Qiu *Advances in Difference Equations* (2016) 2016:10
DOI 10.1186/s13662-016-0742-5

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH **Open Access**

On nonoscillatory solutions tending to zero of third-order nonlinear dynamic equations on time scales

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Abstract

By Kranoselskii's fixed point theorem, we state some sufficient conditions of the existence of nonoscillatory solutions tending to zero as $t \to \infty$ of a class of third-order nonlinear dynamic equations on time scales. Two examples are presented to show the significance of the results.

MSC: 34N05; 34C10; 39A13

Keywords: third-order nonlinear dynamic equations; time scales; existence; nonoscillatory solutions tending to zero; Kranoselskii's fixed point theorem

1 Introduction

In 1988, Hilger introduced the theory of time scales in his Ph.D. thesis [1]; see also [2]. We refer the reader to [3, 4] for details on time scales. We remark that some researchers have studied the existence of nonoscillatory solutions of several kinds of nonlinear dynamic equations on time scales, which can be found in [5–10].

Zhu and Wang [10] studied first-order nonlinear neutral dynamic equations

$$\left[x(t) + p(t)x\big(g(t)\big)\right]^{\Delta} + f\big(t, x\big(h(t)\big)\big) = 0$$

on a time scale $\mathbb{T}$. Then Gao and Wang [7], Deng and Wang [6] discussed second-order nonlinear neutral dynamic equations

$$\left[r(t)\big(x(t) + p(t)x\big(g(t)\big)\big)^{\Delta}\right]^{\Delta} + f\big(t, x\big(h(t)\big)\big) = 0$$

under different conditions successively. Inspired by [6, 7], Qiu [9] considered third-order nonlinear neutral dynamic equations

$$\Big(r_1(t)\Big(r_2(t)\big(x(t) + p(t)x\big(g(t)\big)\big)^{\Delta}\Big)^{\Delta}\Big)^{\Delta} + f\big(t, x\big(h(t)\big)\big) = 0 \tag{1}$$

under the condition $g(t) \le t$, and established the existence of nonoscillatory solutions of equation (1). However, the conditions ensuring the existence of the nonoscillatory solutions tending to zero as $t \to \infty$ of equation (1) were exceptional, and as a result, the applications were within limits (see [9], Theorems 3.5 and 3.6).

In 2011, Mojsej and Tartal'ová [8] studied the asymptotic behavior of nonoscillatory solutions of third-order nonlinear differential equations with quasiderivatives of the form

$$\left(\frac{1}{p(t)}\left(\frac{1}{r(t)}x'(t)\right)'\right)' + q(t)f\big(x(t)\big) = 0, \quad t \ge a, \tag{2}$$

and stated the necessary and sufficient conditions ensuring the existence of nonoscillatory solutions tending to zero as $t \to \infty$.

In this paper, we investigate the existence of nonoscillatory solutions tending to zero as $t \to \infty$ of equation (1) on a time scale $\mathbb{T}$, which satisfies $\inf \mathbb{T} = t_0$ and $\sup \mathbb{T} = \infty$. We state the following conditions, which hold throughout this paper:

(C1) $r_1, r_2 \in C_{rd}(\mathbb{T}, (0,\infty))$.

(C2) $p \in C_{rd}(\mathbb{T}, [0,\infty))$ and there exists a constant p_0 with $0 \le p_0 < 1$ such that $\lim_{t\to\infty} p(t) = p_0$.

(C3) $g, h \in C_{rd}(\mathbb{T}, \mathbb{T})$, $g(t) \ge t$, and $\lim_{t\to\infty} h(t) = \infty$.

(C4) $f \in C(\mathbb{T} \times \mathbb{R}, \mathbb{R})$ and $xf(t,x) > 0$ for $t \in \mathbb{T}$ and $x \neq 0$.

In the sequel, there are two cases of the function f to be considered:

(C4A) There exist $q \in C(\mathbb{T}, (0,\infty))$ and $f_0 \in C(\mathbb{R}, \mathbb{R})$ such that $xf(t,x) \le xq(t)f_0(x)$.

(C4B) $f(t,x)$ is nondecreasing in x.

Motivated by [8, 9], we will establish the existence of nonoscillatory solutions tending to a zero of equation (1) by employing Kranoselskii's fixed point theorem and, finally, two examples are given to show the versatility of the conclusions.

Definition 1.1 A solution x of equation (1) is said to be eventually positive (or eventually negative) if there exists $c \in \mathbb{T}$ such that $x(t) > 0$ (or $x(t) < 0$) for all $t \ge c$ in $\mathbb{T}$. x is said to be nonoscillatory if it is either eventually positive or eventually negative; otherwise, it is oscillatory.

2 Auxiliary results

Let $BC([T_0,\infty)_{\mathbb{T}}, \mathbb{R})$ denote the Banach space of all bounded continuous functions on $[T_0,\infty)_{\mathbb{T}}$ with the norm

$$\|x\| = \sup_{t\in[T_0,\infty)_{\mathbb{T}}} \big|x(t)\big|.$$

Definition 2.1 Let $X \subseteq BC[T_0,\infty)_{\mathbb{T}}$, we say that X is uniformly Cauchy, if for any given $\epsilon > 0$, there exists a $T_1 \in [T_0,\infty)_{\mathbb{T}}$ such that $|x(t_1) - x(t_2)| < \epsilon$ for any $x \in X$ and $t_1, t_2 \in [T_1,\infty)_{\mathbb{T}}$.

Definition 2.2 X is said to be equi-continuous on $[a,b]_{\mathbb{T}}$, if for any given $\epsilon > 0$, there exists $\delta > 0$ such that $|x(t_1) - x(t_2)| < \epsilon$ for any $x \in X$ and $t_1, t_2 \in [a,b]_{\mathbb{T}}$ with $|t_1 - t_2| < \delta$.

Lemma 2.3 ([10], Lemma 4) *Suppose that $X \subseteq BC[T_0,\infty)_{\mathbb{T}}$ is bounded and uniformly Cauchy. Further, suppose that X is equi-continuous on $[T_0,T_1]_{\mathbb{T}}$ for any $T_1 \in [T_0,\infty)_{\mathbb{T}}$. Then X is relatively compact.*

Lemma 2.4 (Kranoselskii's fixed point theorem, see [5]) *Suppose that X is a Banach space and Ω is a bounded, convex, and closed subset of X. Suppose further that there exist two operators $U, S : \Omega \to X$ such that*

(1) *$Ux + Sy \in \Omega$ for all $x, y \in \Omega$;*
(2) *U is a contraction mapping;*
(3) *S is completely continuous.*

Then $U + S$ has a fixed point in Ω.

