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Two generalized q-exponential operators and their applications

Nadia Na Li[*] and Wei Tan

[*]Correspondence:
lina3718@163.com
School of Mathematics and
Statistics, Zhoukou Normal
University, Zhoukou, 466000,
P.R. China

Abstract

In this paper, we construct two generalized q-exponential operators with three parameters and obtain some operator identities. As applications, we give two formal extensions of the q-Gauss sum. An extension of the q-Chu-Vandermonde sums is also obtained by the operator technique. In addition, we also deduce a formal extension of the Askey-Wilson integral and a formal extension of Sears' two-term summation formula. Meanwhile, some curious q-series identities are derived.

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1 Introduction and motivation

Following [1] we define the q-shifted factorial by

$$(a;q)_0 = 1, \qquad (a;q)_n = \prod_{k=0}^{n-1}\left(1 - aq^k\right), \qquad (a;q)_\infty = \prod_{k=0}^{\infty}\left(1 - aq^k\right),$$

and $(a_1, a_2, \ldots, a_m; q)_n = (a_1; q)_n (a_2; q)_n \cdots (a_m; q)_n$, where m is a positive integer and n is a nonnegative integer or ∞. For a complex number α, we shall also use the notation

$$(a;q)_\alpha = (a;q)_\infty / \left(q^\alpha a; q\right)_\infty.$$

The q-binomial coefficients, or the Gauss coefficients, are given by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}.$$

The basic hypergeometric series ${}_r\phi_s$ is defined by

$$ {}_r\phi_s \begin{bmatrix} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_s \end{bmatrix} q; z \end{bmatrix} = \sum_{k=0}^{\infty} \frac{(a_1, a_2, \ldots, a_r; q)_k}{(q, b_1, b_2, \ldots, b_s; q)_k} \left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} z^k.$$

The usual q-differential operator, or q-derivative are defined by

$$D_x\{f(x)\} = \frac{f(x) - f(qx)}{x}, \qquad \theta_x\{f(x)\} = \frac{f(q^{-1}x) - f(x)}{q^{-1}x},$$

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and their Leibniz rules are

$$D_x^n\{f(x)g(x)\} = \sum_{k=0}^{n} q^{k(k-n)} \begin{bmatrix} n \\ k \end{bmatrix} D_x^k\{f(x)\} D_x^{n-k}\{g(xq^k)\}, \tag{1}$$

$$\theta_x^n\{f(x)g(x)\} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \theta_x^k\{f(x)\} \theta_x^{n-k}\{g(xq^{-k})\}, \tag{2}$$

respectively. Based on this, Chen and Liu [2, 3] constructed the following augmentation operator, which is of great significance for deriving identities from its special cases (*cf.* [4–7]):

$$\mathbb{T}(bD_x) = \sum_{n=0}^{\infty} \frac{(bD_x)^n}{(q;q)_n},$$

$$\mathbb{E}(bD_x) = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}(b\theta_x)^n}{(q;q)_n}.$$

Later, Chen and Gu [8] defined the Cauchy augmentation operator,

$$\mathbb{T}(a, bD_x) = \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} (bD_x)^n,$$

$$\mathbb{E}(a, -bD_x) = \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} (-b\theta_x)^n.$$

Fang [9] and Zhang and Yang [10] considered the following finite generalized q-exponential operators with two parameters:

$$\mathbb{T} \begin{bmatrix} q^{-N}, w \\ v \end{bmatrix} q; tD_x \end{bmatrix} = \sum_{n=0}^{N} \frac{(q^{-N}, w; q)_n}{(q, v; q)_n} (tD_x)^n,$$

$$\mathbb{E} \begin{bmatrix} q^{-N}, w \\ v \end{bmatrix} q; t\theta_x \end{bmatrix} = \sum_{n=0}^{N} \frac{(q^{-N}, w; q)_n}{(q, v; q)_n} (t\theta_x)^n.$$

In resent years, some authors applied the aforementioned q-exponential operators to give a system way of obtaining q-series formulas (*cf.* [9–15]). Inspired by their work, in this paper, we construct two generalized q-exponential operator with three parameters and give some operator identities to further investigate the applications of the operator technique.

Our paper is organized as follows. The next section is devoted to the construction of two generalized q-exponential operator with three parameters and we establish some operator identities. The rest of this paper is mainly concerned with applications of these operator identities. In detail, we will give two formal extensions of the q-Gauss sum. An extension of the q-Chu-Vandermonde sums are also obtained by the operator technique. In addition, we also deduce a formal extension of the Askey-Wilson integral and a formal extension of Sears' two-term summation formula. Meanwhile, some curious q-series identities are derived.

2 Two generalized q-exponential operators and their operator identities

In this section, we first of all establish the following two generalized q-exponential operators with three parameters:

$$\mathbb{T}\begin{bmatrix} u, v \\ w \end{bmatrix}\ q; tD_x \end{bmatrix} = \sum_{n=0}^{\infty} \frac{(u, v; q)_n}{(q, w; q)_n} (tD_x)^n, \tag{3}$$

$$\mathbb{E}\begin{bmatrix} u, v \\ w \end{bmatrix}\ q; t\theta_x \end{bmatrix} = \sum_{n=0}^{\infty} \frac{(u, v; q)_n}{(q, w; q)_n} (t\theta_x)^n. \tag{4}$$

Clearly, Zhang and Yang's [10] finite q-exponential operators with two parameters can be considered as special cases of the operator (3) and (4) for $u = q^{-N}$ or $v = q^{-N}$. Based on the definitions for D_x and θ_x and their Leibniz rules (1) and (2), we can easily verify the following explicit formulas by induction.

