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The general solution for impulsive differential equations with Hadamard fractional derivative of order $q \in (1,2)$

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Abstract

In this paper, we find formulas of general solution for a kind of impulsive differential equations with Hadamard fractional derivative of order $q \in (1,2)$ by analysis of the limit case (as the impulse tends to zero) and provide an example to illustrate the importance of our results.

MSC: 34A08; 34A37

Keywords: fractional differential equations; Hadamard fractional derivative; impulse; general solution

1 Introduction

Fractional differential equations are an excellent tool in the modeling of many phenomena in various fields of science and engineering [1–3], and the subject of fractional differential equations is gaining much attention (see [4–11] and the references therein).

Recently, Hadamard fractional derivative was studied in [12–15], and Klimek [16] studied the existence and uniqueness of the solution of a sequential fractional differential equation with Hadamard derivative by using the contraction principle and a new equivalent norm and metric. Ahmad and Ntouyas [17] studied two-dimensional fractional differential systems with Hadamard derivative. Next, Jarad *et al.* [18, 19] presented a Caputo-type modification about Hadamard fractional derivative and developed the fundamental theorem of fractional calculus in the Caputo-Hadamard setting.

Furthermore, impulsive effects exist widely in many processes in which their states can be described by impulsive differential equations, and the subject of impulsive Caputo fractional differential equations is widely studied (see [20–26]); impulsive fractional partial differential equations are also considered (see [27–32]).

Motivated by the above-mentioned works, we consider the following impulsive system with Hadamard fractional derivative:

$$\begin{cases} {}_HD^q_{a^+}u(t) = f(t,u(t)), \quad q \in (1,2), t \in (a,T] \text{ and } t \neq t_k \ (k=1,2,\ldots,m) \\ \quad \text{and } t \neq \bar{t}_l \ (l=1,2,\ldots,n), \\ \Delta({}_H\mathcal{J}^{2-q}_{a^+}u)|_{t=t_k} = {}_H\mathcal{J}^{2-q}_{a^+}u(t_k^+) - {}_H\mathcal{J}^{2-q}_{a^+}u(t_k^-) = \Delta_k(u(t_k^-)), \quad k=1,2,\ldots,m, \\ \Delta({}_HD^{q-1}_{a^+}u)|_{t=\bar{t}_l} = {}_HD^{q-1}_{a^+}u(\bar{t}_l^+) - {}_HD^{q-1}_{a^+}u(\bar{t}_l^-) = \bar{\Delta}_l(u(\bar{t}_l^-)), \quad l=1,2,\ldots,n, \\ {}_H\mathcal{J}^{2-q}_{a^+}u(a^+) = u_2, \qquad {}_HD^{q-1}_{a^+}u(a^+) = u_1, \end{cases} \tag{1.1}$$

where $a > 0$, ${}_HD^q_{a^+}$ denotes left-sided Hadamard fractional derivative of order q, $f : J \times \mathbb{R} \to \mathbb{R}$ is an appropriate continuous function, $a = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$ and $a = \bar{t}_0 < \bar{t}_1 < \cdots < \bar{t}_n < \bar{t}_{n+1} = T$, ${}_H\mathcal{J}^{2-q}_{a^+}$ denotes the left-sided Hadamard fractional integral of order $2 - q$, and ${}_H\mathcal{J}^{2-q}_{a^+}u(t_k^+) = \lim_{\varepsilon\to 0^+} {}_H\mathcal{J}^{2-q}_{a^+}u(t_k + \varepsilon)$ and ${}_H\mathcal{J}^{2-q}_{a^+}u(t_k^-) = \lim_{\varepsilon\to 0^-} {}_H\mathcal{J}^{2-q}_{a^+}u(t_k + \varepsilon)$ represent the right and left limits of ${}_H\mathcal{J}^{2-q}_{a^+}u(t)$ at $t = t_k$, respectively. The derivatives ${}_HD^{q-1}_{a^+}u(\bar{t}_l^+)$ and ${}_HD^{q-1}_{a^+}u(\bar{t}_l^-)$ have a similar meaning for ${}_HD^{q-1}_{a^+}u(t)$. Moreover, $a, t_1, t_2, \ldots, t_m, \bar{t}_1, \bar{t}_2, \ldots, \bar{t}_n, T$ is queued to $a = t'_0 < t'_1 < \cdots < t'_\Omega < t'_{\Omega+1} = T$ so that

$$\operatorname{set}\{t_1, t_2, \ldots, t_m, \bar{t}_1, \bar{t}_2, \ldots, \bar{t}_n\} = \operatorname{set}\{t'_1, t'_2, \ldots, t'_\Omega\}.$$

For each interval $[a, t'_k]$ (here $k = 1, 2, \ldots, \Omega$), suppose that $[a, t_{k_0}] \subseteq [a, t'_k] \subset [a, t_{k_0+1}]$ (here $k_0 \in \{1, 2, \ldots, m\}$) and $[a, \bar{t}_{k_1}] \subseteq [a, t'_k] \subset [a, \bar{t}_{k_1+1}]$ (here $k_1 \in \{1, 2, \ldots, n\}$), respectively.

Next, we simplify system (1.1) to obtain the following system:

$$\begin{cases} {}_HD^q_{a^+}u(t) = f(t, u(t)), \quad q \in (1,2), t \in (a, T] \text{ and } t \neq t_k \ (k = 1, 2, \ldots, m), \\ \Delta({}_H\mathcal{J}^{2-q}_{a^+}u)|_{t=t_k} = {}_H\mathcal{J}^{2-q}_{a^+}u(t_k^+) - {}_H\mathcal{J}^{2-q}_{a^+}u(t_k^-) = \Delta_k(u(t_k^-)), \quad k = 1, 2, \ldots, m, \\ \Delta({}_HD^{q-1}_{a^+}u)|_{t=t_k} = {}_HD^{q-1}_{a^+}u(t_k^+) - {}_HD^{q-1}_{a^+}u(t_k^-) = \bar{\Delta}_k(u(t_k^-)), \quad k = 1, 2, \ldots, m, \\ {}_H\mathcal{J}^{2-q}_{a^+}u(a^+) = u_2, \qquad {}_HD^{q-1}_{a^+}u(a^+) = u_1. \end{cases} \tag{1.2}$$

Let $a = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $J_0 = [a, t_1]$, and $J_k = (t_k, t_{k+1}]$ $(k = 1, 2, \ldots, m)$.

The rest of this paper is organized as follows. In Section 2, some definitions and conclusions are presented. In Section 3, we give formulas of a general solution for a kind of impulsive differential equations with Hadamard fractional derivative of order $q \in (1, 2)$. In Section 4, an example is provided to expound our results.

2 Preliminaries

In this section, we introduce some basic definitions, notation, and lemmas used in this paper.

Definition 2.1 ([2], p.110) The left-sided Hadamard fractional integral of order $q \in \mathbb{R}^+$ of a function $x(t)$ is defined by

$$\left({}_H\mathcal{J}^q_{a^+}x\right)(t) = \frac{1}{\Gamma(q)}\int_a^t \left(\ln\frac{t}{s}\right)^{q-1} x(s)\frac{ds}{s} \quad (0 < a < t \le T),$$

where $\Gamma(\cdot)$ is the gamma function.

Definition 2.2 ([2], p.111) The left-sided Hadamard fractional derivative of order $q \in [n - 1, n]$, $n \in \mathbb{Z}^+$ of a function $x(t)$ is defined by

$$\left({}_HD^q_{a^+}x\right)(t) = \frac{1}{\Gamma(n-q)}\left(t\frac{d}{dt}\right)^n \int_a^t \left(\ln\frac{t}{s}\right)^{n-q-1} x(s)\frac{ds}{s} \quad (0 < a < x \le T).$$

Lemma 2.3 ([2], Theorem 3.28) *Let $q > 0$, $n = -[-q]$, and $0 \le \gamma < 1$. Let G be an open set in $\mathbb{R}$, and $f : (a, b] \times G \to \mathbb{R}$ be a function such that $f(t, x) \in C_{\gamma\cdot\ln}[a, b]$ for any $y \in G$.*

A function $x \in C_{n-q\cdot\ln}[a,b]$ *is a solution of the fractional integral equation*

$$x(t) = \sum_{i=1}^{n} \frac{b_i}{\Gamma(q-i+1)}\left(\ln\frac{t}{a}\right)^{q-i} + \frac{1}{\Gamma(q)}\int_a^t \left(\ln\frac{t}{s}\right)^{q-1} f\bigl(s,x(s)\bigr)\frac{ds}{s} \quad (0 < a < t),$$

if and only if x is a solution of the fractional Cauchy problem

$$\begin{cases} {}_HD^q_{a^+}x(t) = f(t,x(t)), & q \in (n-1,n], t \in (a,b], \\ {}_HD^{q-i}_{a^+}x(a^+) = b_i \in \mathbb{R}, & i = 1,2,\ldots,n; n = -[-q]. \end{cases} \tag{2.1}$$

Lemma 2.4 ([2], Properties 2.26, 2.28, 2.37) *For* $q > 0$, $p > 0$, *and* $0 < a < b < \infty$, *if* $f \in C_{\gamma\cdot\ln}[a,b]$ $(0 \le \gamma < 1)$, *then* ${}_H\mathcal{J}^q_{a^+}{}_H\mathcal{J}^p_{a^+}f = {}_H\mathcal{J}^{q+p}_{a^+}f$ *and* ${}_HD^q_{a^+}{}_H\mathcal{J}^q_{a^+}f = f$.

3 Main results

For system (1.1) we have

$$\lim_{\bar\Delta_1(u(\bar t_1^-))\to 0,\ldots,\bar\Delta_n(u(\bar t_n^-))\to 0}\bigl\{\text{system (1.1)}\bigr\}$$

$$\to \begin{cases} {}_HD^q_{a^+}u(t) = f(t,u(t)), \quad q \in (1,2), t \in (a,T] \text{ and } t \neq t_k\ (k = 1,2,\ldots,m), \\ \Delta({}_H\mathcal{J}^{2-q}_{a^+}u)|_{t=t_k} = {}_H\mathcal{J}^{2-q}_{a^+}u(t_k^+) - {}_H\mathcal{J}^{2-q}_{a^+}u(t_k^-) = \Delta_k(u(t_k^-)), \\ \quad k = 1,2,\ldots,m, \\ {}_H\mathcal{J}^{2-q}_{a^+}u(a^+) = u_2, \qquad {}_HD^{q-1}_{a^+}u(a^+) = u_1. \end{cases} \tag{3.1}$$

$$\lim_{\Delta_1(u(t_1^-))\to 0,\ldots,\Delta_m(u(t_m^-))\to 0}\bigl\{\text{system (1.1)}\bigr\}$$

$$\to \begin{cases} {}_HD^q_{a^+}u(t) = f(t,u(t)), \quad q \in (1,2), t \in (a,T] \text{ and } t \neq \bar t_l\ (l = 1,2,\ldots,n), \\ \Delta({}_HD^{q-1}_{a^+}u)|_{t=\bar t_l} = {}_HD^{q-1}_{a^+}u(\bar t_l^+) - {}_HD^{q-1}_{a^+}u(\bar t_l^-) = \bar\Delta_l(u(\bar t_l^-)), \\ \quad l = 1,2,\ldots,n, \\ {}_H\mathcal{J}^{2-q}_{a^+}u(a^+) = u_2, \qquad {}_HD^{q-1}_{a^+}u(a^+) = u_1. \end{cases} \tag{3.2}$$

$$\lim_{\substack{\bar\Delta_1(u(\bar t_1^-))\to 0,\ldots,\bar\Delta_n(u(\bar t_n^-))\to 0,\\ \Delta_1(u(t_1^-))\to 0,\ldots,\Delta_m(u(t_m^-))\to 0}}\bigl\{\text{system (1.1)}\bigr\}$$

$$\to \begin{cases} {}_HD^q_{a^+}u(t) = f(t,u(t)), \quad q \in (1,2), t \in (a,T], \\ {}_HD^{q-1}_{a^+}u(a^+) = u_1, \qquad {}_H\mathcal{J}^{2-q}_{a^+}u(a^+) = u_2, \end{cases} \tag{3.3}$$

Therefore, the solution of system (1.1) satisfies the following three conditions:

(i) $$\lim_{\bar\Delta_1(u(\bar t_1^-))\to 0,\ldots,\bar\Delta_n(u(\bar t_n^-))\to 0}\bigl\{\text{the solution of system (1.1)}\bigr\} = \bigl\{\text{the solution of system (3.1)}\bigr\},$$

(ii) $$\lim_{\Delta_1(u(t_1^-))\to 0,\ldots,\Delta_m(u(t_m^-))\to 0}\bigl\{\text{the solution of system (1.1)}\bigr\} = \bigl\{\text{the solution of system (3.2)}\bigr\},$$

(iii) $$\lim_{\substack{\Delta_1(u(t_1^-))\to 0,\ldots,\Delta_m(u(t_m^-))\to 0\\ \bar\Delta_1(u(\bar t_1^-))\to 0,\ldots,\bar\Delta_n(u(\bar t_n^-))\to 0}}\bigl\{\text{the solution of system (1.1)}\bigr\} = \bigl\{\text{the solution of system (3.3)}\bigr\}$$

Therefore, we give the definition of a solution of system (1.1).

Definition 3.1 A function $z(t):[a,T]\to\mathbb{R}$ is said to be a solution of the fractional Cauchy problem (1.1) if ${}_H\mathcal{J}_{a^+}^{2-q}z(a^+)=u_2$ and ${}_HD_{a^+}^{q-1}z(a^+)=u_1$, the equation condition ${}_HD_{a^+}^{q}z(t)=f(t,z(t))$ for each $t\in(a,T]$ is satisfied, the impulsive conditions $\Delta({}_H\mathcal{J}_{a^+}^{2-q}z)|_{t=t_k}=\Delta_k(z(t_k^-))$ $(k=1,2,\ldots,m)$ and $\Delta({}_HD_{a^+}^{q-1}z)|_{t=\bar{t}_l}=\bar{\Delta}_l(z(\bar{t}_l^-))$ $(l=1,2,\ldots,n)$ are satisfied, the restriction of $z(\cdot)$ to the interval $(t'_k,t'_{k+1}]$ $(k=0,1,2,\ldots,\Omega)$ is continuous, and conditions (i)-(iii) hold.

Using the equality $\ln\frac{t}{t_k}=\int_{t_k}^t\frac{ds}{s}$ $(k=0,1,2,\ldots,m)$, define the piecewise function

$$\begin{aligned}\tilde{u}(t)={}&\frac{1}{\Gamma(q)}\bigl({}_HD_{a^+}^{q-1}u\bigl(t_k^+\bigr)\bigr)\biggl(\int_{t_k}^t\frac{ds}{s}\biggr)^{q-1}+\frac{1}{\Gamma(q-1)}\bigl({}_H\mathcal{J}_{a^+}^{2-q}u\bigl(t_k^+\bigr)\bigr)\biggl(\int_{t_k}^t\frac{ds}{s}\biggr)^{q-2}\\&+\frac{1}{\Gamma(q)}\int_{t_k}^t\biggl(\ln\frac{t}{s}\biggr)^{q-1}f\bigl(s,u(s)\bigr)\frac{ds}{s}\quad\text{for } t\in(t_k,t_{k+1}]\text{ (where } k=0,1,2,\ldots,m).\end{aligned}$$

By Definition 2.2 we have

$$\begin{aligned}&\bigl[{}_HD_{a^+}^{q}\tilde{u}(t)\bigr]_{t\in(t_k,t_{k+1}]}\\&\quad=\biggl\{\frac{1}{\Gamma(2-q)}\biggl(t\frac{d}{dt}\biggr)^2\int_a^t\biggl(\ln\frac{t}{\eta}\biggr)^{2-q-1}\biggl[\frac{1}{\Gamma(q)}\bigl({}_HD_{a^+}^{q-1}u\bigl(t_k^+\bigr)\bigr)\biggl(\ln\frac{\eta}{t_k}\biggr)^{q-1}\\&\qquad+\frac{1}{\Gamma(q-1)}\bigl({}_H\mathcal{J}_{a^+}^{2-q}u\bigl(t_k^+\bigr)\bigr)\biggl(\ln\frac{\eta}{t_k}\biggr)^{q-2}\\&\qquad+\frac{1}{\Gamma(q)}\int_{t_k}^\eta\biggl(\ln\frac{\eta}{s}\biggr)^{q-1}f\bigl(s,u(s)\bigr)\frac{ds}{s}\biggr]\frac{d\eta}{\eta}\biggr\}_{t\in(t_k,t_{k+1}]}\\&\quad=\biggl\{\frac{1}{\Gamma(2-q)}\biggl(t\frac{d}{dt}\biggr)^2\int_{t_k}^t\biggl(\ln\frac{t}{\eta}\biggr)^{2-q-1}\biggl[\frac{1}{\Gamma(q)}\bigl({}_HD_{a^+}^{q-1}u\bigl(t_k^+\bigr)\bigr)\biggl(\ln\frac{\eta}{t_k}\biggr)^{q-1}\\&\qquad+\frac{1}{\Gamma(q-1)}\bigl({}_H\mathcal{J}_{a^+}^{2-q}u\bigl(t_k^+\bigr)\bigr)\biggl(\ln\frac{\eta}{t_k}\biggr)^{q-2}\\&\qquad+\frac{1}{\Gamma(q)}\int_{t_k}^\eta\biggl(\ln\frac{\eta}{s}\biggr)^{q-1}f\bigl(s,u(s)\bigr)\frac{ds}{s}\biggr]\frac{d\eta}{\eta}\biggr\}_{t\in(t_k,t_{k+1}]}\\&\quad=\biggl\{\frac{1}{\Gamma(2-q)\Gamma(q)}\biggl(t\frac{d}{dt}\biggr)^2\int_{t_k}^t\biggl(\ln\frac{t}{\eta}\biggr)^{2-q-1}\biggl[\int_{t_k}^\eta\biggl(\ln\frac{\eta}{s}\biggr)^{q-1}f\bigl(s,u(s)\bigr)\frac{ds}{s}\biggr]\frac{d\eta}{\eta}\biggr\}_{t\in(t_k,t_{k+1}]}\\&\quad=\biggl\{\biggl(t\frac{d}{dt}\biggr)^2\biggl(\int_{t_k}^t\biggl(\ln\frac{t}{s}\biggr)f\bigl(s,u(s)\bigr)\frac{ds}{s}\biggr)\biggr\}_{t\in(t_k,t_{k+1}]}\\&\quad=f\bigl(t,u(t)\bigr)|_{t\in(t_k,t_{k+1}]}.\end{aligned}$$

So, $\tilde{u}(t)$ satisfies the condition of Hadamard fractional derivative and does not satisfy conditions (i)-(iii). Thus, we will assume that $\tilde{u}(t)$ is an approximate solution to seek the exact solution of system (1.1). First, let us prove two useful conclusions.

Lemma 3.2 *Let $q\in(1,2)$, and let λ be a constant. System* (3.1) *is equivalent to the fractional integral equation*

$$u(t)=\begin{cases}\frac{u_1}{\Gamma(q)}(\int_a^t\frac{ds}{s})^{q-1}+\frac{u_2}{\Gamma(q-1)}(\int_a^t\frac{ds}{s})^{q-2}+\frac{1}{\Gamma(q)}\int_a^t(\ln\frac{t}{s})^{q-1}f(s,u(s))\frac{ds}{s}\\ \quad \textit{for } t\in(a,t_1],\\ \frac{u_1}{\Gamma(q)}(\int_a^t\frac{ds}{s})^{q-1}+\frac{u_2}{\Gamma(q-1)}(\int_a^t\frac{ds}{s})^{q-2}+\frac{1}{\Gamma(q)}\int_a^t(\ln\frac{t}{s})^{q-1}f(s,u(s))\frac{ds}{s}\\ \quad+\sum_{i=1}^k\frac{\Delta_i(u(t_i^-))}{\Gamma(q-1)}(\int_{t_i}^t\frac{ds}{s})^{q-2}-\sum_{i=1}^k\tilde{\lambda}\Delta_i(u(t_i^-))[\frac{u_1}{\Gamma(q)}(\int_a^t\frac{ds}{s})^{q-1}\\ \quad+\frac{u_2}{\Gamma(q-1)}(\int_a^t\frac{ds}{s})^{q-2}+\frac{1}{\Gamma(q)}\int_a^t(\ln\frac{t}{s})^{q-1}f(s,u(s))\frac{ds}{s}\\ \quad-\frac{u_1+\int_a^{t_i}f(s,u(s))\frac{ds}{s}}{\Gamma(q)}(\int_{t_i}^t\frac{ds}{s})^{q-1}-\frac{u_1\ln\frac{t_i}{a}+u_2+\int_a^{t_i}(\ln\frac{t_i}{s})f(s,u(s))\frac{ds}{s}}{\Gamma(q-1)}(\int_{t_i}^t\frac{ds}{s})^{q-2}\\ \quad-\frac{1}{\Gamma(q)}\int_{t_i}^t(\ln\frac{t}{s})^{q-1}f(s,u(s))\frac{ds}{s}]\quad \textit{for } t\in(t_k,t_{k+1}]\end{cases}\tag{3.4}$$

provided that the integral in (3.4) *exists.*

Proof Necessity. We will verify that Eq. (3.4) satisfies the conditions of system (3.1).