Without loss of generality, we shall consider the case that eventually positive solutions of equation (1) in the following. It is because if $x(t)$ is an eventually negative solution of equation (1), then $y(t) = -x(t)$ will satisfy

$$\big(r_1(t)\big(r_2(t)\big(y(t) + p(t)y\big(g(t)\big)\big)^{\Delta}\big)^{\Delta}\big)^{\Delta} - f\big(t, -y\big(h(t)\big)\big) = 0.$$

Note that $\overline{f}(t,u) := -f(t,-u)$ satisfies (C4), and (C4A) or (C4B) similarly as $f(t,u)$.

Define

$$z(t) := x(t) + p(t)x\big(g(t)\big), \tag{3}$$

then we will have the following lemma.

Lemma 2.5 *Suppose that $x(t)$ is an eventually positive solution of equation* (1), *and there exists a constant $a \geq 0$ such that $\lim_{t\to\infty} z(t) = a$. Then we have*

$$\lim_{t\to\infty} x(t) = \frac{a}{1 + p_0}.$$

Proof Suppose that $x(t)$ is an eventually positive solution of equation (1). In view of (C3), there exists $t_1 \in [t_0,\infty)_{\mathbb{T}}$ such that $x(t) > 0$, $x(g(t)) > 0$ for $t \in [t_1,\infty)_{\mathbb{T}}$. We claim that $x(t)$ is bounded on $[t_1,\infty)_{\mathbb{T}}$. Assume not; then we have

$$z(t) = x(t) + p(t)x\big(g(t)\big) \geq x(t) \to \infty$$

which contradicts the fact that $\lim_{t\to\infty} z(t) = a$. Therefore, $x(t)$ is bounded. Then assume that

$$\limsup_{t\to\infty} x(t) = \overline{x} \quad \text{and} \quad \liminf_{t\to\infty} x(t) = \underline{x}.$$

Since $0 \leq p_0 < 1$, we have

$$a \geq \overline{x} + p_0\underline{x} \quad \text{and} \quad a \leq \underline{x} + p_0\overline{x},$$

which implies that $\overline{x} \leq \underline{x}$. So $\overline{x} = \underline{x}$, and we see that $\lim_{t\to\infty} x(t)$ exists and $\lim_{t\to\infty} x(t) = a/(1 + p_0)$. The proof is complete. □

3 Main results

In this section, we state and prove our existence criteria for eventually positive solutions tending to zero as $t \to \infty$ of equation (1).

Theorem 3.1 *Assume that the function f satisfies* (C4A) *and there exists $M_1 > 0$ such that*

$$\int_{t_0}^{\infty} \frac{1}{r_2(t)} \Delta t = M_1 < \infty \tag{4}$$

and

$$\int_{t_0}^{\infty} \int_{t_0}^{s} \frac{q(u)}{r_1(s)} \Delta u \Delta s < \infty. \tag{5}$$

Define $H_1(t) = \int_t^{\infty} \frac{1}{r_2(v)} \Delta v$, which satisfies $\lim_{t\to\infty} \frac{H_1(g(t))}{H_1(t)} = \eta_1 \in (0,1]$. If there exists $L > 0$ such that

$$\left|f(t,x_1) - f(t,x_2)\right| \le L \cdot q(t)|x_1 - x_2|, \quad x_1, x_2 \in [0, 2M_1], \tag{6}$$

then equation (1) *has an eventually positive solution $x(t)$ with $\lim_{t\to\infty} x(t) = 0$, and there exists $T_1 \in [t_0,\infty)_{\mathbb{T}}$ such that*

$$r_2(t)z^{\Delta}(t) < 0, \qquad r_1(t)\big(r_2(t)z^{\Delta}(t)\big)^{\Delta} < 0, \quad t \in [T_1,\infty)_{\mathbb{T}}.$$

Proof From (C2), for $0 \le p_0 < 1$, choose p_1 such that $p_0 < p_1 < (1 + 4p_0)/5 < 1$. By (4) and (5), there exists $T_0 \in [t_0,\infty)_{\mathbb{T}}$ such that

$$\frac{5p_1 - 1}{4} \le p(t) \le p_1 < 1, \qquad p(t)\frac{H_1(g(t))}{H_1(t)} \ge \frac{5p_1 - 1}{4}\eta_1, \quad t \in [T_0,\infty)_{\mathbb{T}}, \tag{7}$$

and

$$\int_{T_0}^{\infty} \int_{T_0}^{s} \frac{q(u)}{r_1(s)} \Delta u \Delta s \le \min\left\{\frac{1 - p_1\eta_1}{4K}, 1\right\}, \tag{8}$$

where $K = \max\{|f_0(x)| : x \in [0, 2M_1]\}$. From (C3), there always exists $T_1 \in (T_0,\infty)_{\mathbb{T}}$ such that

$$h(t) \ge T_0, \quad t \in [T_1,\infty)_{\mathbb{T}}.$$

Define

$$\Omega_1 = \big\{x(t) \in BC[T_0,\infty)_{\mathbb{T}} : H_1(t) \le x(t) \le 2H_1(t)\big\}. \tag{9}$$

It is clear that Ω_1 is a bounded, convex, and closed subset of $BC[T_0,\infty)_{\mathbb{T}}$. For any $x \in \Omega_1$, by (C4A) we have

$$0 < f\big(t, x(h(t))\big) \le q(t) f_0\big(x(h(t))\big), \quad t \in [T_1,\infty)_{\mathbb{T}}.$$

Define the operators U_1 and S_1: $\Omega_1 \to BC[T_0,\infty)_{\mathbb{T}}$ as

$$(U_1x)(t) = \frac{3}{2}p_1\eta_1 H_1(t) - p(t)x\big(g(t)\big), \quad t \in [T_0,\infty)_{\mathbb{T}},$$

and

$$(S_1x)(t) = \begin{cases} \frac{3}{2}H_1(t) + \int_t^{\infty}\int_{T_1}^{v}\int_{T_1}^{s}\frac{f(u,x(h(u)))}{r_1(s)r_2(v)}\Delta u\Delta s\Delta v, & t\in[T_1,\infty)_{\mathbb{T}},\\ (S_1x)(T_1), & t\in[T_0,T_1]_{\mathbb{T}}. \end{cases} \tag{10}$$

Then we prove that U_1 and S_1 satisfy all the conditions in Lemma 2.4.