Lemma 1 *For a nonnegative integer k, we have*

$$\theta_x^k\big\{(xt; q)_\infty\big\} = (-t)^k (xt; q)_\infty,$$

$$D_x^k\left\{\frac{1}{(xt; q)_\infty}\right\} = \frac{t^k}{(xt; q)_\infty},$$

$$\theta_x^k\left\{\frac{(xv; q)_\infty}{(xt; q)_\infty}\right\} = t^k q^{-\binom{k}{2}} (v/t; q)_k \frac{(xv; q)_\infty}{(xtq^{-k}; q)_\infty},$$

$$D_x^k\left\{\frac{(xv; q)_\infty}{(xt; q)_\infty}\right\} = t^k (v/t; q)_k \frac{(xvq^k; q)_\infty}{(xt; q)_\infty},$$

$$\theta_x^k\big\{x^n\big\} = \begin{cases} (-1)^k x^{n-k} q^k (q^{-n}; q)_k & \text{if } 0 \le k \le n, \\ 0 & \text{if } k > n, \end{cases}$$

$$D_x^k\big\{x^n\big\} = \begin{cases} x^{n-k} (q^{n-k+1}; q)_k & \text{if } 0 \le k \le n, \\ 0 & \text{if } k > n. \end{cases}$$

Theorem 2

$$\mathbb{T}\begin{bmatrix} u, v \\ w \end{bmatrix}\ q; tD_x \end{bmatrix} \left\{\frac{(xa; q)_\infty}{(xb, xc; q)_\infty}\right\}$$

$$= \frac{(xa; q)_\infty}{(xb, xc; q)_\infty} \sum_{n,k \ge 0} \frac{(u, v; q)_{n+k}}{(q; q)_n (w; q)_{n+k}} \frac{(a/b, xc; q)_k}{(q, xa; q)_k} (bt)^k (tc)^n.$$

Proof By means of the definition (3) of $\mathbb{T}\begin{bmatrix} u,v \\ w \end{bmatrix} q; tD_x \end{bmatrix}$ operator and the Leibniz rule (1) of the q-derivative operator, we have

$$\mathbb{T}\begin{bmatrix} u, v \\ w \end{bmatrix}\ q; tD_x \end{bmatrix} \left\{\frac{(xa; q)_\infty}{(xb, xc; q)_\infty}\right\}$$

$$= \sum_{n=0}^{\infty} \frac{(u, v; q)_n t^n}{(q, w; q)_n} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k-n)} D_x^k\left\{\frac{(xa; q)_\infty}{(xb; q)_\infty}\right\} D_x^{n-k}\left\{\frac{1}{(q^k xc; q)_\infty}\right\}.$$

In view of Lemma 1, we can evaluate the right-hand side of the last expression in the following form:

$$\sum_{n=0}^{\infty} \frac{(u,v;q)_n t^n}{(q,w;q)_n} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k-n)} b^k (a/b;q)_k \frac{(q^k xa;q)_\infty}{(xb;q)_\infty} \frac{(q^k c)^{n-k}}{(q^k xc;q)_\infty}$$

$$= \frac{(xa;q)_\infty}{(xb,xc;q)_\infty} \sum_{n=0}^{\infty} \frac{(u,v;q)_n (ct)^n}{(q,w;q)_n} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(a/b,xc;q)_k}{(xa;q)_k} (b/c)^k.$$

Interchanging the summation order and changing by $n \to n + k$ on the inner summation index, we get the desired result. □

We remark that when $u = q^{-N}$ and $t = q/c$, the last theorem reduces to Zhang and Yang's conclusion [10]. Similar to the procedure of proving Theorem 2, we establish the following operator identity, which leads to Zhang and Yang's conclusion [10] when $u = q^{-N}$ or $v = q^{-N}$.

Theorem 3

$$\mathbb{E}\begin{bmatrix} u,v \\ w \end{bmatrix} q; t\theta_x \end{bmatrix} \left\{ \frac{(xa,xc;q)_\infty}{(xb;q)_\infty} \right\}$$

$$= \frac{(xa,xc;q)_\infty}{(xb;q)_\infty} \sum_{n,k \geq 0} \frac{(u,v;q)_{n+k}}{(w;q)_{n+k}(q;q)_n} \frac{(a/b,q/xc;q)_k}{(q,q/xb;q)_k} (-1)^n (tc)^{n+k} q^{-\binom{k}{2}-nk}.$$

Letting $c = 0$, the last sum with respect to n in Theorem 2 and Theorem 3 vanishes because 0^n is equal to zero when $n > 0$, therefore we get the following identities, respectively.

Corollary 4

$$\mathbb{T}\begin{bmatrix} u,v \\ w \end{bmatrix} q; tD_x \end{bmatrix} \left\{ \frac{(xa;q)_\infty}{(xb;q)_\infty} \right\} = \frac{(xa;q)_\infty}{(xb;q)_\infty} {}_3\phi_2 \begin{bmatrix} u,v,a/b \\ w,xa \end{bmatrix} q; tb \end{bmatrix}, \tag{5}$$

$$\mathbb{E}\begin{bmatrix} u,v \\ w \end{bmatrix} q; t\theta_x \end{bmatrix} \left\{ \frac{(xa;q)_\infty}{(xb;q)_\infty} \right\} = \frac{(xa;q)_\infty}{(xb;q)_\infty} {}_3\phi_2 \begin{bmatrix} u,v,a/b \\ w,q/xb \end{bmatrix} q; -qt/x \end{bmatrix}. \tag{6}$$

For the symmetry of b and c on both sides of Theorem 2, we have

$$\sum_{n,k \geq 0} \frac{(u,v;q)_{n+k}(a/b,xc;q)_k}{(q;q)_n(w;q)_{n+k}(q,xa;q)_k} (bt)^k (tc)^n = \sum_{n,k \geq 0} \frac{(u,v;q)_{n+k}(a/c,xb;q)_k}{(q;q)_n(w;q)_{n+k}(q,xa;q)_k} (ct)^k (tb)^n.$$