For system (3.1), we have

$$\lim_{\Delta_1(u(t_1^-))\to0,\ldots,\Delta_m(u(t_m^-))\to0}\begin{cases}{}_HD_{a^+}^q u(t)=f(t,u(t)),\quad q\in(1,2),t\in(a,T]\text{ and}\\ \quad t\neq t_k\ (k=1,2,\ldots,m),\\ \Delta({}_H\mathcal{J}_{a^+}^{2-q}u)|_{t=t_k}={}_H\mathcal{J}_{a^+}^{2-q}u(t_k^+)-{}_H\mathcal{J}_{a^+}^{2-q}u(t_k^-)=\Delta_k(u(t_k^-)),\\ \quad k=1,2,\ldots,m,\\ {}_HD_{a^+}^{q-1}u(a^+)=u_1,\qquad {}_H\mathcal{J}_{a^+}^{2-q}u(a^+)=u_2\end{cases}$$

$$\to\begin{cases}{}_HD_{a^+}^q u(t)=f(t,u(t)),\quad q\in(1,2),t\in(a,T],\\ {}_HD_{a^+}^{q-1}u(a^+)=u_1,\qquad {}_H\mathcal{J}_{a^+}^{2-q}u(a^+)=u_2.\end{cases}$$

Therefore,

$$\begin{aligned}&\lim_{\Delta_1(u(t_1^-))\to0,\ldots,\Delta_m(u(t_m^-))\to0}\{\text{the solution of system (3.1)}\}\\ &\quad=\{\text{the solution of system (3.3)}\}.\end{aligned}\tag{3.5}$$

Moreover, we easily verify that Eq. (3.4) satisfies condition (3.5).

Next, taking the Hadamard fractional derivative of Eq. (3.4) for each $t\in(t_k,t_{k+1}]$ ($k=0,1,2,\ldots,m$), we have

$$\begin{aligned}&{}_HD_{a^+}^q u(t)\\ &\quad={}_HD_{a^+}^q\Biggl\{\frac{u_1}{\Gamma(q)}\biggl(\int_a^t\frac{ds}{s}\biggr)^{q-1}+\frac{u_2}{\Gamma(q-1)}\biggl(\int_a^t\frac{ds}{s}\biggr)^{q-2}+\frac{1}{\Gamma(q)}\int_a^t\biggl(\ln\frac{t}{s}\biggr)^{q-1}f\bigl(s,u(s)\bigr)\frac{ds}{s}\\ &\qquad+\sum_{i=1}^k\frac{\Delta_i(u(t_i^-))}{\Gamma(q-1)}\biggl(\int_{t_i}^t\frac{ds}{s}\biggr)^{q-2}\\ &\qquad-\sum_{i=1}^k\tilde{\lambda}\Delta_i\bigl(u(t_i^-)\bigr)\biggl[\frac{u_1}{\Gamma(q)}\biggl(\int_a^t\frac{ds}{s}\biggr)^{q-1}+\frac{u_2}{\Gamma(q-1)}\biggl(\int_a^t\frac{ds}{s}\biggr)^{q-2}\\ &\qquad+\frac{1}{\Gamma(q)}\int_a^t\biggl(\ln\frac{t}{s}\biggr)^{q-1}f\bigl(s,u(s)\bigr)\frac{ds}{s}-\frac{u_1+\int_a^{t_i}f(s,u(s))\frac{ds}{s}}{\Gamma(q)}\biggl(\int_{t_i}^t\frac{ds}{s}\biggr)^{q-1}\\ &\qquad-\frac{u_1\ln\frac{t_i}{a}+u_2+\int_a^{t_i}(\ln\frac{t_i}{s})f(s,u(s))\frac{ds}{s}}{\Gamma(q-1)}\biggl(\int_{t_i}^t\frac{ds}{s}\biggr)^{q-2}\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{\Gamma(q)}\int_{t_i}^{t}\left(\ln\frac{t}{s}\right)^{q-1}f\big(s,u(s)\big)\frac{ds}{s}\bigg]\bigg]\bigg\}\\
&=f\big(t,u(t)\big)_{t\geq a}|_{t\in(t_k,t_{k+1}]}-\left\{\sum_{i=1}^{k}\big(\lambda\Delta_i\big(u\big(t_i^-\big)\big)\big[f\big(t,u(t)\big)_{t\geq a}-f\big(t,u(t)\big)_{t\geq t_i}\big]\big)\right\}_{t\in(t_k,t_{k+1}]}\\
&=f\big(t,u(t)\big)|_{t\in(t_k,t_{k+1}]}.
\end{aligned}
$$

So, Eq. (3.4) satisfies the Hadamard fractional derivative of system (3.1).

Finally, for each t_k ($k = 1, 2, \ldots, m$) in Eq. (3.4), we have

$$
\begin{aligned}
&{}_H\mathcal{J}_{a^+}^{2-q}u\big(t_k^+\big)-{}_H\mathcal{J}_{a^+}^{2-q}u\big(t_k^-\big)\\
&=\left\{\frac{1}{\Gamma(2-q)}\int_a^t\left(\ln\frac{t}{\eta}\right)^{2-q-1}u(\eta)\frac{d\eta}{\eta}\right\}_{t\to t_k^+}-\left\{\frac{1}{\Gamma(2-q)}\int_a^t\left(\ln\frac{t}{\eta}\right)^{2-q-1}u(\eta)\frac{d\eta}{\eta}\right\}_{t=t_k^-}\\
&=\Delta_k\big(u\big(t_k^-\big)\big).
\end{aligned}
$$

Therefore, Eq. (3.4) satisfies the impulsive conditions of (3.1). Thus, Eq. (3.4) satisfies all conditions of system (3.1).

Sufficiency. We will prove that the solutions of system (3.1) satisfy Eq. (3.4) by mathematical induction. By Definitions 2.1 and 2.2 the solution of system (3.1) satisfies

$$
\begin{aligned}
u(t)&=\frac{u_1}{\Gamma(q)}\left(\ln\frac{t}{a}\right)^{q-1}+\frac{u_2}{\Gamma(q-1)}\left(\ln\frac{t}{a}\right)^{q-2}+\frac{1}{\Gamma(q)}\int_a^t\left(\ln\frac{t}{s}\right)^{q-1}f\big(s,u(s)\big)\frac{ds}{s}\\
&=\frac{u_1}{\Gamma(q)}\left(\int_a^t\frac{ds}{s}\right)^{q-1}+\frac{u_2}{\Gamma(q-1)}\left(\int_a^t\frac{ds}{s}\right)^{q-2}\\
&\quad+\frac{1}{\Gamma(q)}\int_a^t\left(\ln\frac{t}{s}\right)^{q-1}f\big(s,u(s)\big)\frac{ds}{s}\quad\text{for } t\in(a,t_1].
\end{aligned}
\tag{3.6}
$$

Using (3.6) and Definitions 2.1 and 2.2, we get ${}_HD_{a^+}^{q-1}u(t_1^+) = {}_HD_{a^+}^{q-1}u(t_1^-) = u_1 + \int_a^{t_1} f(s, u(s))\frac{ds}{s}$ and ${}_H\mathcal{J}_{a^+}^{2-q}u(t_1^+) = {}_H\mathcal{J}_{a^+}^{2-q}u(t_1^-) + \Delta_1(u(t_1^-)) = u_1 \ln\frac{t_1}{a} + u_2 + \int_a^{t_1}(\ln\frac{t_1}{s})f(s,u(s))\frac{ds}{s} + \Delta_1(u(t_1^-))$. Therefore, the approximate solution for $t \in (t_1, t_2]$ is provided by

$$
\begin{aligned}
\tilde{u}(t)&=\frac{1}{\Gamma(q)}\big({}_HD_{a^+}^{q-1}u\big(t_1^+\big)\big)\left(\int_{t_1}^t\frac{ds}{s}\right)^{q-1}+\frac{1}{\Gamma(q-1)}\big({}_H\mathcal{J}_{a^+}^{2-q}u\big(t_1^+\big)\big)\left(\int_{t_1}^t\frac{ds}{s}\right)^{q-2}\\
&\quad+\frac{1}{\Gamma(q)}\int_{t_1}^t\left(\ln\frac{t}{s}\right)^{q-1}f\big(s,u(s)\big)\frac{ds}{s}\\
&=\frac{u_1+\int_a^{t_1}f(s,u(s))\frac{ds}{s}}{\Gamma(q)}\left(\int_{t_1}^t\frac{ds}{s}\right)^{q-1}\\
&\quad+\frac{u_1\ln\frac{t_1}{a}+u_2+\int_a^{t_1}(\ln\frac{t_1}{s})f(s,u(s))\frac{ds}{s}+\Delta_1(u(t_1^-))}{\Gamma(q-1)}\left(\int_{t_1}^t\frac{ds}{s}\right)^{q-2}\\
&\quad+\frac{1}{\Gamma(q)}\int_{t_1}^t\left(\ln\frac{t}{s}\right)^{q-1}f\big(s,u(s)\big)\frac{ds}{s}\quad\text{for } t\in(t_1,t_2].
\end{aligned}
\tag{3.7}
$$

Let $e_1(t) = u(t) - \tilde{u}(t)$ for $t \in (t_1, t_2]$, where $u(t)$ is the exact solution of system (3.1). Due to

$$\lim_{\Delta_1(u(t_1^-))\to 0} u(t) = \frac{u_1}{\Gamma(q)}\left(\int_a^t \frac{ds}{s}\right)^{q-1} + \frac{u_2}{\Gamma(q-1)}\left(\int_a^t \frac{ds}{s}\right)^{q-2} + \frac{1}{\Gamma(q)}\int_a^t \left(\ln\frac{t}{s}\right)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s} \quad \text{for } t \in (t_1, t_2],$$

we obtain

$$\begin{aligned}
&\lim_{\Delta_1(u(t_1^-))\to 0} e_1(t) \\
&= \lim_{\Delta_1(u(t_1^-))\to 0} \bigl\{u(t) - \tilde{u}(t)\bigr\} \\
&= \frac{u_1}{\Gamma(q)}\left(\int_a^t \frac{ds}{s}\right)^{q-1} + \frac{u_2}{\Gamma(q-1)}\left(\int_a^t \frac{ds}{s}\right)^{q-2} + \frac{1}{\Gamma(q)}\int_a^t \left(\ln\frac{t}{s}\right)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s} \\
&\quad - \frac{u_1 + \int_a^{t_1} f(s,u(s))\frac{ds}{s}}{\Gamma(q)}\left(\int_{t_1}^t \frac{ds}{s}\right)^{q-1} \\
&\quad - \frac{u_1 \ln\frac{t_1}{a} + u_2 + \int_a^{t_1}(\ln\frac{t_1}{s}) f(s,u(s))\frac{ds}{s}}{\Gamma(q-1)}\left(\int_{t_1}^t \frac{ds}{s}\right)^{q-2} \\
&\quad - \frac{1}{\Gamma(q)}\int_{t_1}^t \left(\ln\frac{t}{s}\right)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s} \quad \text{for } t \in (t_1, t_2].
\end{aligned} \tag{3.8}$$

By (3.8) we assume that

$$\begin{aligned}
e_1(t) &= \sigma\bigl(\Delta_1\bigl(u\bigl(t_1^-\bigr)\bigr)\bigr) \lim_{\Delta_1\to 0} e_1(t) \\
&= \sigma\bigl(\Delta_1\bigl(u\bigl(t_1^-\bigr)\bigr)\bigr)\biggl[\frac{u_1}{\Gamma(q)}\left(\int_a^t \frac{ds}{s}\right)^{q-1} + \frac{u_2}{\Gamma(q-1)}\left(\int_a^t \frac{ds}{s}\right)^{q-2} \\
&\quad + \frac{1}{\Gamma(q)}\int_a^t \left(\ln\frac{t}{s}\right)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s} - \frac{u_1 + \int_a^{t_1} f(s,u(s))\frac{ds}{s}}{\Gamma(q)}\left(\int_{t_1}^t \frac{ds}{s}\right)^{q-1} \\
&\quad - \frac{u_1 \ln\frac{t_1}{a} + u_2 + \int_a^{t_1}(\ln\frac{t_1}{s}) f(s,u(s))\frac{ds}{s}}{\Gamma(q-1)}\left(\int_{t_1}^t \frac{ds}{s}\right)^{q-2} \\
&\quad - \frac{1}{\Gamma(q)}\int_{t_1}^t \left(\ln\frac{t}{s}\right)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s}\biggr] \quad \text{for } t \in (t_1, t_2],
\end{aligned}$$

where the function $\sigma(\cdot)$ is an undetermined function with $\sigma(0) = 1$. Then,

$$\begin{aligned}
u(t) &= \tilde{u}(t) + e_1(t) \\
&= \frac{u_1}{\Gamma(q)}\left(\int_a^t \frac{ds}{s}\right)^{q-1} + \frac{u_2}{\Gamma(q-1)}\left(\int_a^t \frac{ds}{s}\right)^{q-2} + \frac{1}{\Gamma(q)}\int_a^t \left(\ln\frac{t}{s}\right)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s} \\
&\quad + \frac{\Delta_1(u(t_1^-))}{\Gamma(q-1)}\left(\int_{t_1}^t \frac{ds}{s}\right)^{q-2} \\
&\quad - \bigl(1 - \sigma\bigl(\Delta_1\bigl(u\bigl(t_1^-\bigr)\bigr)\bigr)\bigr)\biggl[\frac{u_1}{\Gamma(q)}\left(\int_a^t \frac{ds}{s}\right)^{q-1} + \frac{u_2}{\Gamma(q-1)}\left(\int_a^t \frac{ds}{s}\right)^{q-2}
\end{aligned}$$

$$+\frac{1}{\Gamma(q)}\int_a^t\left(\ln\frac{t}{s}\right)^{q-1}f\big(s,u(s)\big)\frac{ds}{s}-\frac{u_1+\int_a^{t_1}f(s,u(s))\frac{ds}{s}}{\Gamma(q)}\left(\int_{t_1}^t\frac{ds}{s}\right)^{q-1}$$
$$-\frac{u_1\ln\frac{t_1}{a}+u_2+\int_a^{t_1}(\ln\frac{t_1}{s})f(s,u(s))\frac{ds}{s}}{\Gamma(q-1)}\left(\int_{t_1}^t\frac{ds}{s}\right)^{q-2}$$
$$-\frac{1}{\Gamma(q)}\int_{t_1}^t\left(\ln\frac{t}{s}\right)^{q-1}f\big(s,u(s)\big)\frac{ds}{s}\Bigg]\quad\text{for }t\in(t_1,t_2].\tag{3.9}$$

Using (3.9), we obtain ${}_HD_{a^+}^{q-1}u(t_2^+)={}_HD_{a^+}^{q-1}u(t_2^-)=u_1+\int_a^{t_2}f(s,u(s))\frac{ds}{s}$ and

$$\begin{aligned}{}_H\mathcal{J}_{a^+}^{2-q}u\big(t_2^+\big)&={}_H\mathcal{J}_{a^+}^{2-q}u\big(t_2^-\big)+\Delta_2\big(u\big(t_2^-\big)\big)\\&=u_1\ln\frac{t_2}{a}+u_2+\int_a^{t_2}\left(\ln\frac{t_2}{s}\right)f\big(s,u(s)\big)\frac{ds}{s}+\Delta_1\big(u\big(t_1^-\big)\big)+\Delta_2\big(u\big(t_2^-\big)\big).\end{aligned}$$

So, the approximate solution for $t\in(t_2,t_3]$ is given by

$$\begin{aligned}\tilde{u}(t)={}&\frac{1}{\Gamma(q)}\big({}_HD_{a^+}^{q-1}u\big(t_2^+\big)\big)\left(\int_{t_2}^t\frac{ds}{s}\right)^{q-1}+\frac{1}{\Gamma(q-1)}\big({}_H\mathcal{J}_{a^+}^{2-q}u\big(t_2^+\big)\big)\left(\int_{t_2}^t\frac{ds}{s}\right)^{q-2}\\&+\frac{1}{\Gamma(q)}\int_{t_2}^t\left(\ln\frac{t}{s}\right)^{q-1}f\big(s,u(s)\big)\frac{ds}{s}\\={}&\frac{u_1+\int_a^{t_2}f(s,u(s))\frac{ds}{s}}{\Gamma(q)}\left(\int_{t_2}^t\frac{ds}{s}\right)^{q-1}\\&+\frac{u_1\ln\frac{t_2}{a}+u_2+\int_a^{t_2}(\ln\frac{t_2}{s})f(s,u(s))\frac{ds}{s}+\Delta_1(u(t_1^-))+\Delta_2(u(t_2^-))}{\Gamma(q-1)}\left(\int_{t_2}^t\frac{ds}{s}\right)^{q-2}\\&+\frac{1}{\Gamma(q)}\int_{t_2}^t\left(\ln\frac{t}{s}\right)^{q-1}f\big(s,u(s)\big)\frac{ds}{s}\quad\text{for }t\in(t_2,t_3].\end{aligned}\tag{3.10}$$

Let $e_2(t)=u(t)-\tilde{u}(t)$ for $t\in(t_2,t_3]$. By (3.6) and (3.9) the exact solution $u(t)$ of system (3.1) satisfies

$$\begin{aligned}\lim_{\substack{\Delta_1(u(t_1^-))\to0,\\\Delta_2(u(t_2^-))\to0}}u(t)={}&\frac{u_1}{\Gamma(q)}\left(\int_a^t\frac{ds}{s}\right)^{q-1}+\frac{u_2}{\Gamma(q-1)}\left(\int_a^t\frac{ds}{s}\right)^{q-2}\\&+\frac{1}{\Gamma(q)}\int_a^t\left(\ln\frac{t}{s}\right)^{q-1}f\big(s,u(s)\big)\frac{ds}{s}\quad\text{for }t\in(t_2,t_3],\end{aligned}$$

$$\begin{aligned}&\lim_{\Delta_1(u(t_1^-))\to0}u(t)\\&=\frac{u_1}{\Gamma(q)}\left(\int_a^t\frac{ds}{s}\right)^{q-1}+\frac{u_2}{\Gamma(q-1)}\left(\int_a^t\frac{ds}{s}\right)^{q-2}\\&\quad+\frac{1}{\Gamma(q)}\int_a^t\left(\ln\frac{t}{s}\right)^{q-1}f\big(s,u(s)\big)\frac{ds}{s}+\frac{\Delta_2(u(t_2^-))}{\Gamma(q-1)}\left(\int_{t_2}^t\frac{ds}{s}\right)^{q-2}\\&\quad-\big(1-\sigma\big(\Delta_2\big(u\big(t_2^-\big)\big)\big)\big)\Bigg\{\frac{u_1}{\Gamma(q)}\left(\int_a^t\frac{ds}{s}\right)^{q-1}+\frac{u_2}{\Gamma(q-1)}\left(\int_a^t\frac{ds}{s}\right)^{q-2}\\&\quad+\frac{1}{\Gamma(q)}\int_a^t\left(\ln\frac{t}{s}\right)^{q-1}f\big(s,u(s)\big)\frac{ds}{s}-\frac{u_1+\int_a^{t_2}f(s,u(s))\frac{ds}{s}}{\Gamma(q)}\left(\int_{t_2}^t\frac{ds}{s}\right)^{q-1}\end{aligned}$$