(1) For any $x,y\in\Omega_1$ and $t\in[T_1,\infty)_{\mathbb{T}}$, by (7)-(9) we obtain

$$\begin{aligned}
&(U_1x)(t) + (S_1y)(t)\\
&\quad\le \frac{3(1+p_1\eta_1)}{2}H_1(t) - \frac{5p_1-1}{4}\eta_1H_1(t) + \int_t^{\infty}\int_{T_1}^{v}\int_{T_1}^{s}\frac{q(u)f_0(y(h(u)))}{r_1(s)r_2(v)}\Delta u\Delta s\Delta v\\
&\quad\le \frac{3(1+p_1\eta_1)}{2}H_1(t) - \frac{5p_1-1}{4}\eta_1H_1(t) + KH_1(t)\int_{T_1}^{\infty}\int_{T_1}^{s}\frac{q(u)}{r_1(s)}\Delta u\Delta s\\
&\quad\le \frac{3(1+p_1\eta_1)}{2}H_1(t) - \frac{5p_1-1}{4}\eta_1H_1(t) + \frac{1-p_1\eta_1}{4}H_1(t)\\
&\quad= \frac{7+\eta_1}{4}H_1(t) \le 2H_1(t)
\end{aligned}$$

and

$$\begin{aligned}
&(U_1x)(t) + (S_1y)(t)\\
&\quad= \frac{3(1+p_1\eta_1)}{2}H_1(t) - p(t)x\big(g(t)\big) + \int_t^{\infty}\int_{T_1}^{v}\int_{T_1}^{s}\frac{f(u,y(h(u)))}{r_1(s)r_2(v)}\Delta u\Delta s\Delta v\\
&\quad\ge \frac{3(1+p_1\eta_1)}{2}H_1(t) - 2p_1\eta_1H_1(t) = \frac{3-p_1\eta_1}{2}H_1(t) > H_1(t).
\end{aligned}$$

Similarly, $H_1(t)\le(U_1x)(t)+(S_1y)(t)\le 2H_1(t)$ also holds for any $x,y\in\Omega_1$ and $t\in[T_0,T_1]_{\mathbb{T}}$. It follows that $U_1x+S_1y\in\Omega_1$ for any $x,y\in\Omega_1$.

(2) For any $x,y\in\Omega_1$ and $t\in[T_0,\infty)_{\mathbb{T}}$, we always have

$$\big|(U_1x)(t)-(U_1y)(t)\big| = \big|p(t)\big(x\big(g(t)\big)-y\big(g(t)\big)\big)\big| \le p_1\sup_{t\in[T_0,\infty)_{\mathbb{T}}}\big|x(t)-y(t)\big|.$$

It follows that $\|U_1x-U_1y\|\le p_1\|x-y\|$ for any $x,y\in\Omega_1$. That is, U_1 is a contraction mapping.

(3) For $t\in[T_0,\infty)_{\mathbb{T}}$, we have

$$(S_1x)(t)\le\frac{3}{2}H_1(t)+\frac{1-p_1\eta_1}{4}H_1(t)=\frac{7-p_1\eta_1}{4}H_1(t)<2H_1(t)$$

and

$$(S_1x)(t)>\frac{3}{2}H_1(t)>H_1(t).$$

It is easy to see that S_1 maps Ω_1 into Ω_1.

For any $x\in\Omega_1$ and $t\in[T_0,\infty)_{\mathbb{T}}$, let $x_n\in\Omega_1$ and $\|x_n-x\|\to 0$ as $n\to\infty$. For $t\in[T_1,\infty)_{\mathbb{T}}$, by (4), (6), and (8), we have

$$\begin{aligned}
&\left|(S_1x_n)(t)-(S_1x)(t)\right| \\
&\quad\le \int_t^\infty\int_{T_1}^v\int_{T_1}^s \frac{|f(u,x_n(h(u)))-f(u,x(h(u)))|}{r_1(s)r_2(v)}\Delta u\Delta s\Delta v \\
&\quad\le L\int_{T_1}^\infty \frac{1}{r_2(v)}\Delta v\cdot\int_{T_1}^\infty\int_{T_1}^s \frac{q(u)|x_n(h(u))-x(h(u))|}{r_1(s)}\Delta u\Delta s \\
&\quad\le LM_1\cdot\sup_{t\in[T_0,\infty)_{\mathbb{T}}}\left|x_n(t)-x(t)\right|.
\end{aligned}$$

For $t\in[T_0,T_1]_{\mathbb{T}}$, we also have the result above. By Lebesgue's dominated convergence theorem (see Chapter 5 in [11]), it follows that

$$\|S_1x_n-S_1x\|\to 0,\quad n\to\infty.$$

Therefore, S_1 is continuous.

It is obvious that $S_1\Omega_1$ is bounded. On the other hand, by (4), for any $\epsilon>0$ there exists $T_2\in[T_1,\infty)_{\mathbb{T}}$ such that

$$H_1(T_2)=\int_{T_2}^\infty \frac{1}{r_2(t)}\Delta t<\frac{\epsilon}{2K+3}.$$

For $t_1,t_2\in[T_0,T_1]_{\mathbb{T}}$, we always have $|(S_1x)(t_1)-(S_1x)(t_2)|=0$. Then, for any $x\in\Omega_1$ and $t_1,t_2\in[T_2,\infty)_{\mathbb{T}}$, we have

$$\begin{aligned}
&\left|(S_1x)(t_1)-(S_1x)(t_2)\right| \\
&\quad\le\left|\int_{t_1}^\infty\int_{T_1}^v\int_{T_1}^s\frac{f(u,x(h(u)))}{r_1(s)r_2(v)}\Delta u\Delta s\Delta v-\int_{t_2}^\infty\int_{T_1}^v\int_{T_1}^s\frac{f(u,x(h(u)))}{r_1(s)r_2(v)}\Delta u\Delta s\Delta v\right| \\
&\qquad+\frac{3}{2}\left|H_1(t_1)-H_1(t_2)\right| \\
&\quad\le 2\int_{T_2}^\infty\int_{T_1}^v\int_{T_1}^s\frac{f(u,x(h(u)))}{r_1(s)r_2(v)}\Delta u\Delta s\Delta v+3H_1(T_2) \\
&\quad\le 2KH_1(T_2)\int_{T_1}^\infty\int_{T_1}^s\frac{q(u)}{r_1(s)}\Delta u\Delta s+3H_1(T_2) \\
&\quad\le(2K+3)H_1(T_2)<\epsilon.
\end{aligned}$$

It is clear that $S_1\Omega_1$ is uniformly Cauchy.