Similarly, by means of the symmetry of a and c, we get from Theorem 3

$$\sum_{n,k \geq 0} \frac{(u,v;q)_{n+k}}{(w;q)_{n+k}(q;q)_n} (tc)^{n+k} \frac{(a/b,q/xc;q)_k}{(q,q/xb;q)_k} (-1)^n q^{-\binom{k}{2}-nk}$$

$$= \sum_{n,k \geq 0} \frac{(u,v;q)_{n+k}}{(w;q)_{n+k}(q;q)_n} (ta)^{n+k} \frac{(b/c,q/xa;q)_k}{(q,q/xb;q)_k} (-1)^n q^{-\binom{k}{2}-nk}.$$

Letting $n + k = m$, and extracting the coefficients of $\frac{(u,v;q)_m}{(w;q)_m}t^m$ from both members of the last two identities, respectively, we get the following curious transformation formulas.

Example 1

$$
{}_3\phi_1\left[\begin{array}{c} q^{-m}, a/b, xc \\ xa \end{array}\,\middle|\, q; q^m b/c\right] = (b/c)^m\,{}_3\phi_1\left[\begin{array}{c} q^{-m}, a/c, xb \\ xa \end{array}\,\middle|\, q; q^m c/b\right],
$$

$$
{}_3\phi_2\left[\begin{array}{c} q^{-m}, a/b, q/xc \\ 0, q/xb \end{array}\,\middle|\, q; q\right] = (a/c)^m\,{}_3\phi_2\left[\begin{array}{c} q^{-m}, c/b, q/xa \\ 0, q/xb \end{array}\,\middle|\, q; q\right].
$$

Letting $t = -w/uvc$, the sum with respect to n in Theorem 3 reduces to $\frac{(w/u,w/v;q)_\infty}{(q^k w, q^{-k} w/uv;q)_\infty}$ by means of the q-Gauss sum

$$
{}_2\phi_1\left[\begin{array}{c} a, b \\ c \end{array}\,\middle|\, q; c/ab\right] = \frac{(c/a, c/b;q)_\infty}{(c, c/ab;q)_\infty}, \tag{7}
$$

which results consequently in the following operator identity.

Corollary 5

$$
\mathbb{E}\left[\begin{array}{c} u, v \\ w \end{array}\,\middle|\, q;\frac{-w}{uvc}\theta_x\right]\left\{\frac{(xa, xc;q)_\infty}{(xb;q)_\infty}\right\}
$$

$$
= \frac{(xa, xc, w/u, w/b;q)_\infty}{(xb, w, w/uv;q)_\infty}\,{}_4\phi_3\left[\begin{array}{c} a/b, q/xc, u, v \\ 0, q/xb, quv/w \end{array}\,\middle|\, q; q\right].
$$

Recalling the q-Chu-Vandermonde sums [1], II.6,

$$
{}_2\phi_1\left[\begin{array}{c} q^{-n}, x \\ y \end{array}\,\middle|\, q; q\right] = x^n\frac{(y/x;q)_n}{(y;q)_n}, \tag{8}
$$

and substituting $x \to xa$ into the above equation, we derive

$$
\sum_{k=0}^{n}\frac{(q^{-n};q)_k q^k}{(q, y;q)_k}\frac{(xc;q)_\infty}{(xaq^k, xb;q)_\infty} = \frac{(xa)^n (y/xa;q)_n (xc;q)_\infty}{(y;q)_n (xa, xb;q)_\infty}. \tag{9}
$$

Then applying the operator $\mathbb{T}\left[\begin{smallmatrix} u, v \\ w \end{smallmatrix}\,\middle|\, q; tD_x\right]$ to both sides of (9) with respect to the variable x, it is trivial to see that

$$
\sum_{k=0}^{n}\frac{(q^{-n};q)_k q^k}{(q, y;q)_k}\mathbb{T}\left[\begin{array}{c} u, v \\ w \end{array}\,\middle|\, q; tD_x\right]\left\{\frac{(xc;q)_\infty}{(xaq^k, xb;q)_\infty}\right\}
$$

$$
= \frac{a^n}{(y;q)_n}\mathbb{T}\left[\begin{array}{c} u, v \\ w \end{array}\,\middle|\, q; tD_x\right]\left\{\frac{x^n(y/xa;q)_n(xc;q)_\infty}{(xa, xb;q)_\infty}\right\}.
$$

Evaluating the left-hand side of the last expression in terms of Theorem 2, we show from the last equation the following identity.

Theorem 6 *For a nonnegative integer n, we have*

$$
\mathbb{T}\begin{bmatrix} u,v \\ w \end{bmatrix} q;tD_x \left\{ \frac{P_n(x,y/a)(xc;q)_\infty}{(xa,xb;q)_\infty} \right\}
$$

$$
= \frac{(y;q)_n}{a^n} \frac{(xc;q)_\infty}{(xa,xb;q)_\infty} \sum_{k=0}^n \frac{(q^{-n},xa;q)_k q^k}{(q,y;q)_k} \sum_{i,j\geq 0} \frac{(u,v;q)_{i+j}}{(q;q)_i(w;q)_{i+j}} \frac{(c/b,q^k xa;q)_j}{(q,xc;q)_j}(tb)^j\big(q^k ta\big)^i,
$$

where $P_n(a,b) = a^n(b/a;q)_n$.

When $y = 0$, Theorem 6 reduces to the following corollary.