$$
\begin{aligned}
&- \frac{u_1 \ln \frac{t_2}{a} + u_2 + \int_a^{t_2} (\ln \frac{t_2}{s}) f(s,u(s)) \frac{ds}{s}}{\Gamma(q-1)} \left(\int_{t_2}^t \frac{ds}{s} \right)^{q-2} \\
&- \frac{1}{\Gamma(q)} \int_{t_2}^t \left(\ln \frac{t}{s} \right)^{q-1} f\big(s,u(s)\big) \frac{ds}{s} \Bigg\} \quad \text{for } t \in (t_2, t_3], \\
\lim_{\Delta_2(u(t_2^-)) \to 0} u(t) &= \frac{u_1}{\Gamma(q)} \left(\int_a^t \frac{ds}{s} \right)^{q-1} + \frac{u_2}{\Gamma(q-1)} \left(\int_a^t \frac{ds}{s} \right)^{q-2} \\
&\quad + \frac{1}{\Gamma(q)} \int_a^t \left(\ln \frac{t}{s} \right)^{q-1} f\big(s,u(s)\big) \frac{ds}{s} + \frac{\Delta_1(u(t_1^-))}{\Gamma(q-1)} \left(\int_{t_1}^t \frac{ds}{s} \right)^{q-2} \\
&\quad - \big(1 - \sigma\big(\Delta_1\big(u\big(t_1^-\big)\big)\big)\big) \Bigg\{ \frac{u_1}{\Gamma(q)} \left(\int_a^t \frac{ds}{s} \right)^{q-1} + \frac{u_2}{\Gamma(q-1)} \left(\int_a^t \frac{ds}{s} \right)^{q-2} \\
&\quad + \frac{1}{\Gamma(q)} \int_a^t \left(\ln \frac{t}{s} \right)^{q-1} f\big(s,u(s)\big) \frac{ds}{s} - \frac{u_1 + \int_a^{t_1} f(s,u(s)) \frac{ds}{s}}{\Gamma(q)} \left(\int_{t_1}^t \frac{ds}{s} \right)^{q-1} \\
&\quad - \frac{u_1 \ln \frac{t_1}{a} + u_2 + \int_a^{t_1} (\ln \frac{t_1}{s}) f(s,u(s)) \frac{ds}{s}}{\Gamma(q-1)} \left(\int_{t_1}^t \frac{ds}{s} \right)^{q-2} \\
&\quad - \frac{1}{\Gamma(q)} \int_{t_1}^t \left(\ln \frac{t}{s} \right)^{q-1} f\big(s,u(s)\big) \frac{ds}{s} \Bigg\} \quad \text{for } t \in (t_2, t_3].
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\lim_{\substack{\Delta_1(u(t_1^-)) \to 0, \\ \Delta_2(u(t_2^-)) \to 0}} e_2(t) &= \lim_{\substack{\Delta_1(u(t_1^-)) \to 0, \\ \Delta_2(u(t_2^-)) \to 0}} \big\{ u(t) - \tilde{u}(t) \big\} \\
&= \frac{u_1}{\Gamma(q)} \left(\int_a^t \frac{ds}{s} \right)^{q-1} + \frac{u_2}{\Gamma(q-1)} \left(\int_a^t \frac{ds}{s} \right)^{q-2} \\
&\quad + \frac{1}{\Gamma(q)} \int_a^t \left(\ln \frac{t}{s} \right)^{q-1} f\big(s,u(s)\big) \frac{ds}{s} - \frac{u_1 + \int_a^{t_2} f(s,u(s)) \frac{ds}{s}}{\Gamma(q)} \left(\int_{t_2}^t \frac{ds}{s} \right)^{q-1} \\
&\quad - \frac{u_1 \ln \frac{t_2}{a} + u_2 + \int_a^{t_2} (\ln \frac{t_2}{s}) f(s,u(s)) \frac{ds}{s}}{\Gamma(q-1)} \left(\int_{t_2}^t \frac{ds}{s} \right)^{q-2} \\
&\quad - \frac{1}{\Gamma(q)} \int_{t_2}^t \left(\ln \frac{t}{s} \right)^{q-1} f\big(s,u(s)\big) \frac{ds}{s},
\end{aligned} \tag{3.11}
$$

$$
\begin{aligned}
\lim_{\Delta_1(u(t_1^-)) \to 0} e_2(t) &= \lim_{\Delta_1(u(t_1^-)) \to 0} \big\{ u(t) - \tilde{u}(t) \big\} \\
&= \sigma\big(\Delta_2\big(u\big(t_2^-\big)\big)\big) \Bigg\{ \frac{u_1}{\Gamma(q)} \left(\int_a^t \frac{ds}{s} \right)^{q-1} + \frac{u_2}{\Gamma(q-1)} \left(\int_a^t \frac{ds}{s} \right)^{q-2} \\
&\quad + \frac{1}{\Gamma(q)} \int_a^t \left(\ln \frac{t}{s} \right)^{q-1} f\big(s,u(s)\big) \frac{ds}{s} \\
&\quad - \frac{u_1 + \int_a^{t_2} f(s,u(s)) \frac{ds}{s}}{\Gamma(q)} \left(\int_{t_2}^t \frac{ds}{s} \right)^{q-1} \\
&\quad - \frac{u_1 \ln \frac{t_2}{a} + u_2 + \int_a^{t_2} (\ln \frac{t_2}{s}) f(s,u(s)) \frac{ds}{s}}{\Gamma(q-1)} \left(\int_{t_2}^t \frac{ds}{s} \right)^{q-2} \\
&\quad - \frac{1}{\Gamma(q)} \int_{t_2}^t \left(\ln \frac{t}{s} \right)^{q-1} f\big(s,u(s)\big) \frac{ds}{s} \Bigg\},
\end{aligned} \tag{3.12}
$$

$$\begin{aligned}
\lim_{\Delta_2(u(t_2^-))\to 0} e_2(t) &= \lim_{\Delta_2(u(t_2^-))\to 0}\bigl\{u(t)-\tilde{u}(t)\bigr\} \\
&= \frac{u_1}{\Gamma(q)}\left(\int_a^t \frac{ds}{s}\right)^{q-1} + \frac{u_2}{\Gamma(q-1)}\left(\int_a^t \frac{ds}{s}\right)^{q-2} \\
&\quad + \frac{1}{\Gamma(q)}\int_a^t \left(\ln\frac{t}{s}\right)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s} + \frac{\Delta_1(u(t_1^-))}{\Gamma(q-1)}\left(\int_{t_1}^t \frac{ds}{s}\right)^{q-2} \\
&\quad - \frac{\Delta_1(u(t_1^-))}{\Gamma(q-1)}\left(\int_{t_2}^t \frac{ds}{s}\right)^{q-2} - \frac{u_1 + \int_a^{t_2} f(s,u(s))\frac{ds}{s}}{\Gamma(q)}\left(\int_{t_2}^t \frac{ds}{s}\right)^{q-1} \\
&\quad - \frac{u_1\ln\frac{t_2}{a} + u_2 + \int_a^{t_2}(\ln\frac{t_2}{s})f(s,u(s))\frac{ds}{s}}{\Gamma(q-1)}\left(\int_{t_2}^t \frac{ds}{s}\right)^{q-2} \\
&\quad - \frac{1}{\Gamma(q)}\int_{t_2}^t \left(\ln\frac{t}{s}\right)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s} \\
&\quad - \bigl(1-\sigma\bigl(\Delta_1\bigl(u(t_1^-)\bigr)\bigr)\bigr)\left\{\frac{u_1}{\Gamma(q)}\left(\int_a^t \frac{ds}{s}\right)^{q-1} + \frac{u_2}{\Gamma(q-1)}\left(\int_a^t \frac{ds}{s}\right)^{q-2}\right. \\
&\quad + \frac{1}{\Gamma(q)}\int_a^t \left(\ln\frac{t}{s}\right)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s} - \frac{u_1 + \int_a^{t_1} f(s,u(s))\frac{ds}{s}}{\Gamma(q)}\left(\int_{t_1}^t \frac{ds}{s}\right)^{q-1} \\
&\quad - \frac{u_1\ln\frac{t_1}{a} + u_2 + \int_a^{t_1}(\ln\frac{t_1}{s})f(s,u(s))\frac{ds}{s}}{\Gamma(q-1)}\left(\int_{t_1}^t \frac{ds}{s}\right)^{q-2} \\
&\quad \left. - \frac{1}{\Gamma(q)}\int_{t_1}^t \left(\ln\frac{t}{s}\right)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s}\right\}. \qquad (3.13)
\end{aligned}$$

So, by (3.11)-(3.13) we obtain

$$\begin{aligned}
e_2(t) &= \sigma\bigl(\Delta_2\bigl(u(t_2^-)\bigr)\bigr)\left[\frac{u_1}{\Gamma(q)}\left(\int_a^t \frac{ds}{s}\right)^{q-1} + \frac{u_2}{\Gamma(q-1)}\left(\int_a^t \frac{ds}{s}\right)^{q-2}\right. \\
&\quad + \frac{1}{\Gamma(q)}\int_a^t \left(\ln\frac{t}{s}\right)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s} - \frac{u_1 + \int_a^{t_2} f(s,u(s))\frac{ds}{s}}{\Gamma(q)}\left(\int_{t_2}^t \frac{ds}{s}\right)^{q-1} \\
&\quad - \frac{u_1\ln\frac{t_2}{a} + u_2 + \int_a^{t_2}(\ln\frac{t_2}{s})f(s,u(s))\frac{ds}{s}}{\Gamma(q-1)}\left(\int_{t_2}^t \frac{ds}{s}\right)^{q-2} \\
&\quad \left. - \frac{1}{\Gamma(q)}\int_{t_2}^t \left(\ln\frac{t}{s}\right)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s}\right] + \frac{\Delta_1(u(t_1^-))}{\Gamma(q-1)}\left(\int_{t_1}^t \frac{ds}{s}\right)^{q-2} \\
&\quad - \frac{\Delta_1(u(t_1^-))}{\Gamma(q-1)}\left(\int_{t_2}^t \frac{ds}{s}\right)^{q-2} \\
&\quad - \bigl(1-\sigma\bigl(\Delta_1\bigl(u(t_1^-)\bigr)\bigr)\bigr)\left[\frac{u_1}{\Gamma(q)}\left(\int_a^t \frac{ds}{s}\right)^{q-1} + \frac{u_2}{\Gamma(q-1)}\left(\int_a^t \frac{ds}{s}\right)^{q-2}\right. \\
&\quad + \frac{1}{\Gamma(q)}\int_a^t \left(\ln\frac{t}{s}\right)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s} - \frac{u_1 + \int_a^{t_1} f(s,u(s))\frac{ds}{s}}{\Gamma(q)}\left(\int_{t_1}^t \frac{ds}{s}\right)^{q-1} \\
&\quad - \frac{u_1\ln\frac{t_1}{a} + u_2 + \int_a^{t_1}(\ln\frac{t_1}{s})f(s,u(s))\frac{ds}{s}}{\Gamma(q-1)}\left(\int_{t_1}^t \frac{ds}{s}\right)^{q-2} \\
&\quad \left. - \frac{1}{\Gamma(q)}\int_{t_1}^t \left(\ln\frac{t}{s}\right)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s}\right] \quad \text{for } t\in(t_2,t_3].
\end{aligned}$$

Thus,

$$
\begin{aligned}
u(t) &= \tilde{u}(t) + e_2(t) \\
&= \frac{u_1}{\Gamma(q)}\left(\int_a^t \frac{ds}{s}\right)^{q-1} + \frac{u_2}{\Gamma(q-1)}\left(\int_a^t \frac{ds}{s}\right)^{q-2} \\
&\quad + \frac{1}{\Gamma(q)}\int_a^t \left(\ln\frac{t}{s}\right)^{q-1} f\big(s,u(s)\big)\frac{ds}{s} \\
&\quad + \frac{\Delta_1(u(t_1^-))}{\Gamma(q-1)}\left(\int_{t_1}^t \frac{ds}{s}\right)^{q-2} + \frac{\Delta_2(u(t_2^-))}{\Gamma(q-1)}\left(\int_{t_2}^t \frac{ds}{s}\right)^{q-2} \\
&\quad - \big(1-\sigma\big(\Delta_1\big(u(t_1^-)\big)\big)\big)\left[\frac{u_1}{\Gamma(q)}\left(\int_a^t \frac{ds}{s}\right)^{q-1} + \frac{u_2}{\Gamma(q-1)}\left(\int_a^t \frac{ds}{s}\right)^{q-2}\right. \\
&\quad + \frac{1}{\Gamma(q)}\int_a^t \left(\ln\frac{t}{s}\right)^{q-1} f\big(s,u(s)\big)\frac{ds}{s} - \frac{u_1 + \int_a^{t_1} f(s,u(s))\frac{ds}{s}}{\Gamma(q)}\left(\int_{t_1}^t \frac{ds}{s}\right)^{q-1} \\
&\quad - \frac{u_1 \ln\frac{t_1}{a} + u_2 + \int_a^{t_1} (\ln\frac{t_1}{s}) f(s,u(s))\frac{ds}{s}}{\Gamma(q-1)}\left(\int_{t_1}^t \frac{ds}{s}\right)^{q-2} \\
&\quad \left. - \frac{1}{\Gamma(q)}\int_{t_1}^t \left(\ln\frac{t}{s}\right)^{q-1} f\big(s,u(s)\big)\frac{ds}{s}\right] \\
&\quad - \big(1-\sigma\big(\Delta_2\big(u(t_2^-)\big)\big)\big)\left[\frac{u_1}{\Gamma(q)}\left(\int_a^t \frac{ds}{s}\right)^{q-1} + \frac{u_2}{\Gamma(q-1)}\left(\int_a^t \frac{ds}{s}\right)^{q-2}\right. \\
&\quad + \frac{1}{\Gamma(q)}\int_a^t \left(\ln\frac{t}{s}\right)^{q-1} f\big(s,u(s)\big)\frac{ds}{s} - \frac{u_1 + \int_a^{t_2} f(s,u(s))\frac{ds}{s}}{\Gamma(q)}\left(\int_{t_2}^t \frac{ds}{s}\right)^{q-1} \\
&\quad - \frac{u_1 \ln\frac{t_2}{a} + u_2 + \int_a^{t_2} (\ln\frac{t_2}{s}) f(s,u(s))\frac{ds}{s}}{\Gamma(q-1)}\left(\int_{t_2}^t \frac{ds}{s}\right)^{q-2} \\
&\quad \left. - \frac{1}{\Gamma(q)}\int_{t_2}^t \left(\ln\frac{t}{s}\right)^{q-1} f\big(s,u(s)\big)\frac{ds}{s}\right] \quad \text{for } t \in (t_2, t_3].
\end{aligned}
\tag{3.14}
$$

Letting $t_2 \to t_1$, we have

$$
\lim_{t_2 \to t_1}
\begin{cases}
{}_H D_{a^+}^q u(t) = f(t,u(t)), \quad q \in (1,2), t \in (a,t_3] \text{ and } t \neq t_1 \text{ and } t \neq t_2, \\
\Delta({}_H \mathcal{J}_{a^+}^{2-q} u)|_{t=t_k} = {}_H \mathcal{J}_{a^+}^{2-q} u(t_k^+) - {}_H \mathcal{J}_{a^+}^{2-q} u(t_k^-) = \Delta_k(u(t_k^-)), \quad k = 1,2, \\
{}_H D_{a^+}^{q-1} u(a^+) = u_1, \qquad {}_H \mathcal{J}_{a^+}^{2-q} u(a^+) = u_2,
\end{cases}
\tag{3.15}
$$

$$
\to
\begin{cases}
{}_H D_{a^+}^q u(t) = f(t,u(t)), \quad q \in (0,1), t \in (a,t_3] \text{ and } t \neq t_1, \\
\Delta({}_H \mathcal{J}_{a^+}^{2-q} u)|_{t=t_1} = \Delta_1(u(t_1^-)) + \Delta_2(u(t_2^-)), \\
{}_H D_{a^+}^{q-1} u(a^+) = u_1, \qquad {}_H \mathcal{J}_{a^+}^{2-q} u(a^+) = u_2.
\end{cases}
\tag{3.16}
$$

Using (3.9) and (3.14) in systems (3.16) and (3.15), respectively, we obtain

$$
1 - \sigma(\Delta_1 + \Delta_2) = 1 - \sigma(\Delta_1) + 1 - \sigma(\Delta_2) \quad \forall \Delta_1, \Delta_2 \in \mathbb{R}.
$$

Letting $\rho(z) = 1 - \sigma(z)$ ($\forall z \in \mathbb{R}$), we have $\rho(z + w) = \rho(z) + \rho(w)$ ($\forall z, w \in \mathbb{R}$). Therefore, $\rho(z) = \lambda z$, where λ is a constant. So, by (3.9) and (3.14) we get

$$
\begin{aligned}
u(t) = {} & \frac{u_1}{\Gamma(q)}\left(\int_a^t \frac{ds}{s}\right)^{q-1} + \frac{u_2}{\Gamma(q-1)}\left(\int_a^t \frac{ds}{s}\right)^{q-2} \\
& + \frac{1}{\Gamma(q)}\int_a^t \left(\ln\frac{t}{s}\right)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s} + \frac{\Delta_1(u(t_1^-))}{\Gamma(q-1)}\left(\int_{t_1}^t \frac{ds}{s}\right)^{q-2} \\
& - \hat{\lambda}\Delta_1\bigl(u(t_1^-)\bigr)\biggl[\frac{u_1}{\Gamma(q)}\left(\int_a^t \frac{ds}{s}\right)^{q-1} + \frac{u_2}{\Gamma(q-1)}\left(\int_a^t \frac{ds}{s}\right)^{q-2} \\
& + \frac{1}{\Gamma(q)}\int_a^t \left(\ln\frac{t}{s}\right)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s} - \frac{u_1 + \int_a^{t_1} f(s,u(s))\frac{ds}{s}}{\Gamma(q)}\left(\int_{t_1}^t \frac{ds}{s}\right)^{q-1} \\
& - \frac{u_1 \ln\frac{t_1}{a} + u_2 + \int_a^{t_1}(\ln\frac{t_1}{s})f(s,u(s))\frac{ds}{s}}{\Gamma(q-1)}\left(\int_{t_1}^t \frac{ds}{s}\right)^{q-2} \\
& - \frac{1}{\Gamma(q)}\int_{t_1}^t \left(\ln\frac{t}{s}\right)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s}\biggr] \quad \text{for } t \in (t_1,t_2],
\end{aligned}
\tag{3.17}
$$

and

$$
\begin{aligned}
u(t) = {} & \frac{u_1}{\Gamma(q)}\left(\int_a^t \frac{ds}{s}\right)^{q-1} + \frac{u_2}{\Gamma(q-1)}\left(\int_a^t \frac{ds}{s}\right)^{q-2} + \frac{1}{\Gamma(q)}\int_a^t \left(\ln\frac{t}{s}\right)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s} \\
& + \frac{\Delta_1(u(t_1^-))}{\Gamma(q-1)}\left(\int_{t_1}^t \frac{ds}{s}\right)^{q-2} + \frac{\Delta_2(u(t_2^-))}{\Gamma(q-1)}\left(\int_{t_2}^t \frac{ds}{s}\right)^{q-2} \\
& - \hat{\lambda}\Delta_1\bigl(u(t_1^-)\bigr)\biggl[\frac{u_1}{\Gamma(q)}\left(\int_a^t \frac{ds}{s}\right)^{q-1} + \frac{u_2}{\Gamma(q-1)}\left(\int_a^t \frac{ds}{s}\right)^{q-2} \\
& + \frac{1}{\Gamma(q)}\int_a^t \left(\ln\frac{t}{s}\right)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s} - \frac{u_1 + \int_a^{t_1} f(s,u(s))\frac{ds}{s}}{\Gamma(q)}\left(\int_{t_1}^t \frac{ds}{s}\right)^{q-1} \\
& - \frac{u_1 \ln\frac{t_1}{a} + u_2 + \int_a^{t_1}(\ln\frac{t_1}{s})f(s,u(s))\frac{ds}{s}}{\Gamma(q-1)}\left(\int_{t_1}^t \frac{ds}{s}\right)^{q-2} \\
& - \frac{1}{\Gamma(q)}\int_{t_1}^t \left(\ln\frac{t}{s}\right)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s}\biggr] \\
& - \hat{\lambda}\Delta_2\bigl(u(t_2^-)\bigr)\biggl[\frac{u_1}{\Gamma(q)}\left(\int_a^t \frac{ds}{s}\right)^{q-1} + \frac{u_2}{\Gamma(q-1)}\left(\int_a^t \frac{ds}{s}\right)^{q-2} \\
& + \frac{1}{\Gamma(q)}\int_a^t \left(\ln\frac{t}{s}\right)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s} - \frac{u_1 + \int_a^{t_2} f(s,u(s))\frac{ds}{s}}{\Gamma(q)}\left(\int_{t_2}^t \frac{ds}{s}\right)^{q-1} \\
& - \frac{u_1 \ln\frac{t_2}{a} + u_2 + \int_a^{t_2}(\ln\frac{t_2}{s})f(s,u(s))\frac{ds}{s}}{\Gamma(q-1)}\left(\int_{t_2}^t \frac{ds}{s}\right)^{q-2} \\
& - \frac{1}{\Gamma(q)}\int_{t_2}^t \left(\ln\frac{t}{s}\right)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s}\biggr] \quad \text{for } t \in (t_2,t_3].
\end{aligned}
\tag{3.18}
$$

For $t \in (t_k, t_{k+1}]$, suppose that

$$
\begin{aligned}
u(t) = {} & \frac{u_1}{\Gamma(q)}\left(\int_a^t \frac{ds}{s}\right)^{q-1} + \frac{u_2}{\Gamma(q-1)}\left(\int_a^t \frac{ds}{s}\right)^{q-2} + \frac{1}{\Gamma(q)}\int_a^t \left(\ln\frac{t}{s}\right)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s} \\
& + \sum_{i=1}^{k} \frac{\Delta_i(u(t_i^-))}{\Gamma(q-1)}\left(\int_{t_i}^t \frac{ds}{s}\right)^{q-2}
\end{aligned}
$$

$$
\begin{aligned}
&-\sum_{i=1}^{k}\tilde{\lambda}\Delta_i\big(u\big(t_i^-\big)\big)\bigg[\frac{u_1}{\Gamma(q)}\bigg(\int_a^t\frac{ds}{s}\bigg)^{q-1}+\frac{u_2}{\Gamma(q-1)}\bigg(\int_a^t\frac{ds}{s}\bigg)^{q-2}\\
&+\frac{1}{\Gamma(q)}\int_a^t\bigg(\ln\frac{t}{s}\bigg)^{q-1}f\big(s,u(s)\big)\frac{ds}{s}-\frac{u_1+\int_a^{t_i}f(s,u(s))\frac{ds}{s}}{\Gamma(q)}\bigg(\int_{t_i}^t\frac{ds}{s}\bigg)^{q-1}\\
&-\frac{u_1\ln\frac{t_i}{a}+u_2+\int_a^{t_i}(\ln\frac{t_i}{s})f(s,u(s))\frac{ds}{s}}{\Gamma(q-1)}\bigg(\int_{t_i}^t\frac{ds}{s}\bigg)^{q-2}\\
&-\frac{1}{\Gamma(q)}\int_{t_i}^t\bigg(\ln\frac{t}{s}\bigg)^{q-1}f\big(s,u(s)\big)\frac{ds}{s}\bigg]\quad\text{for } t\in(t_k,t_{k+1}].
\end{aligned}
\tag{3.19}
$$