For $x\in\Omega_1$ and $t_1,t_2\in[\min\{T_1-1,T_0\},T_2+1]_{\mathbb{T}}$, we have

$$\begin{aligned}
&\left|(S_1x)(t_1)-(S_1x)(t_2)\right| \\
&\quad\le\left|\int_{t_1}^{t_2}\int_{T_1}^v\int_{T_1}^s\frac{f(u,x(h(u)))}{r_1(s)r_2(v)}\Delta u\Delta s\Delta v\right|+\frac{3}{2}\left|\int_{t_1}^{t_2}\frac{1}{r_2(t)}\Delta t\right| \\
&\quad\le K\int_{T_1}^\infty\int_{T_1}^s\frac{q(u)}{r_1(s)}\Delta u\Delta s\left|\int_{t_1}^{t_2}\frac{1}{r_2(t)}\Delta t\right|+\frac{3}{2}\left|\int_{t_1}^{t_2}\frac{1}{r_2(t)}\Delta t\right| \\
&\quad\le\left(K+\frac{3}{2}\right)\left|H_1(t_1)-H_1(t_2)\right|.
\end{aligned}$$

Therefore, there exists $0 < \delta < 1$ such that $|(S_1x)(t_1) - (S_1x)(t_2)| < \epsilon$ for any $t_1, t_2 \in [T_0, T_2 + 1]_{\mathbb{T}}$ with $|t_1 - t_2| < \delta$. We conclude that $S_1\Omega_1$ is equi-continuous.

According to Lemma 2.3, it follows that $S_1\Omega_1$ is relatively compact and S_1 is completely continuous. Then by Lemma 2.4, there exists $x \in \Omega_1$ such that $(U_1 + S_1)x = x$. It means that $x(t)$ is an eventually positive solution of equation (1), and for $t \in [T_1, \infty)_{\mathbb{T}}$, we have

$$x(t) = \frac{3(1 + p_1\eta_1)}{2}H_1(t) - p(t)x\big(g(t)\big) + \int_t^\infty \int_{T_1}^v \int_{T_1}^s \frac{f(u, x(h(u)))}{r_1(s)r_2(v)} \Delta u \Delta s \Delta v.$$

As

$$\int_t^\infty \int_{T_1}^v \int_{T_1}^s \frac{f(u, x(h(u)))}{r_1(s)r_2(v)} \Delta u \Delta s \Delta v \leq KH_1(t) \int_{T_1}^\infty \int_{T_1}^s \frac{q(u)}{r_1(s)} \Delta u \Delta s$$

for $t \in [T_1, \infty)_{\mathbb{T}}$, and

$$\lim_{t\to\infty} KH_1(t) \int_{T_1}^\infty \int_{T_1}^s \frac{q(u)}{r_1(s)} \Delta u \Delta s = 0,$$

it follows that $\lim_{t\to\infty} z(t) = 0$. By Lemma 2.5, we obtain $\lim_{t\to\infty} x(t) = 0$. Furthermore, for $t \in [T_1, \infty)_{\mathbb{T}}$, we have

$$r_2(t)z^\Delta(t) = -\frac{3(1 + p_1\eta_1)}{2} - \int_{T_1}^t \int_{T_1}^s \frac{f(u, x(h(u)))}{r_1(s)} \Delta u \Delta s < 0$$

and

$$r_1(t)\big(r_2(t)z^\Delta(t)\big)^\Delta = -\int_{T_1}^t f\big(u, x\big(h(u)\big)\big) \Delta u < 0.$$

The proof is complete. □

Theorem 3.2 *Assume that*

$$\int_{t_0}^\infty \frac{1}{r_2(t)} \Delta t = \infty \quad or \quad \int_{t_0}^\infty \int_{t_0}^v \frac{1}{r_1(s)r_2(v)} \Delta s \Delta v = \infty.$$

Then equation (1) *has no eventually positive solution* $x(t)$, *for which* $r_2(t)z^\Delta(t)$ *and* $r_1(t)(r_2(t)z^\Delta(t))^\Delta$ *are both eventually negative.*

Proof Suppose that $x(t)$ is an eventually positive solution of equation (1), and there exists $T_0 \in [t_0, \infty)_{\mathbb{T}}$ such that

$$r_2(t)z^\Delta(t) < 0, \qquad r_1(t)\big(r_2(t)z^\Delta(t)\big)^\Delta < 0, \quad t \in [T_0, \infty)_{\mathbb{T}}.$$

By (C3), there exists $T_1 \in (T_0, \infty)_{\mathbb{T}}$ such that $h(t) \geq T_0$ for $t \in [T_1, \infty)_{\mathbb{T}}$. Integrating (1) from T_1 to $s \in [\sigma(T_1), \infty)_{\mathbb{T}}$, by (C4) we obtain

$$r_1(s)\big(r_2(s)z^\Delta(s)\big)^\Delta - r_1(T_1)\big(r_2(T_1)z^\Delta(T_1)\big)^\Delta = -\int_{T_1}^s f\big(u, x\big(h(u)\big)\big) \Delta u < 0,$$

which implies that

$$\big(r_2(s)z^{\Delta}(s)\big)^{\Delta} < \frac{r_1(T_1)(r_2(T_1)z^{\Delta}(T_1))^{\Delta}}{r_1(s)}. \tag{11}$$