Corollary 7 *For a nonnegative integer n, we have*

$$
\mathbb{T}\begin{bmatrix} u,v \\ w \end{bmatrix} q;tD_x \left\{ \frac{x^n(xc;q)_\infty}{(xa,xb;q)_\infty} \right\}
$$

$$
= \frac{1}{a^n} \frac{(xc;q)_\infty}{(xa,xb;q)_\infty} \sum_{k=0}^n \frac{(q^{-n},xa;q)_k}{(q;q)_k} q^k \sum_{i,j\geq 0} \frac{(u,v;q)_{i+j}}{(q;q)_i(w;q)_{i+j}} \frac{(c/b,q^k xa;q)_j}{(q,xc;q)_j}(tb)^j\big(q^k ta\big)^i.
$$

Analogously, substituting $x \to xa$ into (7), we can reformulate it as

$$
\sum_{k=0}^n \frac{(q^{-n};q)_k q^k}{(q,y;q)_k} \frac{(xb,xc;q)_\infty}{(xaq^k;q)_\infty} = \frac{(xa)^n(y/xa;q)_n(xb,xc;q)_\infty}{(y;q)_n(xa;q)_\infty}.
$$

Applying the operator $\mathbb{E}\big[\begin{smallmatrix} u,v \\ w \end{smallmatrix} \mid q;t\theta_x\big]$ to both sides of the last expression with respect to the variable x, we confirm that

$$
\sum_{k=0}^n \frac{(q^{-n};q)_k q^k}{(q,y;q)_k} \mathbb{E}\begin{bmatrix} u,v \\ w \end{bmatrix} q;t\theta_x \left\{ \frac{(xb,xc;q)_\infty}{(xaq^k;q)_\infty} \right\}
$$

$$
= \frac{a^n}{(y;q)_n} \mathbb{E}\begin{bmatrix} u,v \\ w \end{bmatrix} q;t\theta_x \left\{ \frac{x^n(y/xa;q)_n(xb,xc;q)_\infty}{(xa;q)_\infty} \right\}.
$$

Rewriting the left-hand side of the last expression by means of Theorem 3, we obtain the following identity.

Theorem 8 *For a nonnegative integer n, we have*

$$
\mathbb{E}\begin{bmatrix} u,v \\ w \end{bmatrix} q;t\theta_x \left\{ \frac{P_n(x,y/a)(xb,xc;q)_\infty}{(xa;q)_\infty} \right\}
$$

$$
= \frac{(y;q)_n}{a^n} \frac{(xc,xb;q)_\infty}{(xa;q)_\infty} \sum_{k=0}^n \frac{(q^{-n},xa;q)_k q^k}{(q,y;q)_k}
$$

$$
\times \sum_{i,j\geq 0} \frac{(u,v;q)_{i+j}}{(q;q)_i(w;q)_{i+j}}(bt)^{i+j} \frac{(cq^{-k}/a,q/xb;q)_j}{(q,q^{1-k}/xa;q)_j}(-1)^i q^{-\binom{j}{2}-ij},
$$

where $P_n(a,b) = a^n(b/a;q)_n$.

When $b = q/x$, the inner sum with respect to j vanishes, and then changing the consequent summation order, in view of the q-Chu-Vandermonde sum (8), we get from Theorem 8

$$\mathbb{E}\left[\begin{matrix} u, v \\ w \end{matrix} \,\middle|\, q; t\theta_x\right]\left\{\frac{P_n(x, y/a)(xc; q)_\infty}{(xa; q)_\infty}\right\}$$
$$= \frac{x^n(y/xa; q)_n(xc; q)_\infty}{(xa; q)_\infty}\,{}_2\phi_1\left[\begin{matrix} u, v \\ w \end{matrix} \,\middle|\, q; -qt/x\right].$$

For $t = -wx/uvq$, the right-hand side of the last identity can be evaluated by (7). We therefore establish the following operator identity.

Corollary 9

$$\mathbb{E}\left[\begin{matrix} u, v \\ w \end{matrix} \,\middle|\, q; -\frac{wx}{uvq}\theta_x\right]\left\{\frac{P_n(x, y/a)(xc; q)_\infty}{(xa; q)_\infty}\right\} = \frac{x^n(y/xa; q)_n(xc, w/u, w/v; q)_\infty}{(xa, w, w/uv; q)_\infty}.$$

3 Some applications

Throughout this section, we address the applications of the theorems and corollaries obtained in the previous section.

3.1 Two formal generalizations of the q-Gauss sum

Recalling the q-binomial theorem [1], II.3,

$$\sum_{n=0}^{\infty}\frac{(a; q)_n}{(q; q)_n}x^n = \frac{(xa; q)_\infty}{(x; q)_\infty}, \tag{10}$$

multiplying $\frac{(xc; q)_\infty}{(xb; q)_\infty}$ to both sides of the last identity, and applying $\mathbb{T}\left[\begin{smallmatrix} u, v \\ w \end{smallmatrix} \,\middle|\, q; tD_x\right]$ to both sides of the consequent identity with respect to the parameter x, we have

$$\sum_{n=0}^{\infty}\frac{(a; q)_n}{(q; q)_n}\mathbb{T}\left[\begin{matrix} u, v \\ w \end{matrix} \,\middle|\, q; tD_x\right]\left\{\frac{x^n(xc; q)_\infty}{(xa, xb; q)_\infty}\right\} = \mathbb{T}\left[\begin{matrix} u, v \\ w \end{matrix} \,\middle|\, q; tD_x\right]\left\{\frac{(xc; q)_\infty}{(x, xb; q)_\infty}\right\}.$$

By means of Theorem 2 and Corollary 7, we have the following theorem.