Using (3.19), we obtain ${}_HD_{a^+}^{q-1}u(t_{k+1}^+)={}_HD_{a^+}^{q-1}u(t_{k+1}^-)=u_1+\int_a^{t_{k+1}}f(s,u(s))\frac{ds}{s}$ and

$$
\begin{aligned}
{}_H\mathcal{J}_{a^+}^{2-q}u\big(t_{k+1}^+\big)&={}_H\mathcal{J}_{a^+}^{2-q}u\big(t_{k+1}^-\big)+\Delta_{k+1}\big(u\big(t_{k+1}^-\big)\big)\\
&=u_1\ln\frac{t_{k+1}}{a}+u_2+\int_a^{t_{k+1}}\bigg(\ln\frac{t_{k+1}}{s}\bigg)f\big(s,u(s)\big)\frac{ds}{s}+\sum_{i=1}^{k+1}\Delta_i\big(u\big(t_i^-\big)\big).
\end{aligned}
$$

So, the approximate solution for $t\in(t_{k+1},t_{k+2}]$ is provided by

$$
\begin{aligned}
\tilde{u}(t)={}&\frac{1}{\Gamma(q)}\big({}_HD_{a^+}^{q-1}u\big(t_{k+1}^+\big)\big)\bigg(\int_{t_{k+1}}^t\frac{ds}{s}\bigg)^{q-1}+\frac{1}{\Gamma(q-1)}\big({}_H\mathcal{J}_{a^+}^{2-q}u\big(t_{k+1}^+\big)\big)\bigg(\int_{t_{k+1}}^t\frac{ds}{s}\bigg)^{q-2}\\
&+\frac{1}{\Gamma(q)}\int_{t_{k+1}}^t\bigg(\ln\frac{t}{s}\bigg)^{q-1}f\big(s,u(s)\big)\frac{ds}{s}\\
={}&\frac{u_1+\int_a^{k+1}f(s,u(s))\frac{ds}{s}}{\Gamma(q)}\bigg(\int_{t_{k+1}}^t\frac{ds}{s}\bigg)^{q-1}\\
&+\frac{u_1\ln\frac{t_{k+1}}{a}+u_2+\int_a^{t_{k+1}}(\ln\frac{t_{k+1}}{s})f(s,u(s))\frac{ds}{s}+\sum_{i=1}^{k+1}\Delta_i(u(t_i^-))}{\Gamma(q-1)}\bigg(\int_{t_{k+1}}^t\frac{ds}{s}\bigg)^{q-2}\\
&+\frac{1}{\Gamma(q)}\int_{t_{k+1}}^t\bigg(\ln\frac{t}{s}\bigg)^{q-1}f\big(s,u(s)\big)\frac{ds}{s}\quad\text{for } t\in(t_{k+1},t_{k+2}].
\end{aligned}
\tag{3.20}
$$

Let $e_{k+1}(t)=u(t)-\tilde{u}(t)$ for $t\in(t_{k+1},t_{k+2}]$. By (3.19) the exact solution $u(t)$ of system (3.1) satisfies

$$
\begin{aligned}
\lim_{\Delta_1(u(t_1^-))\to0,\ldots,\Delta_{k+1}(u(t_{k+1}^-))\to0}u(t)={}&\frac{u_1}{\Gamma(q)}\bigg(\int_a^t\frac{ds}{s}\bigg)^{q-1}+\frac{u_2}{\Gamma(q-1)}\bigg(\int_a^t\frac{ds}{s}\bigg)^{q-2}\\
&+\frac{1}{\Gamma(q)}\int_a^t\bigg(\ln\frac{t}{s}\bigg)^{q-1}f\big(s,u(s)\big)\frac{ds}{s}\quad\text{for } t\in(t_{k+1},t_{k+2}],
\end{aligned}
$$

$$
\begin{aligned}
&\lim_{\Delta_j(u(t_j^-))\to0}u(t)\\
&=\frac{u_1}{\Gamma(q)}\bigg(\int_a^t\frac{ds}{s}\bigg)^{q-1}+\frac{u_2}{\Gamma(q-1)}\bigg(\int_a^t\frac{ds}{s}\bigg)^{q-2}+\frac{1}{\Gamma(q)}\int_a^t\bigg(\ln\frac{t}{s}\bigg)^{q-1}f\big(s,u(s)\big)\frac{ds}{s}\\
&\quad+\sum_{\substack{1\le i\le k+1,\\ \text{and } i\neq j}}\frac{\Delta_i(u(t_i^-))}{\Gamma(q-1)}\bigg(\int_{t_i}^t\frac{ds}{s}\bigg)^{q-2}
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{\substack{1\le i\le k+1,\\ \text{and } i\neq j}} \lambda \Delta_i\big(u(t_i^-)\big)\bigg[\frac{u_1}{\Gamma(q)}\bigg(\int_a^t \frac{ds}{s}\bigg)^{q-1} + \frac{u_2}{\Gamma(q-1)}\bigg(\int_a^t \frac{ds}{s}\bigg)^{q-2} \\
&+ \frac{1}{\Gamma(q)}\int_a^t \bigg(\ln\frac{t}{s}\bigg)^{q-1} f\big(s,u(s)\big)\frac{ds}{s} - \frac{u_1 + \int_a^{t_i} f(s,u(s))\frac{ds}{s}}{\Gamma(q)}\bigg(\int_{t_i}^t \frac{ds}{s}\bigg)^{q-1} \\
&- \frac{u_1 \ln\frac{t_i}{a} + u_2 + \int_a^{t_i}(\ln\frac{t_i}{s})f(s,u(s))\frac{ds}{s}}{\Gamma(q-1)}\bigg(\int_{t_i}^t \frac{ds}{s}\bigg)^{q-2} \\
&- \frac{1}{\Gamma(q)}\int_{t_i}^t \bigg(\ln\frac{t}{s}\bigg)^{q-1} f\big(s,u(s)\big)\frac{ds}{s}\bigg] \quad \text{for } t\in(t_{k+1},t_{k+2}] \text{ and } 1\le j\le k+1.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\lim_{\Delta_j(u(t_j^-))\to 0} e_{k+1}(t) \\
&\quad= \lim_{\Delta_j(u(t_j^-))\to 0}\big\{u(t)-\tilde{u}(t)\big\} \\
&\quad= \sum_{\substack{1\le i\le k+1,\\ \text{and } i\neq j}} \frac{\Delta_i(u(t_i^-))}{\Gamma(q-1)}\bigg[\bigg(\int_{t_i}^t \frac{ds}{s}\bigg)^{q-2} - \bigg(\int_{t_{k+1}}^t \frac{ds}{s}\bigg)^{q-2}\bigg] \\
&\quad+ \bigg\{\frac{u_1}{\Gamma(q)}\bigg(\int_a^t \frac{ds}{s}\bigg)^{q-1} + \frac{u_2}{\Gamma(q-1)}\bigg(\int_a^t \frac{ds}{s}\bigg)^{q-2} \\
&\quad+ \frac{1}{\Gamma(q)}\int_a^t \bigg(\ln\frac{t}{s}\bigg)^{q-1} f\big(s,u(s)\big)\frac{ds}{s} - \frac{u_1 + \int_a^{k+1} f(s,u(s))\frac{ds}{s}}{\Gamma(q)}\bigg(\int_{t_{k+1}}^t \frac{ds}{s}\bigg)^{q-1} \\
&\quad- \frac{u_1 \ln\frac{t_{k+1}}{a} + u_2 + \int_a^{t_{k+1}}(\ln\frac{t_{k+1}}{s})f(s,u(s))\frac{ds}{s}}{\Gamma(q-1)}\bigg(\int_{t_{k+1}}^t \frac{ds}{s}\bigg)^{q-2} \\
&\quad- \frac{1}{\Gamma(q)}\int_{t_{k+1}}^t \bigg(\ln\frac{t}{s}\bigg)^{q-1} f\big(s,u(s)\big)\frac{ds}{s}\bigg\} \\
&\quad- \sum_{\substack{1\le i\le k+1,\\ \text{and } i\neq j}} \lambda \Delta_i\big(u(t_i^-)\big)\bigg[\frac{u_1}{\Gamma(q)}\bigg(\int_a^t \frac{ds}{s}\bigg)^{q-1} + \frac{u_2}{\Gamma(q-1)}\bigg(\int_a^t \frac{ds}{s}\bigg)^{q-2} \\
&\quad+ \frac{1}{\Gamma(q)}\int_a^t \bigg(\ln\frac{t}{s}\bigg)^{q-1} f\big(s,u(s)\big)\frac{ds}{s} - \frac{u_1 + \int_a^{t_i} f(s,u(s))\frac{ds}{s}}{\Gamma(q)}\bigg(\int_{t_i}^t \frac{ds}{s}\bigg)^{q-1} \\
&\quad- \frac{u_1 \ln\frac{t_i}{a} + u_2 + \int_a^{t_i}(\ln\frac{t_i}{s})f(s,u(s))\frac{ds}{s}}{\Gamma(q-1)}\bigg(\int_{t_i}^t \frac{ds}{s}\bigg)^{q-2} \\
&\quad- \frac{1}{\Gamma(q)}\int_{t_i}^t \bigg(\ln\frac{t}{s}\bigg)^{q-1} f\big(s,u(s)\big)\frac{ds}{s}\bigg] \quad \text{for } t\in(t_{k+1},t_{k+2}],
\end{aligned}
\tag{3.21}
$$

$$
\begin{aligned}
&\lim_{\Delta_1(u(t_1^-))\to 0,\dots,\Delta_{k+1}(u(t_{k+1}^-))\to 0} e_{k+1}(t) \\
&\quad= \lim_{\Delta_1(u(t_1^-))\to 0,\dots,\Delta_{k+1}(u(t_{k+1}^-))\to 0}\big\{u(t)-\tilde{u}(t)\big\} \\
&\quad= \frac{u_1}{\Gamma(q)}\bigg(\int_a^t \frac{ds}{s}\bigg)^{q-1} + \frac{u_2}{\Gamma(q-1)}\bigg(\int_a^t \frac{ds}{s}\bigg)^{q-2}
\end{aligned}
$$

$$\begin{aligned}&+\frac{1}{\Gamma(q)}\int_a^t\left(\ln\frac{t}{s}\right)^{q-1}f\big(s,u(s)\big)\frac{ds}{s}-\frac{u_1+\int_a^{k+1}f(s,u(s))\frac{ds}{s}}{\Gamma(q)}\left(\int_{t_{k+1}}^t\frac{ds}{s}\right)^{q-1}\\&-\frac{u_1\ln\frac{t_{k+1}}{a}+u_2+\int_a^{t_{k+1}}(\ln\frac{t_{k+1}}{s})f(s,u(s))\frac{ds}{s}}{\Gamma(q-1)}\left(\int_{t_{k+1}}^t\frac{ds}{s}\right)^{q-2}\\&-\frac{1}{\Gamma(q)}\int_{t_{k+1}}^t\left(\ln\frac{t}{s}\right)^{q-1}f\big(s,u(s)\big)\frac{ds}{s}\quad\text{for } t\in(t_{k+1},t_{k+2}].\end{aligned}\tag{3.22}$$

So, by (3.21) and (3.22) we obtain

$$\begin{aligned}e_{k+1}(t)={}&\sum_{i=1}^{k+1}\frac{\Delta_i(u(t_i^-))}{\Gamma(q-1)}\left[\left(\int_{t_i}^t\frac{ds}{s}\right)^{q-2}-\left(\int_{t_{k+1}}^t\frac{ds}{s}\right)^{q-2}\right]\\&+\frac{u_1}{\Gamma(q)}\left(\int_a^t\frac{ds}{s}\right)^{q-1}+\frac{u_2}{\Gamma(q-1)}\left(\int_a^t\frac{ds}{s}\right)^{q-2}\\&+\frac{1}{\Gamma(q)}\int_a^t\left(\ln\frac{t}{s}\right)^{q-1}f\big(s,u(s)\big)\frac{ds}{s}-\frac{u_1+\int_a^{k+1}f(s,u(s))\frac{ds}{s}}{\Gamma(q)}\left(\int_{t_{k+1}}^t\frac{ds}{s}\right)^{q-1}\\&-\frac{u_1\ln\frac{t_{k+1}}{a}+u_2+\int_a^{t_{k+1}}(\ln\frac{t_{k+1}}{s})f(s,u(s))\frac{ds}{s}}{\Gamma(q-1)}\left(\int_{t_{k+1}}^t\frac{ds}{s}\right)^{q-2}\\&-\frac{1}{\Gamma(q)}\int_{t_{k+1}}^t\left(\ln\frac{t}{s}\right)^{q-1}f\big(s,u(s)\big)\frac{ds}{s}\\&-\sum_{i=1}^{k+1}\lambda\Delta_i\big(u(t_i^-)\big)\left[\frac{u_1}{\Gamma(q)}\left(\int_a^t\frac{ds}{s}\right)^{q-1}+\frac{u_2}{\Gamma(q-1)}\left(\int_a^t\frac{ds}{s}\right)^{q-2}\right.\\&+\frac{1}{\Gamma(q)}\int_a^t\left(\ln\frac{t}{s}\right)^{q-1}f\big(s,u(s)\big)\frac{ds}{s}-\frac{u_1+\int_a^{t_i}f(s,u(s))\frac{ds}{s}}{\Gamma(q)}\left(\int_{t_i}^t\frac{ds}{s}\right)^{q-1}\\&-\frac{u_1\ln\frac{t_i}{a}+u_2+\int_a^{t_i}(\ln\frac{t_i}{s})f(s,u(s))\frac{ds}{s}}{\Gamma(q-1)}\left(\int_{t_i}^t\frac{ds}{s}\right)^{q-2}\\&\left.-\frac{1}{\Gamma(q)}\int_{t_i}^t\left(\ln\frac{t}{s}\right)^{q-1}f\big(s,u(s)\big)\frac{ds}{s}\right]\quad\text{for } t\in(t_{k+1},t_{k+2}].\end{aligned}$$

Thus,

$$\begin{aligned}u(t)={}&\tilde{u}(t)+e_{k+1}(t)\\={}&\frac{u_1}{\Gamma(q)}\left(\int_a^t\frac{ds}{s}\right)^{q-1}+\frac{u_2}{\Gamma(q-1)}\left(\int_a^t\frac{ds}{s}\right)^{q-2}\\&+\frac{1}{\Gamma(q)}\int_a^t\left(\ln\frac{t}{s}\right)^{q-1}f\big(s,u(s)\big)\frac{ds}{s}+\sum_{i=1}^{k+1}\frac{\Delta_i(u(t_i^-))}{\Gamma(q-1)}\left(\int_{t_i}^t\frac{ds}{s}\right)^{q-2}\\&-\sum_{i=1}^{k+1}\lambda\Delta_i\big(u(t_i^-)\big)\left[\frac{u_1}{\Gamma(q)}\left(\int_a^t\frac{ds}{s}\right)^{q-1}+\frac{u_2}{\Gamma(q-1)}\left(\int_a^t\frac{ds}{s}\right)^{q-2}\right.\\&+\frac{1}{\Gamma(q)}\int_a^t\left(\ln\frac{t}{s}\right)^{q-1}f\big(s,u(s)\big)\frac{ds}{s}-\frac{u_1+\int_a^{t_i}f(s,u(s))\frac{ds}{s}}{\Gamma(q)}\left(\int_{t_i}^t\frac{ds}{s}\right)^{q-1}\end{aligned}$$

$$- \frac{u_1 \ln \frac{t_i}{a} + u_2 + \int_a^{t_i} (\ln \frac{t_i}{s}) f(s,u(s)) \frac{ds}{s}}{\Gamma(q-1)} \left(\int_{t_i}^t \frac{ds}{s} \right)^{q-2}$$
$$- \frac{1}{\Gamma(q)} \int_{t_i}^t \left(\ln \frac{t}{s} \right)^{q-1} f\bigl(s,u(s)\bigr) \frac{ds}{s} \Bigg] \quad \text{for } t \in (t_{k+1}, t_{k+2}].$$

So, the solution of system (3.1) satisfies Eq. (3.4).

So, system (3.1) is equivalent to Eq. (3.4). The proof is now completed. □

Lemma 3.3 *Let $q \in (1,2)$, and let $\hbar$ be a constant. System* (3.2) *is equivalent to the fractional integral equation*

$$u(t) = \begin{cases} \frac{u_1}{\Gamma(q)}(\int_a^t \frac{ds}{s})^{q-1} + \frac{u_2}{\Gamma(q-1)}(\int_a^t \frac{ds}{s})^{q-2} \\ \quad + \frac{1}{\Gamma(q)} \int_a^t (\ln \frac{t}{s})^{q-1} f(s,u(s)) \frac{ds}{s} \quad \text{for } t \in (a, \bar{t}_1], \\ \frac{u_1}{\Gamma(q)}(\int_a^t \frac{ds}{s})^{q-1} + \frac{u_2}{\Gamma(q-1)}(\int_a^t \frac{ds}{s})^{q-2} + \frac{1}{\Gamma(q)} \int_a^t (\ln \frac{t}{s})^{q-1} f(s,u(s)) \frac{ds}{s} \\ \quad + \sum_{j=1}^l \frac{\bar{\Delta}_j(u(\bar{t}_j^-))}{\Gamma(q)} (\int_{\bar{t}_j}^t \frac{ds}{s})^{q-1} - \sum_{j=1}^l \hbar \bar{\Delta}_j(u(\bar{t}_j^-)) [\frac{u_1}{\Gamma(q)}(\int_a^t \frac{ds}{s})^{q-1} \\ \quad + \frac{u_2}{\Gamma(q-1)}(\int_a^t \frac{ds}{s})^{q-2} + \frac{1}{\Gamma(q)} \int_a^t (\ln \frac{t}{s})^{q-1} f(s,u(s)) \frac{ds}{s} \\ \quad - \frac{u_1 + \int_a^{\bar{t}_j} f(s,u(s)) \frac{ds}{s}}{\Gamma(q)} (\int_{\bar{t}_j}^t \frac{ds}{s})^{q-1} - \frac{u_1 \ln \frac{\bar{t}_j}{a} + u_2 + \int_a^{\bar{t}_j} (\ln \frac{\bar{t}_j}{s}) f(s,u(s)) \frac{ds}{s}}{\Gamma(q-1)} (\int_{\bar{t}_j}^t \frac{ds}{s})^{q-2} \\ \quad - \frac{1}{\Gamma(q)} \int_{\bar{t}_j}^t (\ln \frac{t}{s})^{q-1} f(s,u(s)) \frac{ds}{s}] \quad \text{for } t \in (\bar{t}_l, \bar{t}_{l+1}], 1 \le l \le n \end{cases} \tag{3.23}$$

provided that the integral in (3.23) *exists.*

Proof Necessity. We will verify that Eq. (3.23) satisfies the conditions of system (3.2).

For system (3.2), there exists an implicit condition

$$\lim_{\bar{\Delta}_1(u(\bar{t}_1^-)) \to 0, \ldots, \bar{\Delta}_n(u(\bar{t}_n^-)) \to 0} \begin{cases} {}_H D_{a^+}^q u(t) = f(t,u(t)), \quad q \in (1,2), t \in (a,T] \text{ and} \\ \quad t \neq \bar{t}_l \ (l = 1,2,\ldots,n), \\ \Delta({}_H D_{a^+}^{q-1} u)|_{t=\bar{t}_l} = {}_H D_{a^+}^{q-1} u(\bar{t}_l^+) - {}_H D_{a^+}^{q-1} u(\bar{t}_l^-) = \bar{\Delta}_l(u(\bar{t}_l^-)), \\ \quad l = 1,2,\ldots,n, \\ {}_H D_{a^+}^{q-1} u(a^+) = u_1, \qquad {}_H \mathcal{J}_{a^+}^{2-q} u(a^+) = u_2, \end{cases}$$
$$\to \begin{cases} {}_H D_{a^+}^q u(t) = f(t,u(t)), \quad q \in (1,2), t \in (a,T], \\ {}_H D_{a^+}^{q-1} u(a^+) = u_1, \qquad {}_H \mathcal{J}_{a^+}^{2-q} u(a^+) = u_2, \end{cases}$$

that is,

$$\lim_{\bar{\Delta}_1(u(\bar{t}_1^-)) \to 0, \ldots, \bar{\Delta}_n(u(\bar{t}_n^-)) \to 0} \bigl\{\text{the solution of system (3.2)}\bigr\}$$
$$= \bigl\{\text{the solution of system (3.3)}\bigr\}. \tag{3.24}$$

Obviously, Eq. (3.23) satisfies condition (3.24).