Integrating (11) from T_1 to $\nu \in [\sigma(T_1),\infty)_{\mathbb{T}}$, we obtain

$$r_2(\nu)z^{\Delta}(\nu) - r_2(T_1)z^{\Delta}(T_1) < r_1(T_1)\big(r_2(T_1)z^{\Delta}(T_1)\big)^{\Delta}\int_{T_1}^{\nu}\frac{1}{r_1(s)}\Delta s$$

or

$$z^{\Delta}(\nu) < \frac{r_2(T_1)z^{\Delta}(T_1)}{r_2(\nu)} + \frac{r_1(T_1)(r_2(T_1)z^{\Delta}(T_1))^{\Delta}}{r_2(\nu)}\int_{T_1}^{\nu}\frac{1}{r_1(s)}\Delta s. \tag{12}$$

Integrating (12) from T_1 to $t \in [\sigma(T_1),\infty)_{\mathbb{T}}$, we obtain

$$\begin{aligned} z(t) &< z(T_1) + r_2(T_1)z^{\Delta}(T_1)\int_{T_1}^{t}\frac{1}{r_2(\nu)}\Delta\nu \\ &\quad + r_1(T_1)\big(r_2(T_1)z^{\Delta}(T_1)\big)^{\Delta}\int_{T_1}^{t}\int_{T_1}^{\nu}\frac{1}{r_1(s)r_2(\nu)}\Delta s\Delta\nu. \end{aligned}$$

Letting $t \to \infty$, we have $z(t) \to -\infty$. It is a contradiction because $z(t) = x(t) + p(t)x(g(t))$ is eventually positive. The proof is complete. □

By Theorems 3.1 and 3.2, we have the following corollary.

Corollary 3.3 *Assume that*

$$\int_{t_0}^{\infty}\int_{t_0}^{s}\frac{q(u)}{r_1(s)}\Delta u\Delta s < \infty.$$

If the function f satisfies (C4A) *and there exists $L > 0$ such that*

$$\big|f(t,x_1) - f(t,x_2)\big| \le L \cdot q(t)|x_1 - x_2|, \quad x_1, x_2 \in \left[0, 2\int_{t_0}^{\infty}\frac{1}{r_2(t)}\Delta t\right],$$

then $\int_{t_0}^{\infty}\frac{1}{r_2(t)}\Delta t < \infty$ is a necessary and sufficient condition for equation (1) *to have an eventually positive solution $x(t)$ satisfying the requirement that $\lim_{t\to\infty} x(t) = 0$ and $r_2(t)z^{\Delta}(t)$, $r_1(t)(r_2(t)z^{\Delta}(t))^{\Delta}$ are both eventually negative.*

Theorem 3.4 *Assume that the function f satisfies* (C4A) *and there exists $M_2 > 0$ such that*

$$\int_{t_0}^{\infty}\int_{\nu}^{\infty}\frac{1}{r_1(s)r_2(\nu)}\Delta s\Delta\nu = M_2 < \infty \tag{13}$$

and

$$\int_{t_0}^{\infty} q(t)\Delta t < \infty. \tag{14}$$

Define $H_2(t) = \int_t^\infty \int_v^\infty \frac{1}{r_1(s)r_2(v)} \Delta s \Delta v$, *which satisfies* $\lim_{t\to\infty} \frac{H_2(g(t))}{H_2(t)} = \eta_2 \in (0,1]$. *If there exists* $L > 0$ *such that*

$$\bigl|f(t,x_1) - f(t,x_2)\bigr| \le L \cdot q(t)|x_1 - x_2|, \quad x_1, x_2 \in [0, 2M_2], \tag{15}$$

then equation (1) *has an eventually positive solution* $x(t)$ *with* $\lim_{t\to\infty} x(t) = 0$, *and there exists* $T_1 \in [t_0,\infty)_{\mathbb{T}}$ *such that*

$$r_2(t)z^\Delta(t) < 0, \qquad r_1(t)\bigl(r_2(t)z^\Delta(t)\bigr)^\Delta > 0, \quad t \in [T_1,\infty)_{\mathbb{T}}.$$

Proof From (C2), for $0 \le p_0 < 1$, choose p_1 such that $p_0 < p_1 < (1 + 4p_0)/5 < 1$. By (13) and (14), there exists $T_0 \in [t_0,\infty)_{\mathbb{T}}$ such that

$$\frac{5p_1 - 1}{4} \le p(t) \le p_1 < 1, \qquad p(t)\frac{H_2(g(t))}{H_2(t)} \ge \frac{5p_1 - 1}{4}\eta_2, \quad t \in [T_0,\infty)_{\mathbb{T}}, \tag{16}$$

and

$$\int_{T_0}^\infty q(t)\Delta t \le \min\left\{\frac{1 - p_1\eta_2}{4K}, 1\right\},$$

where $K = \max\{|f_0(x)| : x \in [0, 2M_2]\}$. Similarly, there always exists $T_1 \in (T_0,\infty)_{\mathbb{T}}$ such that $h(t) \ge T_0$ for $t \in [T_1,\infty)_{\mathbb{T}}$.

Define

$$\Omega_2 = \bigl\{x(t) \in BC[T_0,\infty)_{\mathbb{T}} : H_2(t) \le x(t) \le 2H_2(t)\bigr\}.$$

Then Ω_2 is also a bounded, convex, and closed subset of $BC[T_0,\infty)_{\mathbb{T}}$. For any $x \in \Omega_2$, by (C4A) we have

$$0 < f\bigl(t, x\bigl(h(t)\bigr)\bigr) \le q(t)f_0\bigl(x\bigl(h(t)\bigr)\bigr), \quad t \in [T_1,\infty)_{\mathbb{T}}.$$

Define the operators U_2 and $S_2 : \Omega_2 \to BC[T_0,\infty)_{\mathbb{T}}$ as

$$(U_2x)(t) = \frac{3}{2}p_1\eta_2 H_2(t) - p(t)x\bigl(g(t)\bigr), \quad t \in [T_0,\infty)_{\mathbb{T}},$$

and

$$(S_2x)(t) = \begin{cases} \frac{3}{2}H_2(t) + \int_t^\infty \int_v^\infty \int_s^\infty \frac{f(u,x(h(u)))}{r_1(s)r_2(v)} \Delta u \Delta s \Delta v, & t \in [T_1,\infty)_{\mathbb{T}}, \\ (S_2x)(T_1), & t \in [T_0, T_1]_{\mathbb{T}}. \end{cases}$$