Theorem 10

$$\sum_{n=0}^{\infty}\frac{(a; q)_n}{(q; q)_n}a^{-n}\sum_{k=0}^{n}\frac{(q^{-n}, xa; q)_k}{(q; q)_k}q^k\sum_{i,j\geq 0}\frac{(u, v; q)_{i+j}}{(q; q)_i(w; q)_{i+j}}\frac{(c/b, q^kxa; q)_j}{(q, xc; q)_j}(bt)^j(q^kta)^i$$
$$= \frac{(xa; q)_\infty}{(x; q)_\infty}\sum_{i,j\geq 0}\frac{(u, v; q)_{i+j}}{(q; q)_i(w; q)_{i+j}}\frac{(c, xb; q)_j}{(q, xc; q)_j}t^j(tb)^i.$$

If $c = 1$ and $t = w/uvb$, the sum with respect to j on the right-hand side of the last theorem vanished and the sum with respect to i can be evaluated by the q-Gauss sum (7). We find consequently the following curious summation formula where the parameter b only exists in the left-hand side.

Corollary 11

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} a^{-n} \sum_{k=0}^{n} \frac{(q^{-n},xa;q)_k}{(q;q)_k} q^k$$

$$\times \sum_{i,j \geq 0} \frac{(u,v;q)_{i+j}}{(q;q)_i (w;q)_{i+j}} \frac{(1/b,q^k xa;q)_j}{(q,x;q)_j} \left(\frac{w}{uv}\right)^j \left(\frac{q^k wa}{uvb}\right)^i$$

$$= \frac{(xa,w/u,w/v;q)_\infty}{(x,w,w/uv;q)_\infty}.$$

Further, if we specify with $a \mapsto 1$ in the last corollary, we find the following curious summation formula, which reduces to the q-Gauss sum (7) when $b = 1$.

Corollary 12

$$\sum_{i,j \geq 0} \frac{(u,v;q)_{i+j}}{(q;q)_i (w;q)_{i+j}} \frac{(1/b;q)_j}{(q;q)_j} \left(\frac{w}{uv}\right)^j \left(\frac{w}{uvb}\right)^i = \frac{(w/u,w/v;q)_\infty}{(w,w/uv;q)_\infty}.$$

Letting $i + j = m$, and then extracting the coefficients of $\frac{(u,v;q)_m}{(w;q)_m} t^m$ from two members of Theorem 10, we get the following result.

Corollary 13

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} a^{m-n} \sum_{k=0}^{n} \frac{(q^{-n},xb;q)_k}{(q;q)_k} q^{(1+m)k}$$

$$\times \sum_{j=0}^{m} \frac{(q^{-m},c/b,q^k xa;q)_j}{(q,xc;q)_j} (-1)^j q^{-\binom{j}{2}} \left(bq^{m-k}/a\right)^j$$

$$= b^m \frac{(xa;q)_\infty}{(x;q)_\infty} {}_3\phi_1 \left[\begin{matrix} q^{-m},c,xb \\ xc \end{matrix} \middle| q; q^m/b \right].$$

When $a = 1$, the last sums with respect to n and k vanish because of the q-shift factorial $(1;q)_n$ is equal to zero when $n > 0$. Then we get the following transformation formula.

Example 2

$$_3\phi_1 \left[\begin{matrix} q^{-m},c/b,x \\ xc \end{matrix} \middle| q; q^m b \right] = b^m {}_3\phi_1 \left[\begin{matrix} q^{-m},c,xb \\ xc \end{matrix} \middle| q; q^m/b \right].$$

Similarly, multiplying $\frac{(xb;q)_\infty}{(xc;q)_\infty}$ to both sides of equation (10) and applying $\mathbb{E}\left[\begin{matrix} u,v \\ w \end{matrix} \middle| q; t\theta_x \right]$ to both sides of the consequent identity with respect to the parameter x, we have

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} \mathbb{E}\left[\begin{matrix} u,v \\ w \end{matrix} \middle| q; t\theta_x \right] \left\{\frac{x^n (x,xb;q)_\infty}{(xc;q)_\infty}\right\}$$

$$= \mathbb{E}\left[\begin{matrix} u,v \\ w \end{matrix} \middle| q; t\theta_x \right] \left\{\frac{(xa,xb;q)_\infty}{(xc;q)_\infty}\right\}. \tag{11}$$

On the one hand, letting $y = 0$ and substituting with $a \to c$, $c \to 1$ in Theorem 8, we have

$$
\mathbb{E}\begin{bmatrix} u, v \\ w \end{bmatrix} q; t\theta_x \Bigg] \left\{ \frac{x^n (x, xb; q)_\infty}{(xc; q)_\infty} \right\}
$$

$$
= \frac{1}{c^n} \frac{(xb, x; q)_\infty}{(xc; q)_\infty} \sum_{k=0}^{n} \frac{(q^{-n}, xc; q)_k q^k}{(q; q)_k}
$$

$$
\times \sum_{i,j \geq 0} \frac{(u, v; q)_{i+j}}{(q; q)_i (w; q)_{i+j}} (bt)^{i+j} \frac{(q^{-k}/c, q/xb; q)_j}{(q, q^{1-k}/xc; q)_j} (-1)^i q^{-\binom{j}{2} - ij}.
$$

On the other hand, in view of Theorem 3, we get

$$
\mathbb{E}\begin{bmatrix} u, v \\ w \end{bmatrix} q; t\theta_x \Bigg] \left\{ \frac{(xa, xb; q)_\infty}{(xc; q)_\infty} \right\}
$$

$$
= \frac{(xa, xb; q)_\infty}{(xc; q)_\infty} \sum_{n,k \geq 0} \frac{(u, v; q)_{n+k}}{(w; q)_{n+k} (q; q)_n} (tb)^{n+k} \frac{(a/c, q/xb; q)_k}{(q, q/xc; q)_k} (-1)^n q^{-\binom{k}{2} - nk}.
$$

Combining the last two expressions with (11), we get the following theorem.

