible to use a unique equation of the shape operator for both discrete and continuous geometry. We hope that our study will be useful for the literature of the geometry on time scales.

Competing interests
The author declares that they have no competing interests.

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