Next, taking the Hadamard fractional derivative to Eq. (3.23) for each $t \in (\bar{t}_l, \bar{t}_{l+1}]$ ($l = 0,1,2,\ldots,n$), we have

$${}_H D_{a^+}^q u(t) = {}_H D_{a^+}^q \Bigg\{ \frac{u_1}{\Gamma(q)} \left(\int_a^t \frac{ds}{s} \right)^{q-1} + \frac{u_2}{\Gamma(q-1)} \left(\int_a^t \frac{ds}{s} \right)^{q-2}$$
$$+ \frac{1}{\Gamma(q)} \int_a^t \left(\ln \frac{t}{s} \right)^{q-1} f\bigl(s,u(s)\bigr) \frac{ds}{s} + \sum_{j=1}^l \frac{\bar{\Delta}_j(u(\bar{t}_j^-))}{\Gamma(q)} \left(\int_{\bar{t}_j}^t \frac{ds}{s} \right)^{q-1}$$

$$
\begin{aligned}
&-\sum_{j=1}^{l}\hbar\bar{\Delta}_j\bigl(u(\bar{t}_j^-)\bigr)\biggl[\frac{u_1}{\Gamma(q)}\biggl(\int_a^t\frac{ds}{s}\biggr)^{q-1}+\frac{u_2}{\Gamma(q-1)}\biggl(\int_a^t\frac{ds}{s}\biggr)^{q-2}\\
&+\frac{1}{\Gamma(q)}\int_a^t\biggl(\ln\frac{t}{s}\biggr)^{q-1}f\bigl(s,u(s)\bigr)\frac{ds}{s}-\frac{u_1+\int_a^{\bar{t}_j}f(s,u(s))\frac{ds}{s}}{\Gamma(q)}\biggl(\int_{\bar{t}_j}^t\frac{ds}{s}\biggr)^{q-1}\\
&-\frac{u_1\ln\frac{\bar{t}_j}{a}+u_2+\int_a^{\bar{t}_j}(\ln\frac{\bar{t}_j}{s})f(s,u(s))\frac{ds}{s}}{\Gamma(q-1)}\biggl(\int_{\bar{t}_j}^t\frac{ds}{s}\biggr)^{q-2}\\
&-\frac{1}{\Gamma(q)}\int_{\bar{t}_j}^t\biggl(\ln\frac{t}{s}\biggr)^{q-1}f\bigl(s,u(s)\bigr)\frac{ds}{s}\biggr]\biggr\}\\
=&f\bigl(t,u(t)\bigr)_{t\geq a}|_{t\in(\bar{t}_l,\bar{t}_{l+1}]}\\
&-\biggl\{\sum_{j=1}^{l}\hbar\bar{\Delta}_j\bigl(u(\bar{t}_j^-)\bigr)\bigl[f\bigl(t,u(t)\bigr)_{t\geq a}-f\bigl(t,u(t)\bigr)_{t\geq\bar{t}_j}\bigr]\biggr\}_{t\in(\bar{t}_l,\bar{t}_{l+1}]}\\
=&f\bigl(t,u(t)\bigr)|_{t\in(\bar{t}_l,\bar{t}_{l+1}]}.
\end{aligned}
$$

So, Eq. (3.23) satisfies the Hadamard fractional derivative of system (3.2).

Finally, for each $\bar{t}_l$ ($l=1,2,\ldots,n$) in Eq. (3.23), we have

$$
\begin{aligned}
&{}_HD_{a^+}^{q-1}u(\bar{t}_l^+)-{}_HD_{a^+}^{q-1}u(\bar{t}_l^-)\\
&=\biggl\{\frac{1}{\Gamma(2-q)}\biggl(t\frac{d}{dt}\biggr)\int_a^t\biggl(\ln\frac{t}{\eta}\biggr)^{1-q}u(\eta)\frac{d\eta}{\eta}\biggr\}_{t\to\bar{t}_l^+}\\
&\quad-\biggl\{\frac{1}{\Gamma(2-q)}\biggl(t\frac{d}{dt}\biggr)\int_a^t\biggl(\ln\frac{t}{\eta}\biggr)^{1-q}u(\eta)\frac{d\eta}{\eta}\biggr\}_{t=\bar{t}_l^-}\\
&=\bar{\Delta}_l\bigl(u(\bar{t}_l^-)\bigr).
\end{aligned}
$$

Therefore, Eq. (3.23) satisfies the impulsive conditions of (3.2). Thus, Eq. (3.23) satisfies all conditions of (3.2).

Sufficiency. We will prove that the solutions of system (3.2) satisfy Eq. (3.23) by mathematical induction. For $t\in(a,t_1]$, by Definitions 2.1 and 2.2, the solution of system (3.2) satisfies

$$
\begin{aligned}
u(t)&=\frac{u_1}{\Gamma(q)}\biggl(\ln\frac{t}{a}\biggr)^{q-1}+\frac{u_2}{\Gamma(q-1)}\biggl(\ln\frac{t}{a}\biggr)^{q-2}+\frac{1}{\Gamma(q)}\int_a^t\biggl(\ln\frac{t}{s}\biggr)^{q-1}f\bigl(s,u(s)\bigr)\frac{ds}{s}\\
&=\frac{u_1}{\Gamma(q)}\biggl(\int_a^t\frac{ds}{s}\biggr)^{q-1}+\frac{u_2}{\Gamma(q-1)}\biggl(\int_a^t\frac{ds}{s}\biggr)^{q-2}\\
&\quad+\frac{1}{\Gamma(q)}\int_a^t\biggl(\ln\frac{t}{s}\biggr)^{q-1}f\bigl(s,u(s)\bigr)\frac{ds}{s}\quad\text{for } t\in(a,\bar{t}_1].
\end{aligned}
\tag{3.25}
$$

By (3.25) and Definitions 2.1 and 2.2 we have

$$
{}_HD_{a^+}^{q-1}u(\bar{t}_1^+)={}_HD_{a^+}^{q-1}u(\bar{t}_1^-)=u_1+\int_a^{\bar{t}_1}f\bigl(s,u(s)\bigr)\frac{ds}{s}+\bar{\Delta}_1\bigl(u(\bar{t}_1^-)\bigr)
$$

and

$$ {}_H\mathcal{J}_{a^+}^{2-q}u(\bar{t}_1^+) = {}_H\mathcal{J}_{a^+}^{2-q}u(\bar{t}_1^-) = u_1\left(\int_a^{\bar{t}_1}\frac{ds}{s}\right) + u_2 + \int_a^{\bar{t}_1}\left(\ln\frac{\bar{t}_1}{s}\right)f(s,u(s))\frac{ds}{s}. $$

Therefore, the approximate solution for $t \in (\bar{t}_1, \bar{t}_2]$ is provided by

$$ \begin{aligned} \bar{u}(t) &= \frac{1}{\Gamma(q)}\left({}_HD_{a^+}^{q-1}u(\bar{t}_1^+)\right)\left(\int_{\bar{t}_1}^t\frac{ds}{s}\right)^{q-1} + \frac{1}{\Gamma(q-1)}\left({}_H\mathcal{J}_{a^+}^{2-q}u(\bar{t}_1^+)\right)\left(\int_{\bar{t}_1}^t\frac{ds}{s}\right)^{q-2} \\ &\quad + \frac{1}{\Gamma(q)}\int_{\bar{t}_1}^t\left(\ln\frac{t}{s}\right)^{q-1}f(s,u(s))\frac{ds}{s} \\ &= \frac{u_1 + \int_a^{\bar{t}_1}f(s,u(s))\frac{ds}{s} + \bar{\Delta}_1(u(\bar{t}_1^-))}{\Gamma(q)}\left(\int_{\bar{t}_1}^t\frac{ds}{s}\right)^{q-1} \\ &\quad + \frac{u_1\ln\frac{\bar{t}_1}{a} + u_2 + \int_a^{\bar{t}_1}(\ln\frac{\bar{t}_1}{s})f(s,u(s))\frac{ds}{s}}{\Gamma(q-1)}\left(\int_{\bar{t}_1}^t\frac{ds}{s}\right)^{q-2} \\ &\quad + \frac{1}{\Gamma(q)}\int_{\bar{t}_1}^t\left(\ln\frac{t}{s}\right)^{q-1}f(s,u(s))\frac{ds}{s} \quad \text{for } t \in (\bar{t}_1, \bar{t}_2]. \end{aligned} \tag{3.26} $$

Let $\bar{e}_1(t) = u(t) - \bar{u}(t)$ for $t \in (\bar{t}_1, \bar{t}_2]$. Due to

$$ \begin{aligned} \lim_{\bar{\Delta}_1(u(\bar{t}_1^-))\to 0} u(t) &= \frac{u_1}{\Gamma(q)}\left(\int_a^t\frac{ds}{s}\right)^{q-1} + \frac{u_2}{\Gamma(q-1)}\left(\int_a^t\frac{ds}{s}\right)^{q-2} \\ &\quad + \frac{1}{\Gamma(q)}\int_a^t\left(\ln\frac{t}{s}\right)^{q-1}f(s,u(s))\frac{ds}{s} \quad \text{for } t \in (\bar{t}_1, \bar{t}_2], \end{aligned} $$

we have

$$ \begin{aligned} &\lim_{\bar{\Delta}_1(u(\bar{t}_1^-))\to 0} \bar{e}_1(t) \\ &\quad= \lim_{\bar{\Delta}_1(u(\bar{t}_1^-))\to 0}\left\{u(t) - \bar{u}(t)\right\} \\ &\quad= \frac{u_1}{\Gamma(q)}\left(\int_a^t\frac{ds}{s}\right)^{q-1} + \frac{u_2}{\Gamma(q-1)}\left(\int_a^t\frac{ds}{s}\right)^{q-2} \\ &\qquad + \frac{1}{\Gamma(q)}\int_a^t\left(\ln\frac{t}{s}\right)^{q-1}f(s,u(s))\frac{ds}{s} - \frac{u_1 + \int_a^{\bar{t}_1}f(s,u(s))\frac{ds}{s}}{\Gamma(q)}\left(\int_{\bar{t}_1}^t\frac{ds}{s}\right)^{q-1} \\ &\qquad - \frac{u_1\ln\frac{\bar{t}_1}{a} + u_2 + \int_a^{\bar{t}_1}(\ln\frac{\bar{t}_1}{s})f(s,u(s))\frac{ds}{s}}{\Gamma(q-1)}\left(\int_{\bar{t}_1}^t\frac{ds}{s}\right)^{q-2} \\ &\qquad - \frac{1}{\Gamma(q)}\int_{\bar{t}_1}^t\left(\ln\frac{t}{s}\right)^{q-1}f(s,u(s))\frac{ds}{s} \quad \text{for } t \in (\bar{t}_1, \bar{t}_2]. \end{aligned} $$

Therefore, we assume that

$$ \begin{aligned} \bar{e}_1(t) &= \delta\left(\bar{\Delta}_1(u(\bar{t}_1^-))\right)\lim_{\bar{\Delta}_1(u(\bar{t}_1^-))\to 0}\bar{e}_1(t) \\ &= \delta\left(\bar{\Delta}_1(u(\bar{t}_1^-))\right)\left[\frac{u_1}{\Gamma(q)}\left(\int_a^t\frac{ds}{s}\right)^{q-1} + \frac{u_2}{\Gamma(q-1)}\left(\int_a^t\frac{ds}{s}\right)^{q-2}\right. \end{aligned} $$

$$+\frac{1}{\Gamma(q)}\int_a^t\left(\ln\frac{t}{s}\right)^{q-1}f\big(s,u(s)\big)\frac{ds}{s}-\frac{u_1+\int_a^{\bar{t}_1}f(s,u(s))\frac{ds}{s}}{\Gamma(q)}\left(\int_{\bar{t}_1}^t\frac{ds}{s}\right)^{q-1}$$
$$-\frac{u_1\ln\frac{\bar{t}_1}{a}+u_2+\int_a^{\bar{t}_1}(\ln\frac{\bar{t}_1}{s})f(s,u(s))\frac{ds}{s}}{\Gamma(q-1)}\left(\int_{\bar{t}_1}^t\frac{ds}{s}\right)^{q-2}$$
$$-\frac{1}{\Gamma(q)}\int_{\bar{t}_1}^t\left(\ln\frac{t}{s}\right)^{q-1}f\big(s,u(s)\big)\frac{ds}{s}\bigg],$$

where $\delta(\cdot)$ is an undetermined function with $\delta(0)=1$. So,

$$\begin{aligned}u(t)&=\bar{u}(t)+\bar{e}_1(t)\\&=\frac{u_1}{\Gamma(q)}\left(\int_a^t\frac{ds}{s}\right)^{q-1}+\frac{u_2}{\Gamma(q-1)}\left(\int_a^t\frac{ds}{s}\right)^{q-2}\\&\quad+\frac{1}{\Gamma(q)}\int_a^t\left(\ln\frac{t}{s}\right)^{q-1}f\big(s,u(s)\big)\frac{ds}{s}+\frac{\bar{\Delta}_1(u(\bar{t}_1^-))}{\Gamma(q)}\left(\int_{\bar{t}_1}^t\frac{ds}{s}\right)^{q-1}\\&\quad-\big(1-\delta\big(\bar{\Delta}_1\big(u\big(\bar{t}_1^-\big)\big)\big)\big)\bigg[\frac{u_1}{\Gamma(q)}\left(\int_a^t\frac{ds}{s}\right)^{q-1}+\frac{u_2}{\Gamma(q-1)}\left(\int_a^t\frac{ds}{s}\right)^{q-2}\\&\quad+\frac{1}{\Gamma(q)}\int_a^t\left(\ln\frac{t}{s}\right)^{q-1}f\big(s,u(s)\big)\frac{ds}{s}-\frac{u_1+\int_a^{\bar{t}_1}f(s,u(s))\frac{ds}{s}}{\Gamma(q)}\left(\int_{\bar{t}_1}^t\frac{ds}{s}\right)^{q-1}\\&\quad-\frac{u_1\ln\frac{\bar{t}_1}{a}+u_2+\int_a^{\bar{t}_1}(\ln\frac{\bar{t}_1}{s})f(s,u(s))\frac{ds}{s}}{\Gamma(q-1)}\left(\int_{\bar{t}_1}^t\frac{ds}{s}\right)^{q-2}\\&\quad-\frac{1}{\Gamma(q)}\int_{\bar{t}_1}^t\left(\ln\frac{t}{s}\right)^{q-1}f\big(s,u(s)\big)\frac{ds}{s}\bigg]\quad\text{for }t\in(\bar{t}_1,\bar{t}_2].\end{aligned}\tag{3.27}$$

By (3.27) we obtain

$${}_HD_{a^+}^{q-1}u\big(\bar{t}_2^+\big)={}_HD_{a^+}^{q-1}u\big(\bar{t}_2^-\big)+\bar{\Delta}_2\big(u\big(\bar{t}_2^-\big)\big)=u_1+\int_a^{\bar{t}_2}f\big(s,u(s)\big)\frac{ds}{s}+\bar{\Delta}_1\big(u\big(\bar{t}_1^-\big)\big)+\bar{\Delta}_2\big(u\big(\bar{t}_2^-\big)\big)$$

and

$${}_H\mathcal{J}_{a^+}^{2-q}u\big(\bar{t}_2^+\big)={}_H\mathcal{J}_{a^+}^{2-q}u\big(\bar{t}_2^-\big)=u_1\ln\frac{\bar{t}_2}{a}+u_2+\bar{\Delta}_1\big(u\big(\bar{t}_1^-\big)\big)\ln\frac{\bar{t}_2}{\bar{t}_1}+\int_a^{\bar{t}_2}\ln\frac{\bar{t}_2}{s}f\big(s,u(s)\big)\frac{ds}{s}.$$

So, the approximate solution for $t\in(\bar{t}_2,\bar{t}_3]$ is given by

$$\begin{aligned}\bar{u}(t)&=\frac{1}{\Gamma(q)}\big({}_HD_{a^+}^{q-1}u\big(\bar{t}_2^+\big)\big)\left(\int_{\bar{t}_2}^t\frac{ds}{s}\right)^{q-1}+\frac{1}{\Gamma(q-1)}\big({}_H\mathcal{J}_{a^+}^{2-q}u\big(\bar{t}_2^+\big)\big)\left(\int_{\bar{t}_2}^t\frac{ds}{s}\right)^{q-2}\\&\quad+\frac{1}{\Gamma(q)}\int_{\bar{t}_2}^t\left(\ln\frac{t}{s}\right)^{q-1}f\big(s,u(s)\big)\frac{ds}{s}\\&=\frac{u_1+\int_a^{\bar{t}_2}f(s,u(s))\frac{ds}{s}+\bar{\Delta}_1(u(\bar{t}_1^-))+\bar{\Delta}_2(u(\bar{t}_2^-))}{\Gamma(q)}\left(\int_{\bar{t}_2}^t\frac{ds}{s}\right)^{q-1}\\&\quad+\frac{u_1\ln\frac{\bar{t}_2}{a}+u_2+\bar{\Delta}_1(u(\bar{t}_1^-))\ln\frac{\bar{t}_2}{\bar{t}_1}+\int_a^{\bar{t}_2}(\ln\frac{\bar{t}_2}{s})f(s,u(s))\frac{ds}{s}}{\Gamma(q-1)}\left(\int_{\bar{t}_2}^t\frac{ds}{s}\right)^{q-2}\\&\quad+\frac{1}{\Gamma(q)}\int_{\bar{t}_2}^t\left(\ln\frac{t}{s}\right)^{q-1}f\big(s,u(s)\big)\frac{ds}{s}\quad\text{for }t\in(\bar{t}_2,\bar{t}_3].\end{aligned}\tag{3.28}$$

Let $\bar{e}_2(t) = u(t) - \bar{u}(t)$ for $t \in (\bar{t}_2, \bar{t}_3]$. By (3.27) the exact solution $u(t)$ of system (3.2) satisfies

$$\begin{aligned}
&\lim_{\substack{\bar{\Delta}_1(u(\bar{t}_1^-))\to 0,\\ \bar{\Delta}_2(u(\bar{t}_2^-))\to 0}} u(t) = \frac{u_1}{\Gamma(q)}\left(\int_a^t \frac{ds}{s}\right)^{q-1} + \frac{u_2}{\Gamma(q-1)}\left(\int_a^t \frac{ds}{s}\right)^{q-2} \\
&\qquad + \frac{1}{\Gamma(q)}\int_a^t \left(\ln\frac{t}{s}\right)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s} \quad \text{for } t \in (\bar{t}_2, \bar{t}_3],
\end{aligned}$$

$$\begin{aligned}
&\lim_{\bar{\Delta}_1(u(\bar{t}_1^-))\to 0} u(t) \\
&= \frac{u_1}{\Gamma(q)}\left(\int_a^t \frac{ds}{s}\right)^{q-1} + \frac{u_2}{\Gamma(q-1)}\left(\int_a^t \frac{ds}{s}\right)^{q-2} \\
&\quad + \frac{1}{\Gamma(q)}\int_a^t \left(\ln\frac{t}{s}\right)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s} + \frac{\bar{\Delta}_2(u(\bar{t}_2^-))}{\Gamma(q)}\left(\int_{\bar{t}_2}^t \frac{ds}{s}\right)^{q-1} \\
&\quad - \bigl(1-\delta\bigl(\bar{\Delta}_2\bigl(u\bigl(\bar{t}_2^-\bigr)\bigr)\bigr)\bigr)\left[\frac{u_1}{\Gamma(q)}\left(\int_a^t \frac{ds}{s}\right)^{q-1} + \frac{u_2}{\Gamma(q-1)}\left(\int_a^t \frac{ds}{s}\right)^{q-2}\right. \\
&\quad + \frac{1}{\Gamma(q)}\int_a^t \left(\ln\frac{t}{s}\right)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s} - \frac{u_1 + \int_a^{\bar{t}_2} f(s,u(s))\frac{ds}{s}}{\Gamma(q)}\left(\int_{\bar{t}_2}^t \frac{ds}{s}\right)^{q-1} \\
&\quad - \frac{u_1 \ln\frac{\bar{t}_2}{a} + u_2 + \int_a^{\bar{t}_2}(\ln\frac{\bar{t}_2}{s}) f(s,u(s))\frac{ds}{s}}{\Gamma(q-1)}\left(\int_{\bar{t}_2}^t \frac{ds}{s}\right)^{q-2} \\
&\quad \left. - \frac{1}{\Gamma(q)}\int_{\bar{t}_2}^t \left(\ln\frac{t}{s}\right)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s}\right] \quad \text{for } t \in (\bar{t}_2, \bar{t}_3],
\end{aligned}$$

$$\begin{aligned}
&\lim_{\bar{\Delta}_2(u(\bar{t}_2^-))\to 0} u(t) \\
&= \frac{u_1}{\Gamma(q)}\left(\int_a^t \frac{ds}{s}\right)^{q-1} + \frac{u_2}{\Gamma(q-1)}\left(\int_a^t \frac{ds}{s}\right)^{q-2} \\
&\quad + \frac{1}{\Gamma(q)}\int_a^t \left(\ln\frac{t}{s}\right)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s} + \frac{\bar{\Delta}_1(u(\bar{t}_1^-))}{\Gamma(q)}\left(\int_{\bar{t}_1}^t \frac{ds}{s}\right)^{q-1} \\
&\quad - \bigl(1-\delta\bigl(\bar{\Delta}_1\bigl(u\bigl(\bar{t}_1^-\bigr)\bigr)\bigr)\bigr)\left[\frac{u_1}{\Gamma(q)}\left(\int_a^t \frac{ds}{s}\right)^{q-1} + \frac{u_2}{\Gamma(q-1)}\left(\int_a^t \frac{ds}{s}\right)^{q-2}\right. \\
&\quad + \frac{1}{\Gamma(q)}\int_a^t \left(\ln\frac{t}{s}\right)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s} - \frac{u_1 + \int_a^{\bar{t}_1} f(s,u(s))\frac{ds}{s}}{\Gamma(q)}\left(\int_{\bar{t}_1}^t \frac{ds}{s}\right)^{q-1} \\
&\quad - \frac{u_1 \ln\frac{\bar{t}_1}{a} + u_2 + \int_a^{\bar{t}_1}(\ln\frac{\bar{t}_1}{s}) f(s,u(s))\frac{ds}{s}}{\Gamma(q-1)}\left(\int_{\bar{t}_1}^t \frac{ds}{s}\right)^{q-2} \\
&\quad \left. - \frac{1}{\Gamma(q)}\int_{\bar{t}_1}^t \left(\ln\frac{t}{s}\right)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s}\right] \quad \text{for } t \in (\bar{t}_2, \bar{t}_3].
\end{aligned}$$