Theorem 14

$$
\sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} \frac{1}{c^n} \sum_{k=0}^{n} \frac{(q^{-n}, xc; q)_k q^k}{(q; q)_k} \sum_{i,j \geq 0} \frac{(u, v; q)_{i+j} (bt)^{i+j}}{(q; q)_i (w; q)_{i+j}} \frac{(q^{-k}/c, q/xb; q)_j}{(q, q^{1-k}/xc; q)_j} (-1)^i q^{-\binom{j}{2} - ij}
$$

$$
= \frac{(xa; q)_\infty}{(x; q)_\infty} \sum_{n,k \geq 0} \frac{(u, v; q)_{n+k}}{(w; q)_{n+k} (q; q)_n} (tb)^{n+k} \frac{(a/c, q/xb; q)_k}{(q, q/xc; q)_k} (-1)^n q^{-\binom{k}{2} - nk}.
$$

If $a = c$, the sum with respect to k on the right-hand side of the above theorem vanishes. Further setting $t = -w/uvb$ in the consequent expression, we get the following summation formula, which returns to the q-Gauss sum (7) when $a = 1$.

Corollary 15

$$
\sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} \frac{1}{a^n} \sum_{k=0}^{n} \frac{(q^{-n}, xa; q)_k q^k}{(q; q)_k} \sum_{i,j \geq 0} \frac{(u, v; q)_{i+j}}{(q; q)_i (w; q)_{i+j}} \frac{(q^{-k}/a, q/xb; q)_j}{(q, q^{1-k}/xa; q)_j}
$$

$$
\times (-w/uv)^{i+j} (-1)^i q^{-\binom{j}{2} - ij}
$$

$$
= \frac{(xa, w/u, w/v; q)_\infty}{(x, w, w/uv; q)_\infty}.
$$

3.2 A formal generalization of q-exponential function
Theorem 16

$$
\sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} x^n}{(q; q)_n} \sum_{k=0}^{n} \frac{(q^{-n}, u, v; q)_k}{(q, w; q)_k} \left(-\frac{qw}{xuv} \right)^k = \frac{(-x, w/u, w/v; q)_\infty}{(w, w/uv; q)_\infty}.
$$

Proof Recalling the q-exponential function [1], II.2

$$
\sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}}{(q; q)_n} x^n = (-x; q)_\infty
$$

and then applying $\mathbb{E}\begin{bmatrix} u,v \\ w \end{bmatrix} q; \frac{w}{uv}\theta_x \end{bmatrix}$ to the both sides of the last identity, we have

$$\sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}}{(q;q)_n} \mathbb{E}\begin{bmatrix} u,v \\ w \end{bmatrix} q; \frac{w}{uv}\theta_x \end{bmatrix}\{x^n\} = \mathbb{E}\begin{bmatrix} u,v \\ w \end{bmatrix} q; \frac{w}{uv}\theta_x \end{bmatrix}\{(-x;q)_\infty\}. \tag{12}$$

On the one hand, letting $b = 0$, $a = -1$ and $t = w/uv$ in (6), we can write the right-hand side of (12) as

$$\mathbb{E}\begin{bmatrix} u,v \\ w \end{bmatrix} q; \frac{w}{uv}\theta_x \end{bmatrix}\{(-x;q)_\infty\} = \frac{(-x, w/u, w/v; q)_\infty}{(w, w/uv; q)_\infty}$$

by means of the q-Gauss sum (7).

On the other hand, in view of the definition (4) and the equation of $\theta_x^k\{x^n\}$ obtained in Lemma 1, we can easily verify that

$$\mathbb{E}\begin{bmatrix} u,v \\ w \end{bmatrix} q; w/uv\theta_x \end{bmatrix}\{x^n\} = x^n {}_3\phi_2\begin{bmatrix} q^{-n}, u, v \\ w, 0 \end{bmatrix} q; -qw/uvx \end{bmatrix}.$$

Inserting the last two expressions into (12), we obtain the desired result which reduces to the q-exponential function when $u = 1$ or $v = 1$. $\qquad\square$

3.3 A generalization of q-Chu-Vandermonde sums

In view of the fact that

$$(a;q)_n = \frac{(a;q)_\infty}{(q^n a;q)_\infty}$$

and

$$(q/a;q)_n = (-a)^{-n} q^{\binom{n+1}{2}} \frac{(q^{-n}a;q)_\infty}{(a;q)_\infty},$$

we can reformulate the q-Chu-Vandermonde sums (8) as

$$\sum_{k=0}^{n} \frac{(q^{-n};q)_k q^k}{(q,c;q)_k} \frac{1}{(xq^k;q)_\infty} = \frac{(-1)^n c^n q^{\binom{n}{2}}}{(c;q)_n} \frac{(q^{1-n}x/c;q)_\infty}{(x, qx/c)_\infty}.$$

Applying $\mathbb{T}\begin{bmatrix} u,v \\ w \end{bmatrix} q; tD_x \end{bmatrix}$ to both sides of the last expression, we have

$$\sum_{k=0}^{n} \frac{(q^{-n};q)_k q^k}{(q,c;q)_k} \mathbb{T}\begin{bmatrix} u,v \\ w \end{bmatrix} q; tD_x \end{bmatrix}\left\{ \frac{1}{(xq^k;q)_\infty} \right\}$$

$$= \frac{(-1)^n c^n q^{\binom{n}{2}}}{(c;q)_n} \mathbb{T}\begin{bmatrix} u,v \\ w \end{bmatrix} q; tD_x \end{bmatrix}\left\{ \frac{(q^{1-n}x/c;q)_\infty}{(x, qx/c)_\infty} \right\}.$$

On the one hand, specifying with $a = 0$ and $b = q^k$, we can restate (5) as

$$\mathbb{T}\begin{bmatrix} u,v \\ w \end{bmatrix} q; tD_x \end{bmatrix}\left\{ \frac{1}{(xq^k;q)_\infty} \right\} = \frac{1}{(q^k x;q)_\infty} {}_2\phi_1\begin{bmatrix} u,v \\ w \end{bmatrix} q; q^k t \end{bmatrix}.$$