Similarly to the proof of Theorem 3.1, we can conclude that U_2 and S_2 satisfy all the conditions in Lemma 2.4. Therefore, there exists $x \in \Omega_2$ such that $(U_2 + S_2)x = x$, which means that $x(t)$ is an eventually positive solution of equation (1). It follows that for $t \in [T_1,\infty)_{\mathbb{T}}$, we have

$$x(t) = \frac{3(1 + p_1\eta_2)}{2}H_2(t) - p(t)x\bigl(g(t)\bigr) + \int_t^\infty \int_v^\infty \int_s^\infty \frac{f(u,x(h(u)))}{r_1(s)r_2(v)} \Delta u \Delta s \Delta v.$$

As

$$\int_t^\infty \int_v^\infty \int_s^\infty \frac{f(u,x(h(u)))}{r_1(s)r_2(v)} \Delta u \Delta s \Delta v \le KH_2(t) \int_{T_1}^\infty q(u)\Delta u$$

for $t \in [T_1, \infty)_{\mathbb{T}}$, and

$$\lim_{t\to\infty} KH_2(t) \int_{T_1}^\infty q(u)\Delta u = 0,$$

we have $\lim_{t\to\infty} z(t) = 0$, which implies that $\lim_{t\to\infty} x(t) = 0$ by Lemma 2.5. For $t \in [T_1, \infty)_{\mathbb{T}}$, we obtain

$$r_2(t)z^\Delta(t) = -\frac{3(1+p_1\eta_2)}{2} \int_t^\infty \frac{1}{r_1(s)}\Delta s - \int_t^\infty \int_s^\infty \frac{f(u,x(h(u)))}{r_1(s)}\Delta u \Delta s < 0$$

and

$$r_1(t)\bigl(r_2(t)z^\Delta(t)\bigr)^\Delta = \frac{3(1+p_1\eta_2)}{2} + \int_t^\infty f\bigl(u,x\bigl(h(u)\bigr)\bigr)\Delta u > 0.$$

The proof is complete. □

Remark 3.5 Actually, (6) in Theorem 3.1 and (15) in Theorem 3.4 will hold especially when $f(t,x) = q(t)f_0(x)$ and $f_0(x)$ satisfies the Lipschitz condition on $[0, 2M_1]$ and $[0, 2M_2]$, respectively.

Remark 3.6 Letting $\mathbb{T} = [a, \infty)$, $r_1(t) = \frac{1}{p(t)}$, $r_2(t) = \frac{1}{r(t)}$, $p(t) = 0$, and the function f satisfying (C4A), equation (1) is simplified as (2). Therefore, Theorems 3.1, 3.2, 3.4, and Corollary 3.3 complement and extend the results in Mojsej and Tartal'ová [8].

In the sequel, we change the condition of the function f from (C4A) to (C4B), and get some similar conclusions as follows.

Theorem 3.7 *Assume that the function f satisfies* (C4B) *and*

$$\int_{t_0}^\infty \frac{1}{r_2(t)}\Delta t < \infty \tag{17}$$

and

$$\int_{t_0}^\infty \int_{t_0}^s \frac{f(u, 2H_1(h(u)))}{r_1(s)} \Delta u \Delta s < \infty, \tag{18}$$

where $H_1(t) = \int_t^\infty \frac{1}{r_2(v)}\Delta v$, which satisfies $\lim_{t\to\infty} \frac{H_1(g(t))}{H_1(t)} = \eta_1 \in (0,1]$. Then equation (1) *has an eventually positive solution $x(t)$ with $\lim_{t\to\infty} x(t) = 0$, and there exists $T_1 \in [t_0, \infty)_{\mathbb{T}}$ such that*

$$r_2(t)z^\Delta(t) < 0, \qquad r_1(t)\bigl(r_2(t)z^\Delta(t)\bigr)^\Delta < 0, \quad t \in [T_1, \infty)_{\mathbb{T}}.$$

Proof From (C2), for $0 \le p_0 < 1$, choose p_1 such that $p_0 < p_1 < (1 + 4p_0)/5 < 1$. By (17) and (18), there exists $T_0 \in [t_0, \infty)_{\mathbb{T}}$ such that (7) holds and

$$\int_{T_0}^{\infty} \int_{T_0}^{s} \frac{f(u, 2H_1(h(u)))}{r_1(s)} \Delta u \Delta s \le \min\left\{\frac{1 - p_1\eta_1}{4}, 1\right\}.$$

Similarly, there always exists $T_1 \in (T_0, \infty)_{\mathbb{T}}$ such that $h(t) \ge T_0$ for $t \in [T_1, \infty)_{\mathbb{T}}$.

Define Ω_1 as in Theorem 3.1. For any $x \in \Omega_1$, by (C4B) we have

$$0 < f\bigl(t, x\bigl(h(t)\bigr)\bigr) \le f\bigl(t, 2H_1\bigl(h(t)\bigr)\bigr), \quad t \in [T_1, \infty)_{\mathbb{T}}.$$

In addition, define U_1 and S_1 as in Theorem 3.1. It is obvious that U_1 and S_1 also satisfy all the conditions in Lemma 2.4. Hence, there exists $x \in \Omega_1$ such that $(U_1 + S_1)x = x$. That is, $x(t)$ is an eventually positive solution of equation (1). For $t \in [T_1, \infty)_{\mathbb{T}}$, it follows that

$$x(t) = \frac{3(1 + p_1\eta_1)}{2} H_1(t) - p(t)x\bigl(g(t)\bigr) + \int_{t}^{\infty} \int_{T_1}^{v} \int_{T_1}^{s} \frac{f(u, x(h(u)))}{r_1(s)r_2(v)} \Delta u \Delta s \Delta v.$$