Therefore,

$$\begin{aligned}
\lim_{\substack{\bar{\Delta}_1(u(\bar{t}_1^-))\to 0,\\ \bar{\Delta}_2(u(\bar{t}_2^-))\to 0}} \bar{e}_2(t) &= \lim_{\substack{\bar{\Delta}_1(u(\bar{t}_1^-))\to 0,\\ \bar{\Delta}_2(u(\bar{t}_2^-))\to 0}} \bigl\{u(t) - \bar{u}(t)\bigr\} \\
&= \frac{u_1}{\Gamma(q)}\left(\int_a^t \frac{ds}{s}\right)^{q-1} + \frac{u_2}{\Gamma(q-1)}\left(\int_a^t \frac{ds}{s}\right)^{q-2}
\end{aligned}$$

$$+\frac{1}{\Gamma(q)}\int_a^t\left(\ln\frac{t}{s}\right)^{q-1}f\big(s,u(s)\big)\frac{ds}{s}-\frac{u_1+\int_a^{\bar{t}_2}f(s,u(s))\frac{ds}{s}}{\Gamma(q)}\left(\int_{\bar{t}_2}^t\frac{ds}{s}\right)^{q-1}$$
$$-\frac{u_1\ln\frac{\bar{t}_2}{a}+u_2+\int_a^{\bar{t}_2}(\ln\frac{\bar{t}_2}{s})f(s,u(s))\frac{ds}{s}}{\Gamma(q-1)}\left(\int_{\bar{t}_2}^t\frac{ds}{s}\right)^{q-2}$$
$$-\frac{1}{\Gamma(q)}\int_{\bar{t}_2}^t\left(\ln\frac{t}{s}\right)^{q-1}f\big(s,u(s)\big)\frac{ds}{s}\quad\text{for } t\in(\bar{t}_2,\bar{t}_3],\tag{3.29}$$

$$\lim_{\bar{\Delta}_2(u(\bar{t}_2^-))\to 0}\bar{e}_2(t)$$
$$=\lim_{\bar{\Delta}_2(u(\bar{t}_2^-))\to 0}\big\{u(t)-\bar{u}(t)\big\}$$
$$=\frac{\bar{\Delta}_1(u(\bar{t}_1^-))}{\Gamma(q)}\left(\int_{\bar{t}_1}^t\frac{ds}{s}\right)^{q-1}-\frac{\bar{\Delta}_1(u(\bar{t}_1^-))}{\Gamma(q)}\left(\int_{\bar{t}_2}^t\frac{ds}{s}\right)^{q-1}-\frac{\bar{\Delta}_1(u(\bar{t}_1^-))}{\Gamma(q-1)}\ln\frac{\bar{t}_2}{\bar{t}_1}\left(\int_{\bar{t}_2}^t\frac{ds}{s}\right)^{q-2}$$
$$+\left[\frac{u_1}{\Gamma(q)}\left(\int_a^t\frac{ds}{s}\right)^{q-1}+\frac{u_2}{\Gamma(q-1)}\left(\int_a^t\frac{ds}{s}\right)^{q-2}+\frac{1}{\Gamma(q)}\int_a^t\left(\ln\frac{t}{s}\right)^{q-1}f\big(s,u(s)\big)\frac{ds}{s}\right.$$
$$-\frac{u_1+\int_a^{\bar{t}_2}f(s,u(s))\frac{ds}{s}}{\Gamma(q)}\left(\int_{\bar{t}_2}^t\frac{ds}{s}\right)^{q-1}$$
$$-\frac{u_1\ln\frac{\bar{t}_2}{a}+u_2+\int_a^{\bar{t}_2}(\ln\frac{\bar{t}_2}{s})f(s,u(s))\frac{ds}{s}}{\Gamma(q-1)}\left(\int_{\bar{t}_2}^t\frac{ds}{s}\right)^{q-2}$$
$$\left.-\frac{1}{\Gamma(q)}\int_{\bar{t}_2}^t\left(\ln\frac{t}{s}\right)^{q-1}f\big(s,u(s)\big)\frac{ds}{s}\right]$$
$$-\big(1-\delta\big(\bar{\Delta}_1\big(u(\bar{t}_1^-)\big)\big)\big)\left[\frac{u_1}{\Gamma(q)}\left(\int_a^t\frac{ds}{s}\right)^{q-1}+\frac{u_2}{\Gamma(q-1)}\left(\int_a^t\frac{ds}{s}\right)^{q-2}\right.$$
$$+\frac{1}{\Gamma(q)}\int_a^t\left(\ln\frac{t}{s}\right)^{q-1}f\big(s,u(s)\big)\frac{ds}{s}-\frac{u_1+\int_a^{\bar{t}_1}f(s,u(s))\frac{ds}{s}}{\Gamma(q)}\left(\int_{\bar{t}_1}^t\frac{ds}{s}\right)^{q-1}$$
$$-\frac{u_1\ln\frac{\bar{t}_1}{a}+u_2+\int_a^{\bar{t}_1}(\ln\frac{\bar{t}_1}{s})f(s,u(s))\frac{ds}{s}}{\Gamma(q-1)}\left(\int_{\bar{t}_1}^t\frac{ds}{s}\right)^{q-2}$$
$$\left.-\frac{1}{\Gamma(q)}\int_{\bar{t}_1}^t\left(\ln\frac{t}{s}\right)^{q-1}f\big(s,u(s)\big)\frac{ds}{s}\right]\quad\text{for } t\in(\bar{t}_2,\bar{t}_3],\tag{3.30}$$

$$\lim_{\bar{\Delta}_1(u(\bar{t}_1^-))\to 0}\bar{e}_2(t)$$
$$=\lim_{\bar{\Delta}_1(u(\bar{t}_1^-))\to 0}\big\{u(t)-\bar{u}(t)\big\}$$
$$=\delta\big(\bar{\Delta}_2\big(u(\bar{t}_2^-)\big)\big)\left[\frac{u_1}{\Gamma(q)}\left(\int_a^t\frac{ds}{s}\right)^{q-1}+\frac{u_2}{\Gamma(q-1)}\left(\int_a^t\frac{ds}{s}\right)^{q-2}\right.$$
$$+\frac{1}{\Gamma(q)}\int_a^t\left(\ln\frac{t}{s}\right)^{q-1}f\big(s,u(s)\big)\frac{ds}{s}-\frac{u_1+\int_a^{\bar{t}_2}f(s,u(s))\frac{ds}{s}}{\Gamma(q)}\left(\int_{\bar{t}_2}^t\frac{ds}{s}\right)^{q-1}$$
$$-\frac{u_1\ln\frac{\bar{t}_2}{a}+u_2+\int_a^{\bar{t}_2}(\ln\frac{\bar{t}_2}{s})f(s,u(s))\frac{ds}{s}}{\Gamma(q-1)}\left(\int_{\bar{t}_2}^t\frac{ds}{s}\right)^{q-2}$$
$$\left.-\frac{1}{\Gamma(q)}\int_{\bar{t}_2}^t\left(\ln\frac{t}{s}\right)^{q-1}f\big(s,u(s)\big)\frac{ds}{s}\right]\quad\text{for } t\in(\bar{t}_2,\bar{t}_3].\tag{3.31}$$

So, by (3.29)-(3.31) we obtain

$$\begin{aligned}
\bar{e}_2(t) ={}& \frac{\bar{\Delta}_1(u(\bar{t}_1^-))}{\Gamma(q)}\left(\int_{\bar{t}_1}^t \frac{ds}{s}\right)^{q-1} - \frac{\bar{\Delta}_1(u(\bar{t}_1^-))}{\Gamma(q)}\left(\int_{\bar{t}_2}^t \frac{ds}{s}\right)^{q-1} - \frac{\Delta_1(u(\bar{t}_1^-))\ln\frac{\bar{t}_2}{\bar{t}_1}}{\Gamma(q-1)}\left(\int_{\bar{t}_2}^t \frac{ds}{s}\right)^{q-2} \\
&+ \delta\bigl(\bar{\Delta}_2\bigl(u\bigl(\bar{t}_2^-\bigr)\bigr)\bigr)\biggl[\frac{u_1}{\Gamma(q)}\left(\int_a^t \frac{ds}{s}\right)^{q-1} + \frac{u_2}{\Gamma(q-1)}\left(\int_a^t \frac{ds}{s}\right)^{q-2} \\
&+ \frac{1}{\Gamma(q)}\int_a^t \left(\ln\frac{t}{s}\right)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s} - \frac{u_1 + \int_a^{\bar{t}_2} f(s,u(s))\frac{ds}{s}}{\Gamma(q)}\left(\int_{\bar{t}_2}^t \frac{ds}{s}\right)^{q-1} \\
&- \frac{u_1 \ln\frac{\bar{t}_2}{a} + u_2 + \int_a^{\bar{t}_2}(\ln\frac{\bar{t}_2}{s}) f(s,u(s))\frac{ds}{s}}{\Gamma(q-1)}\left(\int_{\bar{t}_2}^t \frac{ds}{s}\right)^{q-2} \\
&- \frac{1}{\Gamma(q)}\int_{\bar{t}_2}^t \left(\ln\frac{t}{s}\right)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s}\biggr] \\
&- \bigl(1-\delta\bigl(\bar{\Delta}_1\bigl(u\bigl(\bar{t}_1^-\bigr)\bigr)\bigr)\bigr)\biggl[\frac{u_1}{\Gamma(q)}\left(\int_a^t \frac{ds}{s}\right)^{q-1} + \frac{u_2}{\Gamma(q-1)}\left(\int_a^t \frac{ds}{s}\right)^{q-2} \\
&+ \frac{1}{\Gamma(q)}\int_a^t \left(\ln\frac{t}{s}\right)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s} - \frac{u_1 + \int_a^{\bar{t}_1} f(s,u(s))\frac{ds}{s}}{\Gamma(q)}\left(\int_{\bar{t}_1}^t \frac{ds}{s}\right)^{q-1} \\
&- \frac{u_1 \ln\frac{\bar{t}_1}{a} + u_2 + \int_a^{\bar{t}_1}(\ln\frac{\bar{t}_1}{s}) f(s,u(s))\frac{ds}{s}}{\Gamma(q-1)}\left(\int_{\bar{t}_1}^t \frac{ds}{s}\right)^{q-2} \\
&- \frac{1}{\Gamma(q)}\int_{\bar{t}_1}^t \left(\ln\frac{t}{s}\right)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s}\biggr] \quad \text{for } t \in (\bar{t}_2, \bar{t}_3].
\end{aligned} \tag{3.32}$$

Thus,

$$\begin{aligned}
u(t) ={}& \bar{u}(t) + \bar{e}_2(t) \\
={}& \frac{u_1}{\Gamma(q)}\left(\int_a^t \frac{ds}{s}\right)^{q-1} + \frac{u_2}{\Gamma(q-1)}\left(\int_a^t \frac{ds}{s}\right)^{q-2} + \frac{1}{\Gamma(q)}\int_a^t \left(\ln\frac{t}{s}\right)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s} \\
&+ \frac{\bar{\Delta}_1(u(\bar{t}_1^-))}{\Gamma(q)}\left(\int_{\bar{t}_1}^t \frac{ds}{s}\right)^{q-1} + \frac{\bar{\Delta}_2(u(\bar{t}_2^-))}{\Gamma(q)}\left(\int_{\bar{t}_2}^t \frac{ds}{s}\right)^{q-1} \\
&- \bigl(1-\delta\bigl(\bar{\Delta}_1\bigl(u\bigl(\bar{t}_1^-\bigr)\bigr)\bigr)\bigr)\biggl[\frac{u_1}{\Gamma(q)}\left(\int_a^t \frac{ds}{s}\right)^{q-1} + \frac{u_2}{\Gamma(q-1)}\left(\int_a^t \frac{ds}{s}\right)^{q-2} \\
&+ \frac{1}{\Gamma(q)}\int_a^t \left(\ln\frac{t}{s}\right)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s} - \frac{u_1 + \int_a^{\bar{t}_1} f(s,u(s))\frac{ds}{s}}{\Gamma(q)}\left(\int_{\bar{t}_1}^t \frac{ds}{s}\right)^{q-1} \\
&- \frac{u_1 \ln\frac{\bar{t}_1}{a} + u_2 + \int_a^{\bar{t}_1}(\ln\frac{\bar{t}_1}{s}) f(s,u(s))\frac{ds}{s}}{\Gamma(q-1)}\left(\int_{\bar{t}_1}^t \frac{ds}{s}\right)^{q-2} \\
&- \frac{1}{\Gamma(q)}\int_{\bar{t}_1}^t \left(\ln\frac{t}{s}\right)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s}\biggr] \\
&- \bigl(1-\delta\bigl(\bar{\Delta}_2\bigl(u\bigl(\bar{t}_2^-\bigr)\bigr)\bigr)\bigr)\biggl[\frac{u_1}{\Gamma(q)}\left(\int_a^t \frac{ds}{s}\right)^{q-1} + \frac{u_2}{\Gamma(q-1)}\left(\int_a^t \frac{ds}{s}\right)^{q-2} \\
&+ \frac{1}{\Gamma(q)}\int_a^t \left(\ln\frac{t}{s}\right)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s} - \frac{u_1 + \int_a^{\bar{t}_2} f(s,u(s))\frac{ds}{s}}{\Gamma(q)}\left(\int_{\bar{t}_2}^t \frac{ds}{s}\right)^{q-1}
\end{aligned}$$

$$- \frac{u_1 \ln \frac{\bar{t}_2}{a} + u_2 + \int_a^{\bar{t}_2} (\ln \frac{\bar{t}_2}{s}) f(s,u(s)) \frac{ds}{s}}{\Gamma(q-1)} \left(\int_{\bar{t}_2}^t \frac{ds}{s} \right)^{q-2}$$
$$- \frac{1}{\Gamma(q)} \int_{\bar{t}_2}^t \left(\ln \frac{t}{s} \right)^{q-1} f\big(s,u(s)\big) \frac{ds}{s} \Bigg] \quad \text{for } t \in (\bar{t}_2, \bar{t}_3]. \tag{3.33}$$

Letting $\bar{t}_2 \to \bar{t}_1$, we have

$$\lim_{\bar{t}_2 \to \bar{t}_1} \begin{cases} {}_H D_{a^+}^q u(t) = f(t,u(t)), & q \in (1,2), t \in (a, \bar{t}_3] \text{ and } t \neq \bar{t}_1 \text{ and } t \neq \bar{t}_2, \\ \Delta({}_H \mathcal{J}_{a^+}^{2-q} u)|_{t=\bar{t}_l} = {}_H \mathcal{J}_{a^+}^{2-q} u(\bar{t}_l^+) - {}_H \mathcal{J}_{a^+}^{1-q} u(\bar{t}_l^-) = \bar{\Delta}_l(u(\bar{t}_l^-)), & l = 1,2, \\ {}_H D_{a^+}^{q-1} u(a^+) = u_1, \qquad {}_H \mathcal{J}_{a^+}^{2-q} u(a^+) = u_2 \end{cases} \tag{3.34}$$

$$\to \begin{cases} {}_H D_{a^+}^q u(t) = f(t,u(t)), & q \in (0,1), t \in (a, \bar{t}_3] \text{ and } t \neq \bar{t}_1, \\ \Delta({}_H \mathcal{J}_{a^+}^{2-q} u)|_{t=\bar{t}_1} = \bar{\Delta}_1(u(\bar{t}_1^-)) + \bar{\Delta}_2(u(\bar{t}_2^-)), \\ {}_H D_{a^+}^{q-1} u(a^+) = u_1, \qquad {}_H \mathcal{J}_{a^+}^{2-q} u(a^+) = u_2. \end{cases} \tag{3.35}$$

Using (3.27) and (3.33) in systems (3.35) and (3.34), respectively, we have

$$1 - \delta(\bar{\Delta}_1 + \bar{\Delta}_2) = 1 - \delta(\bar{\Delta}_1) + 1 - \delta(\bar{\Delta}_2) \quad \forall \bar{\Delta}_1, \bar{\Delta}_2 \in \mathbb{R}.$$

Letting $\gamma(z) = 1 - \delta(z)$ ($\forall z \in \mathbb{R}$), we have $\gamma(z + w) = \gamma(z) + \gamma(w)$ ($\forall z, w \in \mathbb{R}$). Therefore, $\gamma(z) = \hbar z$, where $\hbar$ is a constant. So, by (3.27) and (3.33) we get

$$\begin{aligned} u(t) = {} & \frac{u_1}{\Gamma(q)} \left(\int_a^t \frac{ds}{s} \right)^{q-1} + \frac{u_2}{\Gamma(q-1)} \left(\int_a^t \frac{ds}{s} \right)^{q-2} \\ & + \frac{1}{\Gamma(q)} \int_a^t \left(\ln \frac{t}{s} \right)^{q-1} f\big(s,u(s)\big) \frac{ds}{s} + \frac{\bar{\Delta}_1(u(\bar{t}_1^-))}{\Gamma(q)} \left(\int_{\bar{t}_1}^t \frac{ds}{s} \right)^{q-1} \\ & - \hbar \bar{\Delta}_1\big(u(\bar{t}_1^-)\big) \Bigg[\frac{u_1}{\Gamma(q)} \left(\int_a^t \frac{ds}{s} \right)^{q-1} + \frac{u_2}{\Gamma(q-1)} \left(\int_a^t \frac{ds}{s} \right)^{q-2} \\ & + \frac{1}{\Gamma(q)} \int_a^t \left(\ln \frac{t}{s} \right)^{q-1} f\big(s,u(s)\big) \frac{ds}{s} - \frac{u_1 + \int_a^{\bar{t}_1} f(s,u(s)) \frac{ds}{s}}{\Gamma(q)} \left(\int_{\bar{t}_1}^t \frac{ds}{s} \right)^{q-1} \\ & - \frac{u_1 \ln \frac{\bar{t}_1}{a} + u_2 + \int_a^{\bar{t}_1} (\ln \frac{\bar{t}_1}{s}) f(s,u(s)) \frac{ds}{s}}{\Gamma(q-1)} \left(\int_{\bar{t}_1}^t \frac{ds}{s} \right)^{q-2} \\ & - \frac{1}{\Gamma(q)} \int_{\bar{t}_1}^t \left(\ln \frac{t}{s} \right)^{q-1} f\big(s,u(s)\big) \frac{ds}{s} \Bigg] \quad \text{for } t \in (\bar{t}_1, \bar{t}_2], \end{aligned}$$

and

$$\begin{aligned} u(t) = {} & \frac{u_1}{\Gamma(q)} \left(\int_a^t \frac{ds}{s} \right)^{q-1} + \frac{u_2}{\Gamma(q-1)} \left(\int_a^t \frac{ds}{s} \right)^{q-2} + \frac{1}{\Gamma(q)} \int_a^t \left(\ln \frac{t}{s} \right)^{q-1} f\big(s,u(s)\big) \frac{ds}{s} \\ & + \frac{\bar{\Delta}_1(u(\bar{t}_1^-))}{\Gamma(q)} \left(\int_{\bar{t}_1}^t \frac{ds}{s} \right)^{q-1} + \frac{\bar{\Delta}_2(u(\bar{t}_2^-))}{\Gamma(q)} \left(\int_{\bar{t}_2}^t \frac{ds}{s} \right)^{q-1} \\ & - \hbar \bar{\Delta}_1\big(u(\bar{t}_1^-)\big) \Bigg[\frac{u_1}{\Gamma(q)} \left(\int_a^t \frac{ds}{s} \right)^{q-1} + \frac{u_2}{\Gamma(q-1)} \left(\int_a^t \frac{ds}{s} \right)^{q-2} \\ & + \frac{1}{\Gamma(q)} \int_a^t \left(\ln \frac{t}{s} \right)^{q-1} f\big(s,u(s)\big) \frac{ds}{s} - \frac{u_1 + \int_a^{\bar{t}_1} f(s,u(s)) \frac{ds}{s}}{\Gamma(q)} \left(\int_{\bar{t}_1}^t \frac{ds}{s} \right)^{q-1} \end{aligned}$$

$$
\begin{aligned}
&- \frac{u_1 \ln \frac{\bar{t}_1}{a} + u_2 + \int_a^{\bar{t}_1} (\ln \frac{\bar{t}_1}{s}) f(s,u(s)) \frac{ds}{s}}{\Gamma(q-1)} \left(\int_{\bar{t}_1}^t \frac{ds}{s} \right)^{q-2} \\
&- \frac{1}{\Gamma(q)} \int_{\bar{t}_1}^t \left(\ln \frac{t}{s} \right)^{q-1} f\big(s,u(s)\big) \frac{ds}{s} \bigg] \\
&- \hbar \bar{\Delta}_2\big(u(\bar{t}_2^-)\big) \bigg[\frac{u_1}{\Gamma(q)} \left(\int_a^t \frac{ds}{s} \right)^{q-1} + \frac{u_2}{\Gamma(q-1)} \left(\int_a^t \frac{ds}{s} \right)^{q-2} \\
&+ \frac{1}{\Gamma(q)} \int_a^t \left(\ln \frac{t}{s} \right)^{q-1} f\big(s,u(s)\big) \frac{ds}{s} - \frac{u_1 + \int_a^{\bar{t}_2} f(s,u(s)) \frac{ds}{s}}{\Gamma(q)} \left(\int_{\bar{t}_2}^t \frac{ds}{s} \right)^{q-1} \\
&- \frac{u_1 \ln \frac{\bar{t}_2}{a} + u_2 + \int_a^{\bar{t}_2} (\ln \frac{\bar{t}_2}{s}) f(s,u(s)) \frac{ds}{s}}{\Gamma(q-1)} \left(\int_{\bar{t}_2}^t \frac{ds}{s} \right)^{q-2} \\
&- \frac{1}{\Gamma(q)} \int_{\bar{t}_2}^t \left(\ln \frac{t}{s} \right)^{q-1} f\big(s,u(s)\big) \frac{ds}{s} \bigg] \quad \text{for } t \in (\bar{t}_2, \bar{t}_3].
\end{aligned}
$$

Due to similarity of the proof with Lemma 3.2, the rest of proof is omitted.