On the other hand, in view of Theorem 2, it is not hard to derive

$$
\mathbb{T}\begin{bmatrix} u, v \\ w \end{bmatrix} q; tD_x \Bigg] \left\{ \frac{(q^{1-n}x/c; q)_\infty}{(x, qx/c)_\infty} \right\}
$$
$$
= \frac{(q^{1-n}x/c; q)_\infty}{(x, qx/c; q)_\infty} \sum_{i,k \geq 0} \frac{(u, v; q)_{i+k}}{(q; q)_i (w; q)_{i+k}} \frac{(q^{1-n}/c, qx/c; q)_k}{(q, q^{1-n}x/c; q)_k} t^{i+k}(q/c)^i.
$$

Combining the last three expression, we get the following result, which is an extension of the q-Chu-Vandermonde sums (8). We remark that it leads to Zhang and Yang's result ([10], Theorem 3.1), when u or v equals q^{-N} and $t = c$.

Theorem 17

$$
\sum_{k=0}^{n} \frac{(q^{-n}, x; q)_k q^k}{(q, c; q)_k} {}_2\phi_1 \begin{bmatrix} u, v \\ w \end{bmatrix} q; q^k t \Bigg]
$$
$$
= \frac{x^n (c/x; q)_n}{(c; q)_n} \sum_{i,k \geq 0} \frac{(u, v; q)_{i+k}}{(q; q)_i (w; q)_{i+k}} \frac{(q^{1-n}/c, qx/c; q)_k}{(q, q^{1-n}x/c; q)_k} t^{i+k}(q/c)^i.
$$

Supposing $n = 0$ and $t = w/uv$, we can reduce the previous theorem as follows.

Corollary 18

$$
\sum_{i,k \geq 0} \frac{(u, v; q)_{i+k}}{(q; q)_i (w; q)_{i+k}} \frac{(q/c; q)_k}{(q; q)_k} (w/uv)^{i+k}(q/c)^i = \frac{(w/u, w/v; q)_\infty}{(w, w/uv; q)_\infty}.
$$

Letting $i + k = m$, and then extracting the coefficients of $\frac{(u,v;q)_m}{(w;q)_m} t^m$ from both members of Theorem 17, we get the following curious transformation formula, which results in the q-Chu-Vandermonde sums (8) when $m = 0$.

Corollary 19

$$
{}_2\phi_1 \begin{bmatrix} q^{-n}, x \\ c \end{bmatrix} q; q^{m+1} \Bigg] = \frac{(c/x; q)_n}{(c; q)_n} x^n (q/c)^m {}_3\phi_1 \begin{bmatrix} q^{-m}, q^{1-n}/c, qx/c \\ q^{1-n}x/c \end{bmatrix} q; q^{m-1}c \Bigg].
$$

3.4 A formal generalization of Sears' formula

Recalling the Sears two-term summation formula [1], Eq. (2.10.18)

$$
\int_c^d \frac{(qt/c, qt/d, abcdet; q)_\infty}{(at, bt, et; q)_\infty} d_q t = \frac{d(1-q)(q, qd/c, c/d, abcd, acde; q)_\infty}{(ac, ad, bc, bd, ce, de; q)_\infty}, \tag{13}
$$

we can restate it as

$$
\int_c^d \frac{(qt/c, qt/d; q)_\infty}{(bt, et; q)_\infty} \frac{(abcdet; q)_\infty}{(at, acde; q)_\infty} d_q t
$$
$$
= \frac{d(1-q)(q, qd/c, c/d; q)_\infty}{(bd, ce, de; q)_\infty} \frac{(abcd; q)_\infty}{(ac, ad; q)_\infty}.
$$

Further applying $\mathbb{T}\begin{bmatrix} u,v \\ w \end{bmatrix}\mid q;sD_x$ to both sides of the above identity with respect to the variable a, we write down immediately that

$$\int_c^d \frac{(qt/c,qt/d;q)_\infty}{(bt,et;q)_\infty}\mathbb{T}\begin{bmatrix} u,v \\ w \end{bmatrix}\bigg| q;sD_x\bigg]\left\{\frac{(abcdet;q)_\infty}{(at,acde;q)_\infty}\right\}d_qt$$

$$= \frac{d(1-q)(q,qd/c,c/d;q)_\infty}{(bd,ce,de;q)_\infty}\mathbb{T}\begin{bmatrix} u,v \\ w \end{bmatrix}\bigg| q;sD_x\bigg]\left\{\frac{(abcd;q)_\infty}{(ac,ad;q)_\infty}\right\}.$$

Applying Theorem 2 to both sides of the last identity, we therefore establish the following transformation formula.

Theorem 20

$$\int_c^d \frac{(qt/c,qt/d,abcdet;q)_\infty}{(at,bt,et;q)_\infty}\sum_{n,k=0}^\infty \frac{(u,v;q)_{n+k}s^{n+k}}{(q;q)_n(w;q)_{n+k}}\frac{(at,bt;q)_k}{(q,abcdet;q)_k}(cde)^k t^n\, d_qt$$

$$= \frac{d(1-q)(q,qd/c,c/d,abcd,acde;q)_\infty}{(ac,ad,bc,bd,ce,de;q)_\infty}\sum_{n,k=0}^\infty \frac{(u,v;q)_{n+k}s^{n+k}}{(q;q)_n(w;q)_{n+k}}\frac{(bd,ad;q)_k}{(q,abcd;q)_k}c^k d^n.$$

Letting $n+k=m$, and then extracting the coefficients of $\frac{(u,v;q)_m}{(w;q)_m}s^m$ from both members of the last theorem, we derive the following result which leads to the Sears two-term summation formula (13) when $m=0$.