As

$$\int_{t}^{\infty} \int_{T_1}^{v} \int_{T_1}^{s} \frac{f(u, x(h(u)))}{r_1(s)r_2(v)} \Delta u \Delta s \Delta v \le H_1(t) \int_{T_1}^{\infty} \int_{T_1}^{s} \frac{f(u, 2H_1(h(u)))}{r_1(s)} \Delta u \Delta s$$

for $t \in [T_1, \infty)_{\mathbb{T}}$, and

$$\lim_{t\to\infty} H_1(t) \int_{T_1}^{\infty} \int_{T_1}^{s} \frac{f(u, 2H_1(h(u)))}{r_1(s)} \Delta u \Delta s = 0,$$

it follows that $\lim_{t\to\infty} z(t) = 0$. By Lemma 2.5, we obtain $\lim_{t\to\infty} x(t) = 0$. Furthermore, for $t \in [T_1, \infty)_{\mathbb{T}}$, it satisfies

$$r_2(t)z^{\Delta}(t) < 0, \qquad r_1(t)\bigl(r_2(t)z^{\Delta}(t)\bigr)^{\Delta} < 0, \quad t \in [T_1, \infty)_{\mathbb{T}}.$$

The proof is complete. □

Theorem 3.8 *Assume that the function f satisfies* (C4B) *and*

$$\int_{t_0}^{\infty} \int_{v}^{\infty} \frac{1}{r_1(s)r_2(v)} \Delta s \Delta v < \infty \tag{19}$$

and

$$\int_{t_0}^{\infty} f\bigl(t, 2H_2\bigl(h(t)\bigr)\bigr) \Delta t < \infty, \tag{20}$$

where $H_2(t) = \int_t^{\infty} \int_v^{\infty} \frac{1}{r_1(s)r_2(v)} \Delta s \Delta v$, *which satisfies* $\lim_{t\to\infty} \frac{H_2(g(t))}{H_2(t)} = \eta_2 \in (0, 1]$. *Then equation* (1) *has an eventually positive solution* $x(t)$ *with* $\lim_{t\to\infty} x(t) = 0$, *and there exists* $T_1 \in [t_0, \infty)_{\mathbb{T}}$ *such that*

$$r_2(t)z^{\Delta}(t) < 0, \qquad r_1(t)\bigl(r_2(t)z^{\Delta}(t)\bigr)^{\Delta} > 0, \quad t \in [T_1, \infty)_{\mathbb{T}}.$$

Proof From (C2), for $0 \le p_0 < 1$, choose p_1 such that $p_0 < p_1 < (1 + 4p_0)/5 < 1$. By (19) and (20), there exists $T_0 \in [t_0, \infty)_{\mathbb{T}}$ such that (16) holds and

$$\int_{T_0}^{\infty} f\bigl(t, 2H_2\bigl(h(t)\bigr)\bigr)\Delta t \le \min\left\{\frac{1 - p_1\eta_2}{4}, 1\right\}.$$

From (C3), there always exists $T_1 \in (T_0, \infty)_{\mathbb{T}}$ such that $h(t) \ge T_0$ for $t \in [T_1, \infty)_{\mathbb{T}}$.

Define Ω_2 as in Theorem 3.4. For any $x \in \Omega_1$, by (C4B) we have

$$0 < f\bigl(t, x\bigl(h(t)\bigr)\bigr) \le f\bigl(t, 2H_2\bigl(h(t)\bigr)\bigr), \quad t \in [T_1, \infty)_{\mathbb{T}}.$$

In addition, define the operators U_2 and S_2 as in Theorem 3.4. It is obvious that U_2 and S_2 also satisfy all the conditions in Lemma 2.4. Then there exists $x \in \Omega_2$ such that $(U_2 + S_2)x = x$, which means that $x(t)$ is an eventually positive solution of equation (1). For $t \in [T_1, \infty)_{\mathbb{T}}$, it follows that

$$x(t) = \frac{3(1 + p_1\eta_2)}{2} H_2(t) - p(t)x\bigl(g(t)\bigr) + \int_t^{\infty}\int_v^{\infty}\int_s^{\infty} \frac{f(u, x(h(u)))}{r_1(s)r_2(v)} \Delta u \Delta s \Delta v.$$

As

$$\int_t^{\infty}\int_v^{\infty}\int_s^{\infty} \frac{f(u, x(h(u)))}{r_1(s)r_2(v)} \Delta u \Delta s \Delta v \le H_2(t) \int_{T_1}^{\infty} f\bigl(u, 2H_2\bigl(h(u)\bigr)\bigr)\Delta u$$

for $t \in [T_1, \infty)_{\mathbb{T}}$, and

$$\lim_{t\to\infty} H_2(t) \int_{T_1}^{\infty} f\bigl(u, 2H_2\bigl(h(u)\bigr)\bigr)\Delta u = 0,$$

we have $\lim_{t\to\infty} z(t) = 0$, which implies that $\lim_{t\to\infty} x(t) = 0$ by Lemma 2.5. We also obtain

$$r_2(t)z^{\Delta}(t) < 0, \qquad r_1(t)\bigl(r_2(t)z^{\Delta}(t)\bigr)^{\Delta} > 0, \quad t \in [T_1, \infty)_{\mathbb{T}}.$$

The proof is complete. □

Remark 3.9 Indeed, if the function f satisfies both of (C4A) and (C4B), compared to Theorems 3.1 and 3.4, Theorems 3.7 and 3.8 are more convenient to employ, respectively.

4 Examples

In this section, we give two examples to show the application of our results. The first example is presented to illustrate Theorems 3.1 and 3.7.

Example 4.1 Let $\mathbb{T} = \bigcup_{n=1}^{\infty}[2n-1, 2n]$. Consider

$$\left(t^5\left(t^2\left(x(t) + \frac{t-1}{2t}x(t+2)\right)^{\Delta}\right)^{\Delta}\right)^{\Delta} + t^2x(t) + \frac{x^3(t)}{t} = 0, \tag{21}$$

where $r_1(t) = t^5$, $r_2(t) = t^2$, $p(t) = \frac{t-1}{2t}$, $g(t) = t + 2$, $h(t) = t$, $f(t, x) = t^2x + \frac{x^3}{t}$, $t_0 = 1$.