So, system (3.2) is equivalent to Eq. (3.23). The proof is now completed. □

Theorem 3.4 *Let* $q \in (1,2)$, *and let* λ, $\hbar$ *be two constants. System* (1.1) *is equivalent to the fractional integral equation*

$$
u(t) = \begin{cases}
\frac{u_1}{\Gamma(q)} (\int_a^t \frac{ds}{s})^{q-1} + \frac{u_2}{\Gamma(q-1)} (\int_a^t \frac{ds}{s})^{q-2} + \frac{1}{\Gamma(q)} \int_a^t (\ln \frac{t}{s})^{q-1} f(s,u(s)) \frac{ds}{s} \\
\quad \textit{for } t \in (a, t_1'], \\
\frac{u_1}{\Gamma(q)} (\int_a^t \frac{ds}{s})^{q-1} + \frac{u_2}{\Gamma(q-1)} (\int_a^t \frac{ds}{s})^{q-2} + \frac{1}{\Gamma(q)} \int_a^t (\ln \frac{t}{s})^{q-1} f(s,u(s)) \frac{ds}{s} \\
\quad + \sum_{i=1}^{k_0} \frac{\Delta_i(u(t_i^-))}{\Gamma(q-1)} (\int_{t_i}^t \frac{ds}{s})^{q-2} + \sum_{j=1}^{k_1} \frac{\bar{\Delta}_j(u(\bar{t}_j^-))}{\Gamma(q)} (\int_{\bar{t}_j}^t \frac{ds}{s})^{q-1} \\
\quad - \sum_{i=1}^{k_0} \lambda \Delta_i(u(t_i^-)) [\frac{u_1}{\Gamma(q)} (\int_a^t \frac{ds}{s})^{q-1} + \frac{u_2}{\Gamma(q-1)} (\int_a^t \frac{ds}{s})^{q-2} \\
\quad + \frac{1}{\Gamma(q)} \int_a^t (\ln \frac{t}{s})^{q-1} f(s,u(s)) \frac{ds}{s} - \frac{u_1 + \int_a^{t_i} f(s,u(s)) \frac{ds}{s}}{\Gamma(q)} (\int_{t_i}^t \frac{ds}{s})^{q-1} \\
\quad - \frac{u_1 \ln \frac{t_i}{a} + u_2 + \int_a^{t_i} (\ln \frac{t_i}{s}) f(s,u(s)) \frac{ds}{s}}{\Gamma(q-1)} (\int_{t_i}^t \frac{ds}{s})^{q-2} \\
\quad - \frac{1}{\Gamma(q)} \int_{t_i}^t (\ln \frac{t}{s})^{q-1} f(s,u(s)) \frac{ds}{s}] \\
\quad - \sum_{j=1}^{k_1} \hbar \bar{\Delta}_j(u(\bar{t}_j^-)) [\frac{u_1}{\Gamma(q)} (\int_a^t \frac{ds}{s})^{q-1} + \frac{u_2}{\Gamma(q-1)} (\int_a^t \frac{ds}{s})^{q-2} \\
\quad + \frac{1}{\Gamma(q)} \int_a^t (\ln \frac{t}{s})^{q-1} f(s,u(s)) \frac{ds}{s} - \frac{u_1 + \int_a^{\bar{t}_j} f(s,u(s)) \frac{ds}{s}}{\Gamma(q)} (\int_{\bar{t}_j}^t \frac{ds}{s})^{q-1} \\
\quad - \frac{u_1 \ln \frac{\bar{t}_j}{a} + u_2 + \int_a^{\bar{t}_j} (\ln \frac{\bar{t}_j}{s}) f(s,u(s)) \frac{ds}{s}}{\Gamma(q-1)} (\int_{\bar{t}_j}^t \frac{ds}{s})^{q-2} - \frac{1}{\Gamma(q)} \int_{\bar{t}_j}^t (\ln \frac{t}{s})^{q-1} f(s,u(s)) \frac{ds}{s}] \\
\quad \textit{for } t \in (t_k', t_{k+1}'], k = 1, 2, \ldots, \Omega
\end{cases}
\tag{3.36}
$$

provided that the integral in (3.36) *exists.*

Proof Necessity. We will verify that Eq. (3.36) satisfies the conditions of system (1.1).

First, we can easily verify that Eq. (3.36) satisfies the three implicit conditions (i)-(iii) by Lemmas 3.2 and 3.3.

Second, the proof that Eq. (3.36) satisfies the Hadamard fractional derivative and impulsive conditions of system (1.1) is similar to that of Lemma 3.2 and is omitted.

Sufficiency. We will prove that the solutions of system (3.1) satisfy Eq. (3.36) by mathematical induction. For $t \in [a, t_1']$, it is clear that the solution of system (1.1) satisfies

Eq. (3.36) with

$$u(t) = \frac{u_1}{\Gamma(q)}\left(\int_a^t \frac{ds}{s}\right)^{q-1} + \frac{u_2}{\Gamma(q-1)}\left(\int_a^t \frac{ds}{s}\right)^{q-2} + \frac{1}{\Gamma(q)}\int_a^t \left(\ln\frac{t}{s}\right)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s} \quad \text{for } t \in \bigl[a,t_1'\bigr]. \tag{3.37}$$

For $t \in (t_1', t_2']$, there exist three cases $t_1' = t_1 < \bar{t}_1$, $t_1' = \bar{t}_1 < t_1$, and $t_1' = t_1 = \bar{t}_1$. For the two cases ($t_1' = t_1 < \bar{t}_1$ and $t_1' = \bar{t}_1 < t_1$), we can verify that the solution of (1.1) satisfies Eq. (3.36) for $t \in (t_1', t_2']$ by Lemmas 3.2 and 3.3, respectively. Hence, we will consider the case $t_1' = t_1 = \bar{t}_1$. Using (3.37), we have

$$\begin{aligned} {}_H D_{a^+}^{q-1} u\bigl(t_1'^+\bigr) &= {}_H D_{a^+}^{q-1} u\bigl(t_1'^-\bigr) + \bar{\Delta}_1\bigl(u\bigl(t_1'^-\bigr)\bigr) = u_1 + \int_a^{t_1'} f\bigl(s,u(s)\bigr)\frac{ds}{s} + \bar{\Delta}_1\bigl(u\bigl(t_1'^-\bigr)\bigr) \quad \text{and} \\ {}_H \mathcal{J}_{a^+}^{2-q} u\bigl(t_1'^+\bigr) &= {}_H \mathcal{J}_{a^+}^{2-q} u\bigl(t_1'^-\bigr) + \Delta_1\bigl(u\bigl(t_1'^-\bigr)\bigr) \\ &= u_1 \ln\frac{t_1'}{a} + u_2 + \int_a^{t_1'} \ln\frac{t_1'}{s} f\bigl(s,u(s)\bigr)\frac{ds}{s} + \Delta_1\bigl(u\bigl(t_1'^-\bigr)\bigr). \end{aligned}$$

Therefore, the approximate solution of system (1.1) for $t \in (t_1', t_2']$ is given by

$$\begin{aligned} \hat{u}(t) &= \frac{1}{\Gamma(q)}\bigl({}_H D_{a^+}^{q-1} u\bigl(t_1'^+\bigr)\bigr)\left(\int_{t_1'}^t \frac{ds}{s}\right)^{q-1} + \frac{1}{\Gamma(q-1)}\bigl({}_H \mathcal{J}_{a^+}^{2-q} u\bigl(t_1'^+\bigr)\bigr)\left(\int_{t_1'}^t \frac{ds}{s}\right)^{q-2} \\ &\quad + \frac{1}{\Gamma(q)}\int_{t_1'}^t \left(\ln\frac{t}{s}\right)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s} \\ &= \frac{u_1 + \int_a^{t_1'} f(s,u(s))\frac{ds}{s} + \bar{\Delta}_1(u(t_1'^-))}{\Gamma(q)}\left(\int_{t_1'}^t \frac{ds}{s}\right)^{q-1} \\ &\quad + \frac{u_1 \ln\frac{t_1'}{a} + u_2 + \int_a^{t_1'} \ln\frac{t_1'}{s} f(s,u(s))\frac{ds}{s} + \Delta_1(u(t_1'^-))}{\Gamma(q-1)}\left(\int_{t_1'}^t \frac{ds}{s}\right)^{q-2} \\ &\quad + \frac{1}{\Gamma(q)}\int_{t_1'}^t \left(\ln\frac{t}{s}\right)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s} \quad \text{for } t \in \bigl(t_1', t_2'\bigr]. \end{aligned} \tag{3.38}$$

Let $\hat{e}_1(t) = u(t) - \hat{u}(t)$ for $t \in (t_1', t_2']$. By Lemmas 3.2 and 3.3 the exact solution $u(t)$ of system (1.1) satisfies

$$\begin{aligned} &\lim_{\Delta_1(u(t_1'^-))\to 0, \bar{\Delta}_1(u(t_1'^-))\to 0} u(t) \\ &\quad = \frac{u_1}{\Gamma(q)}\left(\int_a^t \frac{ds}{s}\right)^{q-1} + \frac{u_2}{\Gamma(q-1)}\left(\int_a^t \frac{ds}{s}\right)^{q-2} \\ &\qquad + \frac{1}{\Gamma(q)}\int_a^t \left(\ln\frac{t}{s}\right)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s} \quad \text{for } t \in \bigl(t_1', t_2'\bigr], \\ &\lim_{\Delta_1(u(t_1'^-))\to 0} u(t) \\ &\quad = \frac{u_1}{\Gamma(q)}\left(\int_a^t \frac{ds}{s}\right)^{q-1} + \frac{u_2}{\Gamma(q-1)}\left(\int_a^t \frac{ds}{s}\right)^{q-2} \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(q)}\int_a^t\left(\ln\frac{t}{s}\right)^{q-1}f\bigl(s,u(s)\bigr)\frac{ds}{s}+\frac{\bar{\Delta}_1(u(t_1'^-))}{\Gamma(q)}\left(\int_{t_1'}^t\frac{ds}{s}\right)^{q-1}\\
&-\hbar\bar{\Delta}_1\bigl(u\bigl(t_1'^-\bigr)\bigr)\left\{\frac{u_1}{\Gamma(q)}\left(\int_a^t\frac{ds}{s}\right)^{q-1}+\frac{u_2}{\Gamma(q-1)}\left(\int_a^t\frac{ds}{s}\right)^{q-2}\right.\\
&+\frac{1}{\Gamma(q)}\int_a^t\left(\ln\frac{t}{s}\right)^{q-1}f\bigl(s,u(s)\bigr)\frac{ds}{s}-\frac{u_1+\int_a^{t_1'}f(s,u(s))\frac{ds}{s}}{\Gamma(q)}\left(\int_{t_1'}^t\frac{ds}{s}\right)^{q-1}\\
&-\frac{u_1\ln\frac{t_1'}{a}+u_2+\int_a^{t_1'}(\ln\frac{t_1'}{s})f(s,u(s))\frac{ds}{s}}{\Gamma(q-1)}\left(\int_{t_1'}^t\frac{ds}{s}\right)^{q-2}\\
&\left.-\frac{1}{\Gamma(q)}\int_{t_1'}^t\left(\ln\frac{t}{s}\right)^{q-1}f\bigl(s,u(s)\bigr)\frac{ds}{s}\right\}\quad\text{for } t\in\bigl(t_1',t_2'\bigr],
\end{aligned}
$$

$$
\begin{aligned}
&\lim_{\bar{\Delta}_1(u(t_1'^-))\to 0}u(t)\\
&=\frac{u_1}{\Gamma(q)}\left(\int_a^t\frac{ds}{s}\right)^{q-1}+\frac{u_2}{\Gamma(q-1)}\left(\int_a^t\frac{ds}{s}\right)^{q-2}\\
&+\frac{1}{\Gamma(q)}\int_a^t\left(\ln\frac{t}{s}\right)^{q-1}f\bigl(s,u(s)\bigr)\frac{ds}{s}+\frac{\Delta_1(u(t_1'^-))}{\Gamma(q-1)}\left(\int_{t_1'}^t\frac{ds}{s}\right)^{q-2}\\
&-\lambda\Delta_1\bigl(u\bigl(t_1'^-\bigr)\bigr)\left\{\frac{u_1}{\Gamma(q)}\left(\int_a^t\frac{ds}{s}\right)^{q-1}+\frac{u_2}{\Gamma(q-1)}\left(\int_a^t\frac{ds}{s}\right)^{q-2}\right.\\
&+\frac{1}{\Gamma(q)}\int_a^t\left(\ln\frac{t}{s}\right)^{q-1}f\bigl(s,u(s)\bigr)\frac{ds}{s}-\frac{u_1+\int_a^{t_1'}f(s,u(s))\frac{ds}{s}}{\Gamma(q)}\left(\int_{t_1'}^t\frac{ds}{s}\right)^{q-1}\\
&-\frac{u_1\ln\frac{t_1'}{a}+u_2+\int_a^{t_1'}(\ln\frac{t_1'}{s})f(s,u(s))\frac{ds}{s}}{\Gamma(q-1)}\left(\int_{t_1'}^t\frac{ds}{s}\right)^{q-2}\\
&\left.-\frac{1}{\Gamma(q)}\int_{t_1'}^t\left(\ln\frac{t}{s}\right)^{q-1}f\bigl(s,u(s)\bigr)\frac{ds}{s}\right\}\quad\text{for } t\in\bigl(t_1',t_2'\bigr].
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\lim_{\substack{\Delta_1(u(t_1'^-))\to 0,\\ \bar{\Delta}_1(u(t_1'^-))\to 0}}\hat{e}_1(t)&=\frac{u_1}{\Gamma(q)}\left(\int_a^t\frac{ds}{s}\right)^{q-1}+\frac{u_2}{\Gamma(q-1)}\left(\int_a^t\frac{ds}{s}\right)^{q-2}\\
&+\frac{1}{\Gamma(q)}\int_a^t\left(\ln\frac{t}{s}\right)^{q-1}f\bigl(s,u(s)\bigr)\frac{ds}{s}-\frac{u_1+\int_a^{t_1'}f(s,u(s))\frac{ds}{s}}{\Gamma(q)}\left(\int_{t_1'}^t\frac{ds}{s}\right)^{q-1}\\
&-\frac{u_1\ln\frac{t_1'}{a}+u_2+\int_a^{t_1'}\ln\frac{t_1'}{s}f(s,u(s))\frac{ds}{s}}{\Gamma(q-1)}\left(\int_{t_1'}^t\frac{ds}{s}\right)^{q-2}\\
&-\frac{1}{\Gamma(q)}\int_{t_1'}^t\left(\ln\frac{t}{s}\right)^{q-1}f\bigl(s,u(s)\bigr)\frac{ds}{s}\quad\text{for } t\in\bigl(t_1',t_2'\bigr],
\end{aligned}
\tag{3.39}
$$

$$
\begin{aligned}
\lim_{\Delta_1(u(t_1'^-))\to 0}\hat{e}_1(t)&=\bigl[1-\hbar\bar{\Delta}_1\bigl(u\bigl(t_1'^-\bigr)\bigr)\bigr]\left\{\frac{u_1}{\Gamma(q)}\left(\int_a^t\frac{ds}{s}\right)^{q-1}+\frac{u_2}{\Gamma(q-1)}\left(\int_a^t\frac{ds}{s}\right)^{q-2}\right.\\
&+\frac{1}{\Gamma(q)}\int_a^t\left(\ln\frac{t}{s}\right)^{q-1}f\bigl(s,u(s)\bigr)\frac{ds}{s}-\frac{u_1+\int_a^{t_1'}f(s,u(s))\frac{ds}{s}}{\Gamma(q)}\left(\int_{t_1'}^t\frac{ds}{s}\right)^{q-1}
\end{aligned}
$$

$$- \frac{u_1 \ln \frac{t_1'}{a} + u_2 + \int_a^{t_1'} (\ln \frac{t_1'}{s}) f(s,u(s)) \frac{ds}{s}}{\Gamma(q-1)} \left(\int_{t_1'}^t \frac{ds}{s} \right)^{q-2}$$

$$- \frac{1}{\Gamma(q)} \int_{t_1'}^t \left(\ln \frac{t}{s} \right)^{q-1} f\big(s,u(s)\big) \frac{ds}{s} \Bigg\} \quad \text{for } t \in \big(t_1', t_2'\big], \tag{3.40}$$

$$\lim_{\bar{\Delta}_1(u(t_1'^-)) \to 0} \hat{e}_1(t) = \big[1 - \lambda \Delta_1\big(u\big(t_1'^-\big)\big)\big] \Bigg\{ \frac{u_1}{\Gamma(q)} \left(\int_a^t \frac{ds}{s} \right)^{q-1} + \frac{u_2}{\Gamma(q-1)} \left(\int_a^t \frac{ds}{s} \right)^{q-2}$$

$$+ \frac{1}{\Gamma(q)} \int_a^t \left(\ln \frac{t}{s} \right)^{q-1} f\big(s,u(s)\big) \frac{ds}{s} - \frac{u_1 + \int_a^{t_1'} f(s,u(s)) \frac{ds}{s}}{\Gamma(q)} \left(\int_{t_1'}^t \frac{ds}{s} \right)^{q-1}$$

$$- \frac{u_1 \ln \frac{t_1'}{a} + u_2 + \int_a^{t_1'} (\ln \frac{t_1'}{s}) f(s,u(s)) \frac{ds}{s}}{\Gamma(q-1)} \left(\int_{t_1'}^t \frac{ds}{s} \right)^{q-2}$$

$$- \frac{1}{\Gamma(q)} \int_{t_1'}^t \left(\ln \frac{t}{s} \right)^{q-1} f\big(s,u(s)\big) \frac{ds}{s} \Bigg\} \quad \text{for } t \in \big(t_1', t_2'\big]. \tag{3.41}$$