Corollary 21

$$\int_c^d \frac{t^m(qt/c,qt/d,abcdet;q)_\infty}{(at,bt,et;q)_\infty}{}_3\phi_1\begin{bmatrix} q^{-m},at,bt \\ abcdet \end{bmatrix}\bigg| q;q^m cde/t\bigg]d_qt$$

$$= \frac{d^{m+1}(1-q)(q,qd/c,c/d,abcd,acde;q)_\infty}{(ac,ad,bc,bd,ce,de;q)_\infty}{}_3\phi_1\begin{bmatrix} q^{-m},ad,bd \\ abcd \end{bmatrix}\bigg| q;q^m c/d\bigg].$$

3.5 A formal generalization of Askey-Wilson integral

Recalling the Askey-Wilson integral [13]

$$\int_0^\pi \frac{h(\cos 2\theta;1)\,d\theta}{h(\cos\theta;b,c,d,x)} = \frac{2\pi}{(q,bc,bd,cd;q)_\infty}\frac{(xbcd;q)_\infty}{(xb,xc,xd;q)_\infty}, \tag{14}$$

we express it as

$$\int_0^\pi \frac{h(\cos 2\theta;1)}{h(\cos\theta;b,c,d)}\frac{(xb;q)_\infty}{(xe^{i\theta},xe^{-i\theta};q)_\infty}\,d\theta = \frac{2\pi}{(q,bc,bd,cd;q)_\infty}\frac{(xbcd;q)_\infty}{(xc,xd;q)_\infty}.$$

Applying $\mathbb{T}\begin{bmatrix} u,v \\ w \end{bmatrix}\mid q;tD_x$ to both sides of the above identity with respect to the variable x, we get

$$\int_0^\pi \frac{h(\cos 2\theta;1)}{h(\cos\theta;b,c,d)}\mathbb{T}\begin{bmatrix} u,v \\ w \end{bmatrix}\bigg| q;tD_x\bigg]\left\{\frac{(xb;q)_\infty}{(xe^{i\theta},xe^{-i\theta};q)_\infty}\right\}d\theta \tag{15}$$

$$= \frac{2\pi}{(q,bc,bd,cd;q)_\infty}\mathbb{T}\begin{bmatrix} u,v \\ w \end{bmatrix}\bigg| q;tD_x\bigg]\left\{\frac{(xbcd;q)_\infty}{(xc,xd;q)_\infty}\right\}. \tag{16}$$

By means of Theorem 2, we get, respectively,

$$\mathbb{T}\begin{bmatrix} u, v \\ w \end{bmatrix} q; tD_x \left\{ \frac{(xb;q)_\infty}{(xe^{i\theta}, xe^{-i\theta};q)_\infty} \right\}$$

$$= \frac{(xb;q)_\infty}{(xe^{i\theta}, xe^{-i\theta};q)_\infty} \sum_{n,k \geq 0} \frac{(u,v;q)_{n+k}}{(q;q)_n (w;q)_{n+k}} \frac{(be^{-i\theta}, xe^{-i\theta};q)_k}{(q, xb;q)_k} (te^{i\theta})^k (te^{-i\theta})^n$$

and

$$\mathbb{T}\begin{bmatrix} u, v \\ w \end{bmatrix} q; tD_x \left\{ \frac{(xbcd;q)_\infty}{(xc, xd;q)_\infty} \right\}$$

$$= \frac{(xbcd;q)_\infty}{(xc, xd;q)_\infty} \sum_{n,k \geq 0} \frac{(u,v;q)_{n+k}}{(q;q)_n (w;q)_{n+k}} \frac{(bd, xd;q)_k}{(q, xbcd;q)_k} (tc)^k (td)^n.$$

Substituting the last two expressions into the identity (15), we have the following extension of the Askey-Wilson integral.

Theorem 22 *For* $\max\{|b|, |c|, |d|, |x|\} < 1$, *we have*

$$\int_0^\pi \frac{h(\cos 2\theta; 1)}{h(\cos \theta; b, c, d, x)} \sum_{n,k \geq 0} \frac{(u,v;q)_{n+k} t^{n+k}}{(q;q)_n (w;q)_{n+k}} \frac{(be^{-i\theta}, xe^{-i\theta};q)_k}{(q, xb;q)_k} e^{(k-n)i\theta} \, d\theta$$

$$= \frac{2\pi (xbcd;q)_\infty}{(q, bc, bd, cd, xb, xc, xd;q)_\infty} \sum_{n,k \geq 0} \frac{(u,v;q)_{n+k}}{(q;q)_n (w;q)_{n+k}} \frac{(bd, xd;q)_k}{(q, xbcd;q)_k} (tc)^k (td)^n.$$

Letting $n + k = m$, and then extracting the coefficients of $\frac{(u,v;q)_m}{(w;q)_m} t^m$ from both members of the last theorem, we thus give the following transformation formula, which leads to the Askey-Wilson integral (14) when $m = 0$.

Corollary 23

$$\int_0^\pi \frac{h(\cos 2\theta; 1)}{h(\cos \theta; b, c, d, x)} \sum_{k=0}^m \begin{bmatrix} m \\ k \end{bmatrix} \frac{(be^{-i\theta}, xe^{-i\theta};q)_k}{(xb;q)_k} e^{(2k-m)i\theta} \, d\theta$$

$$= \frac{d^m 2\pi (xbcd;q)_\infty}{(q, bc, bd, cd, xb, xc, xd;q)_\infty} \sum_{k=0}^m \begin{bmatrix} m \\ k \end{bmatrix} \frac{(bd, xd;q)_k}{(xbcd;q)_k} (c/d)^k.$$

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