Take $q(t) = t^2$ and $f_0(x) = x + x^3$. It is obvious that the coefficients of equation (21) satisfy (C1)-(C4), (C4A), and (C4B). Since

$$\begin{aligned}
&\int_{t_0}^{\infty} \frac{1}{r_2(t)} \Delta t = \int_1^{\infty} \frac{1}{t^2} \Delta t = M_1 < 2, \\
&H_1(t) = \int_t^{\infty} \frac{1}{r_2(\nu)} \Delta \nu = \int_t^{\infty} \frac{1}{\nu^2} \Delta \nu < 2, \\
&\lim_{t\to\infty} \frac{H_1(g(t))}{H_1(t)} = \lim_{t\to\infty} \frac{\int_{t+2}^{\infty} \frac{1}{\nu^2} \Delta \nu}{\int_t^{\infty} \frac{1}{\nu^2} \Delta \nu} = \lim_{t\to\infty} \frac{t^2}{(t+2)^2} = 1, \\
&\int_{t_0}^{\infty} \int_{t_0}^{s} \frac{q(u)}{r_1(s)} \Delta u \Delta s = \int_1^{\infty} \int_1^{s} \frac{u^2}{s^5} \Delta u \Delta s < \int_1^{\infty} \int_1^{s} \frac{1}{s^3} \Delta u \Delta s < \int_1^{\infty} \frac{1}{s^2} \Delta s < \infty,
\end{aligned}$$

and, for any $x_1, x_2 \in [0, 4]$,

$$\begin{aligned}
\bigl|f(t,x_1) - f(t,x_2)\bigr| &= \left| t^2(x_1 - x_2) + \frac{1}{t}\bigl(x_1^3 - x_2^3\bigr) \right| \\
&\le t^2\bigl(|x_1 - x_2| + \bigl|x_1^3 - x_2^3\bigr|\bigr) \le \bigl(1 + x_1^2 + x_1 x_2 + x_2^2\bigr) \cdot t^2 |x_1 - x_2| \\
&\le 49 \cdot q(t)|x_1 - x_2|,
\end{aligned}$$

by Theorem 3.1 we see that equation (21) has an eventually positive solution $x(t)$ with $\lim_{t\to\infty} x(t) = 0$, and $r_2(t)z^{\Delta}(t)$, $r_1(t)(r_2(t)z^{\Delta}(t))^{\Delta}$ are both eventually negative.

On the other hand, since $H_1(t) < 2$, we obtain

$$f\bigl(u, 2H_1\bigl(h(u)\bigr)\bigr) = u^2 \cdot 2H_1(u) + \frac{8H_1^3(u)}{u} \le u^2\bigl(2H_1(u) + 8H_1^3(u)\bigr) < 68u^2$$

and

$$\begin{aligned}
\int_{t_0}^{\infty} \int_{t_0}^{s} \frac{f(u, 2H_1(h(u)))}{r_1(s)} \Delta u \Delta s &< 68 \int_1^{\infty} \int_1^{s} \frac{u^2}{s^5} \Delta u \Delta s \\
&< 68 \int_1^{\infty} \int_1^{s} \frac{1}{s^3} \Delta u \Delta s < 68 \int_1^{\infty} \frac{1}{s^2} \Delta s < \infty.
\end{aligned}$$

By Theorem 3.7 we have the same conclusion.

Then the second example demonstrates Theorems 3.4 and 3.8.

Example 4.2 Let $\mathbb{T} = \bigcup_{n=1}^{\infty} [2^n - 1, 2^n]$. Consider

$$\left(t^2\left(t\left(\left(1 + \frac{1}{t}\right)x(t)\right)^{\Delta}\right)^{\Delta}\right)^{\Delta} + \frac{x(\frac{t}{2})}{t^2} = 0, \tag{22}$$

where $r_1(t) = t^2$, $r_2(t) = t$, $p(t) = \frac{1}{t}$, $g(t) = t$, $h(t) = \frac{t}{2}$, $f(t,x) = \frac{x}{t^2}$, $t_0 = 1$.

Take $q(t) = \frac{1}{t^2}$ and $f_0(x) = x$. It is obvious that the coefficients of equation (22) satisfy (C1)-(C4), (C4A), and (C4B). Since

$$\begin{aligned}\int_{t_0}^{\infty}\int_{v}^{\infty}\frac{1}{r_1(s)r_2(v)}\Delta s\Delta v &= \int_{1}^{\infty}\int_{v}^{\infty}\frac{1}{s^2 v}\Delta s\Delta v\\ &= \int_{1}^{\infty}\int_{1}^{s}\frac{1}{s^2 v}\Delta v\Delta s = M_2 < \infty,\\ H_2(t) = \int_{t}^{\infty}\int_{v}^{\infty}\frac{1}{r_1(s)r_2(v)}\Delta s\Delta v &= \int_{t}^{\infty}\int_{v}^{\infty}\frac{1}{s^2 v}\Delta s\Delta v \le M_2,\\ \lim_{t\to\infty}\frac{H_2(g(t))}{H_2(t)} &= \lim_{t\to\infty}\frac{H_2(t)}{H_2(t)} = 1,\\ \int_{t_0}^{\infty}q(t)\Delta t &= \int_{1}^{\infty}\frac{1}{t^2}\Delta t < \infty,\end{aligned}$$

and for any $x_1, x_2 \in [0, 2M_2]$,

$$\left|f(t,x_1) - f(t,x_2)\right| = \frac{|x_1 - x_2|}{t^2} = 1\cdot q(t)|x_1 - x_2|,$$

by Theorem 3.4 we see that equation (22) has an eventually positive solution $x(t)$ with $\lim_{t\to\infty} x(t) = 0$, $r_2(t)z^{\Delta}(t)$ is eventually negative and $r_1(t)(r_2(t)z^{\Delta}(t))^{\Delta}$ is eventually positive.

On the other hand, since $H_2(t) \le M_2$, we obtain

$$f\bigl(t, 2H_2\bigl(h(t)\bigr)\bigr) = \frac{2H_2(\frac{t}{2})}{t^2} \le \frac{2M_2}{t^2}$$

and

$$\int_{t_0}^{\infty} f\bigl(t, 2H_2\bigl(h(t)\bigr)\bigr)\Delta t \le 2M_2\int_{1}^{\infty}\frac{1}{t^2}\Delta t < \infty.$$

By Theorem 3.8 we have the same conclusion.

Competing interests

The author declares that he has no competing interests.

Acknowledgements

This project was supported by the NNSF of China (no. 11271379), the Program of Cultivation for Outstanding Young Scholars Sponsored by Guangdong Province (ZX03240302) and the Program of Cultivation for Young Scholars Sponsored by Shunde Polytechnic (2015-KJZX080).

Received: 6 October 2015 Accepted: 4 January 2016 Published online: 13 January 2016

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