By (3.39)-(3.41) we get

$$\hat{e}_1(t) = \big[1 - \hbar \bar{\Delta}_1\big(u\big(t_1'^-\big)\big) - \lambda \Delta_1\big(u\big(t_1'^-\big)\big)\big] \Bigg\{ \frac{u_1}{\Gamma(q)} \left(\int_a^t \frac{ds}{s} \right)^{q-1} + \frac{u_2}{\Gamma(q-1)} \left(\int_a^t \frac{ds}{s} \right)^{q-2}$$

$$+ \frac{1}{\Gamma(q)} \int_a^t \left(\ln \frac{t}{s} \right)^{q-1} f\big(s,u(s)\big) \frac{ds}{s} - \frac{u_1 + \int_a^{t_1'} f(s,u(s)) \frac{ds}{s}}{\Gamma(q)} \left(\int_{t_1'}^t \frac{ds}{s} \right)^{q-1}$$

$$- \frac{u_1 \ln \frac{t_1'}{a} + u_2 + \int_a^{t_1'} (\ln \frac{t_1'}{s}) f(s,u(s)) \frac{ds}{s}}{\Gamma(q-1)} \left(\int_{t_1'}^t \frac{ds}{s} \right)^{q-2}$$

$$- \frac{1}{\Gamma(q)} \int_{t_1'}^t \left(\ln \frac{t}{s} \right)^{q-1} f\big(s,u(s)\big) \frac{ds}{s} \Bigg\} \quad \text{for } t \in \big(t_1', t_2'\big].$$

Then,

$$u(t) = \hat{u}(t) + \hat{e}_1(t)$$

$$= \frac{u_1}{\Gamma(q)} \left(\int_a^t \frac{ds}{s} \right)^{q-1} + \frac{u_2}{\Gamma(q-1)} \left(\int_a^t \frac{ds}{s} \right)^{q-2} + \frac{1}{\Gamma(q)} \int_a^t \left(\ln \frac{t}{s} \right)^{q-1} f\big(s,u(s)\big) \frac{ds}{s}$$

$$+ \frac{\Delta_1(u(t_1'^-))}{\Gamma(q-1)} \left(\int_{t_1'}^t \frac{ds}{s} \right)^{q-2} + \frac{\bar{\Delta}_1(u(t_1'^-))}{\Gamma(q)} \left(\int_{t_1'}^t \frac{ds}{s} \right)^{q-1}$$

$$- \big[\lambda \Delta_1\big(u\big(t_1'^-\big)\big) + \hbar \bar{\Delta}_1\big(u\big(t_1'^-\big)\big)\big] \Bigg\{ \frac{u_1}{\Gamma(q)} \left(\int_a^t \frac{ds}{s} \right)^{q-1} + \frac{u_2}{\Gamma(q-1)} \left(\int_a^t \frac{ds}{s} \right)^{q-2}$$

$$+ \frac{1}{\Gamma(q)} \int_a^t \left(\ln \frac{t}{s} \right)^{q-1} f\big(s,u(s)\big) \frac{ds}{s} - \frac{u_1 + \int_a^{t_1'} f(s,u(s)) \frac{ds}{s}}{\Gamma(q)} \left(\int_{t_1'}^t \frac{ds}{s} \right)^{q-1}$$

$$- \frac{u_1 \ln \frac{t_1'}{a} + u_2 + \int_a^{t_1'} (\ln \frac{t_1'}{s}) f(s,u(s)) \frac{ds}{s}}{\Gamma(q-1)} \left(\int_{t_1'}^t \frac{ds}{s} \right)^{q-2}$$

$$- \frac{1}{\Gamma(q)} \int_{t_1'}^t \left(\ln \frac{t}{s} \right)^{q-1} f\big(s,u(s)\big) \frac{ds}{s} \Bigg\} \quad \text{for } t \in \big(t_1', t_2'\big]. \tag{3.42}$$

Next, for $t \in (t'_k, t'_{k+1}]$ ($k = 1, 2, \dots, \Omega$), suppose that

$$\begin{aligned}
u(t) &= \frac{u_1}{\Gamma(q)}\left(\int_a^t \frac{ds}{s}\right)^{q-1} + \frac{u_2}{\Gamma(q-1)}\left(\int_a^t \frac{ds}{s}\right)^{q-2} \\
&\quad + \frac{1}{\Gamma(q)}\int_a^t \left(\ln\frac{t}{s}\right)^{q-1} f\big(s, u(s)\big)\frac{ds}{s} \\
&\quad + \sum_{i=1}^{k_0} \frac{\Delta_i(u(t_i^-))}{\Gamma(q-1)}\left(\int_{t_i}^t \frac{ds}{s}\right)^{q-2} + \sum_{j=1}^{k_1} \frac{\bar{\Delta}_j(u(\bar{t}_j^-))}{\Gamma(q)}\left(\int_{\bar{t}_j}^t \frac{ds}{s}\right)^{q-1} \\
&\quad - \sum_{i=1}^{k_0} \tilde{\lambda}\Delta_i\big(u(t_i^-)\big)\left[\frac{u_1}{\Gamma(q)}\left(\int_a^t \frac{ds}{s}\right)^{q-1} + \frac{u_2}{\Gamma(q-1)}\left(\int_a^t \frac{ds}{s}\right)^{q-2}\right. \\
&\quad + \frac{1}{\Gamma(q)}\int_a^t \left(\ln\frac{t}{s}\right)^{q-1} f\big(s, u(s)\big)\frac{ds}{s} - \frac{u_1 + \int_a^{t_i} f(s, u(s))\frac{ds}{s}}{\Gamma(q)}\left(\int_{t_i}^t \frac{ds}{s}\right)^{q-1} \\
&\quad - \frac{u_1 \ln\frac{t_i}{a} + u_2 + \int_a^{t_i}(\ln\frac{t_i}{s}) f(s, u(s))\frac{ds}{s}}{\Gamma(q-1)}\left(\int_{t_i}^t \frac{ds}{s}\right)^{q-2} \\
&\quad \left. - \frac{1}{\Gamma(q)}\int_{t_i}^t \left(\ln\frac{t}{s}\right)^{q-1} f\big(s, u(s)\big)\frac{ds}{s}\right] \\
&\quad - \sum_{j=1}^{k_1} \hbar\bar{\Delta}_j\big(u(\bar{t}_j^-)\big)\left[\frac{u_1}{\Gamma(q)}\left(\int_a^t \frac{ds}{s}\right)^{q-1} + \frac{u_2}{\Gamma(q-1)}\left(\int_a^t \frac{ds}{s}\right)^{q-2}\right. \\
&\quad + \frac{1}{\Gamma(q)}\int_a^t \left(\ln\frac{t}{s}\right)^{q-1} f\big(s, u(s)\big)\frac{ds}{s} - \frac{u_1 + \int_a^{\bar{t}_j} f(s, u(s))\frac{ds}{s}}{\Gamma(q)}\left(\int_{\bar{t}_j}^t \frac{ds}{s}\right)^{q-1} \\
&\quad - \frac{u_1 \ln\frac{\bar{t}_j}{a} + u_2 + \int_a^{\bar{t}_j}(\ln\frac{\bar{t}_j}{s}) f(s, u(s))\frac{ds}{s}}{\Gamma(q-1)}\left(\int_{\bar{t}_j}^t \frac{ds}{s}\right)^{q-2} \\
&\quad \left. - \frac{1}{\Gamma(q)}\int_{\bar{t}_j}^t \left(\ln\frac{t}{s}\right)^{q-1} f\big(s, u(s)\big)\frac{ds}{s}\right] \quad \text{for } t \in \big(t'_k, t'_{k+1}\big].
\end{aligned} \tag{3.43}$$

By (3.43) we have

$$\begin{aligned}
{}_H D_{a^+}^{q-1} u\big(t'^+_{k+1}\big) &= {}_H D_{a^+}^{q-1} u\big(t'^-_{k+1}\big) + \sum_{k_1+1}^{(k+1)_1} \bar{\Delta}_j \\
&= u_1 + \int_a^{t'_{k+1}} f\big(s, u(s)\big)\frac{ds}{s} + \sum_{j=1}^{(k+1)_1} \bar{\Delta}_j\big(u(\bar{t}_j^-)\big)
\end{aligned}$$

and

$$\begin{aligned}
{}_H\mathcal{J}_{a^+}^{2-q} u\big(t'^+_{k+1}\big) &= {}_H\mathcal{J}_{a^+}^{2-q} u\big(t'^-_{k+1}\big) + \sum_{k_0+1}^{(k+1)_0} \Delta_i\big(u(t_i^-)\big) \\
&= u_1 \int_a^{t'_{k+1}} \frac{ds}{s} + u_2 + \int_a^{t'_{k+1}} \ln\frac{t'_{k+1}}{s} f\big(s, u(s)\big)\frac{ds}{s} + \sum_{i=1}^{(k+1)_0} \Delta_i\big(u(t_i^-)\big) \\
&\quad + \sum_{j=1}^{k_1} \bar{\Delta}_j\big(u(\bar{t}_j^-)\big) \ln\frac{t'_{k+1}}{\bar{t}_j}.
\end{aligned}$$

Therefore, the approximate solution of system (1.1) for $t \in (t'_{k+1}, t'_{k+2}]$ is given by

$$\begin{aligned}
\hat{u}(t) &= \frac{1}{\Gamma(q)}\bigl({}_HD^{q-1}_{a^+}u\bigl(t'^+_{k+1}\bigr)\bigr)\left(\int_{t'_{k+1}}^t \frac{ds}{s}\right)^{q-1} + \frac{1}{\Gamma(q-1)}\bigl({}_H\mathcal{J}^{2-q}_{a^+}u\bigl(t'^+_{k+1}\bigr)\bigr)\left(\int_{t'_{k+1}}^t \frac{ds}{s}\right)^{q-2} \\
&\quad + \frac{1}{\Gamma(q)}\int_{t'_{k+1}}^t \left(\ln\frac{t}{s}\right)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s} \\
&= \frac{u_1 + \int_a^{t'_{k+1}} f(s,u(s))\frac{ds}{s} + \sum_{j=1}^{(k+1)_1} \bar{\Delta}_j(u(\bar{t}_j^-))}{\Gamma(q)}\left(\int_{t'_{k+1}}^t \frac{ds}{s}\right)^{q-1} \\
&\quad + \frac{u_1 \ln\frac{t'_{k+1}}{a} + u_2 + \int_a^{t'_{k+1}} \ln\frac{t'_{k+1}}{s} f(s,u(s))\frac{ds}{s} + \sum_{i=1}^{(k+1)_0} \Delta_i(u(t_i^-)) + \sum_{j=1}^{k_1} \bar{\Delta}_j(u(\bar{t}_j^-)) \ln\frac{t'_{k+1}}{t_j}}{\Gamma(q-1)} \\
&\quad \times \left(\int_{t'_{k+1}}^t \frac{ds}{s}\right)^{q-2} + \frac{1}{\Gamma(q)}\int_{t'_{k+1}}^t \left(\ln\frac{t}{s}\right)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s} \quad \text{for } t \in \bigl(t'_{k+1}, t'_{k+2}\bigr]. \qquad (3.44)
\end{aligned}$$

Let $\hat{e}_{k+1}(t) = u(t) - \hat{u}(t)$ for $t \in (t'_{k+1}, t'_{k+2}]$. By (3.43) the exact solution of system (1.1) satisfies

$$\begin{aligned}
&\lim_{\substack{\Delta_i(u(t_i^-))\to 0, \bar{\Delta}_j(u(\bar{t}_j^-))\to 0 \\ \text{for all } i \text{ and } j}} u(t) \\
&\quad = \frac{u_1}{\Gamma(q)}\left(\int_a^t \frac{ds}{s}\right)^{q-1} + \frac{u_2}{\Gamma(q-1)}\left(\int_a^t \frac{ds}{s}\right)^{q-2} \\
&\qquad + \frac{1}{\Gamma(q)}\int_a^t \left(\ln\frac{t}{s}\right)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s} \quad \text{for } t \in \bigl(t'_{k+1}, t'_{k+2}\bigr],
\end{aligned}$$

$$\begin{aligned}
&\lim_{\substack{\Delta_i(u(t_i^-))\to 0 \text{ for all } i\in\{l_0+1, l_0+2,\ldots,(l+1)_0\}, \\ \bar{\Delta}_j(u(\bar{t}_j^-))\to 0 \text{ for all } j\in\{l_1+1, l_1+2,\ldots,(l+1)_1\}}} u(t) \\
&\quad = \frac{u_1}{\Gamma(q)}\left(\int_a^t \frac{ds}{s}\right)^{q-1} + \frac{u_2}{\Gamma(q-1)}\left(\int_a^t \frac{ds}{s}\right)^{q-2} + \frac{1}{\Gamma(q)}\int_a^t \left(\ln\frac{t}{s}\right)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s} \\
&\qquad + \sum_{\substack{1\le i\le (k+1)_0 \text{ and} \\ i\notin\{l_0+1, l_0+2,\ldots,(l+1)_0\}}} \frac{\Delta_i(u(t_i^-))}{\Gamma(q-1)}\left(\int_{t_i}^t \frac{ds}{s}\right)^{q-2} \\
&\qquad + \sum_{\substack{1\le j\le (k+1)_1 \text{ and} \\ j\notin\{l_1+1, l_1+2,\ldots,(l+1)_1\}}} \frac{\bar{\Delta}_j(u(\bar{t}_j^-))}{\Gamma(q)}\left(\int_{\bar{t}_j}^t \frac{ds}{s}\right)^{q-1} \\
&\qquad - \sum_{\substack{1\le i\le (k+1)_0 \text{ and} \\ i\notin\{l_0+1, l_0+2,\ldots,(l+1)_0\}}} \lambda\Delta_i\bigl(u\bigl(t_i^-\bigr)\bigr)\left[\frac{u_1}{\Gamma(q)}\left(\int_a^t \frac{ds}{s}\right)^{q-1} + \frac{u_2}{\Gamma(q-1)}\left(\int_a^t \frac{ds}{s}\right)^{q-2}\right. \\
&\qquad + \frac{1}{\Gamma(q)}\int_a^t \left(\ln\frac{t}{s}\right)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s} - \frac{u_1 + \int_a^{t_i} f(s,u(s))\frac{ds}{s}}{\Gamma(q)}\left(\int_{t_i}^t \frac{ds}{s}\right)^{q-1} \\
&\qquad - \frac{u_1 \ln\frac{t_i}{a} + u_2 + \int_a^{t_i} (\ln\frac{t_i}{s}) f(s,u(s))\frac{ds}{s}}{\Gamma(q-1)}\left(\int_{t_i}^t \frac{ds}{s}\right)^{q-2} \\
&\qquad \left. - \frac{1}{\Gamma(q)}\int_{t_i}^t \left(\ln\frac{t}{s}\right)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s}\right]
\end{aligned}$$

$$
\begin{aligned}
&- \sum_{\substack{1\le j\le (k+1)_1 \text{ and}\\ j\notin\{l_1+1,l_1+2,\ldots,(l+1)_1\}}} \hbar\bar{\Delta}_j\bigl(u(\bar{t}_j^-)\bigr)\biggl[\frac{u_1}{\Gamma(q)}\biggl(\int_a^t \frac{ds}{s}\biggr)^{q-1} + \frac{u_2}{\Gamma(q-1)}\biggl(\int_a^t \frac{ds}{s}\biggr)^{q-2} \\
&+ \frac{1}{\Gamma(q)}\int_a^t \biggl(\ln\frac{t}{s}\biggr)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s} - \frac{u_1 + \int_a^{\bar{t}_j} f(s,u(s))\frac{ds}{s}}{\Gamma(q)}\biggl(\int_{\bar{t}_j}^t \frac{ds}{s}\biggr)^{q-1} \\
&- \frac{u_1 \ln\frac{\bar{t}_j}{a} + u_2 + \int_a^{\bar{t}_j}(\ln\frac{\bar{t}_j}{s}) f(s,u(s))\frac{ds}{s}}{\Gamma(q-1)}\biggl(\int_{\bar{t}_j}^t \frac{ds}{s}\biggr)^{q-2} \\
&- \frac{1}{\Gamma(q)}\int_{\bar{t}_j}^t \biggl(\ln\frac{t}{s}\biggr)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s}\biggr] \quad \text{for } t\in\bigl(t'_{k+1},t'_{k+2}\bigr] \text{ and } 1\le l\le k+1.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\lim_{\substack{\Delta_i(u(t_i^-))\to 0, \bar{\Delta}_j(u(\bar{t}_j^-))\to 0\\ \text{for all } i \text{ and } j}} \hat{e}_{k+1}(t) \\
&\quad= \lim_{\substack{\Delta_i(u(t_i^-))\to 0, \bar{\Delta}_j(u(\bar{t}_j^-))\to 0\\ \text{for all } i \text{ and } j}} \bigl\{u(t) - \hat{u}(t)\bigr\} \\
&\quad= \frac{u_1}{\Gamma(q)}\biggl(\int_a^t \frac{ds}{s}\biggr)^{q-1} + \frac{u_2}{\Gamma(q-1)}\biggl(\int_a^t \frac{ds}{s}\biggr)^{q-2} + \frac{1}{\Gamma(q)}\int_a^t\biggl(\ln\frac{t}{s}\biggr)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s} \\
&\qquad- \frac{u_1 + \int_a^{t'_{k+1}} f(s,u(s))\frac{ds}{s}}{\Gamma(q)}\biggl(\int_{t'_{k+1}}^t \frac{ds}{s}\biggr)^{q-1} \\
&\qquad- \frac{u_1\int_a^{t'_{k+1}}\frac{ds}{s} + u_2 + \int_a^{t'_{k+1}} \ln\frac{t'_{k+1}}{s} f(s,u(s))\frac{ds}{s}}{\Gamma(q-1)}\biggl(\int_{t'_{k+1}}^t \frac{ds}{s}\biggr)^{q-2} \\
&\qquad- \frac{1}{\Gamma(q)}\int_{t'_{k+1}}^t \biggl(\ln\frac{t}{s}\biggr)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s} \quad \text{for } t\in\bigl(t'_{k+1},t'_{k+2}\bigr],
\end{aligned}
\tag{3.45}
$$

$$
\begin{aligned}
&\lim_{\substack{\Delta_i(u(t_i^-))\to 0 \text{ for all } i\in\{l_0+1,l_0+2,\ldots,(l+1)_0\},\\ \bar{\Delta}_j(u(\bar{t}_j^-))\to 0 \text{ for all } j\in\{l_1+1,l_1+2,\ldots,(l+1)_1\}}} \hat{e}_{k+1}(t) \\
&\quad= \lim_{\substack{\Delta_i(u(\bar{t}_i^-))\to 0 \text{ for all } i\in\{l_0+1,l_0+2,\ldots,(l+1)_0\},\\ \bar{\Delta}_j(u(\bar{t}_j^-))\to 0 \text{ for all } j\in\{l_1+1,l_1+2,\ldots,(l+1)_1\}}} \bigl\{u(t) - \hat{u}(t)\bigr\} \\
&\quad= \sum_{\substack{1\le i\le (k+1)_0 \text{ and}\\ i\notin\{l_0+1,l_0+2,\ldots,(l+1)_0\}}} \frac{\Delta_i(u(t_i^-))}{\Gamma(q-1)}\biggl[\biggl(\int_{t_i}^t \frac{ds}{s}\biggr)^{q-2} - \biggl(\int_{t'_{k+1}}^t \frac{ds}{s}\biggr)^{q-2}\biggr] \\
&\qquad+ \sum_{\substack{1\le j\le (k+1)_1 \text{ and}\\ j\notin\{l_1+1,l_1+2,\ldots,(l+1)_1\}}} \frac{\bar{\Delta}_j(u(\bar{t}_j^-))}{\Gamma(q)}\biggl[\biggl(\int_{\bar{t}_j}^t \frac{ds}{s}\biggr)^{q-1} - \biggl(\int_{t'_{k+1}}^t \frac{ds}{s}\biggr)^{q-1}\biggr] \\
&\qquad- \sum_{\substack{1\le j\le k_1 \text{ and}\\ j\notin\{l_1+1,l_1+2,\ldots,(l+1)_1\}}} \bar{\Delta}_j\bigl(u(\bar{t}_j^-)\bigr)\ln\frac{t'_{k+1}}{\bar{t}_j}\biggl(\int_{t'_{k+1}}^t \frac{ds}{s}\biggr)^{q-2} \\
&\qquad+ \frac{u_1}{\Gamma(q)}\biggl(\int_a^t \frac{ds}{s}\biggr)^{q-1} + \frac{u_2}{\Gamma(q-1)}\biggl(\int_a^t \frac{ds}{s}\biggr)^{q-2} + \frac{1}{\Gamma(q)}\int_a^t\biggl(\ln\frac{t}{s}\biggr)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s}
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{u_1 + \int_a^{t'_{k+1}} f(s,u(s))\frac{ds}{s}}{\Gamma(q)} \left(\int_{t'_{k+1}}^t \frac{ds}{s}\right)^{q-1} \\
&- \frac{u_1 \int_a^{t'_{k+1}} \frac{ds}{s} + u_2 + \int_a^{t'_{k+1}} \ln \frac{t'_{k+1}}{s} f(s,u(s))\frac{ds}{s}}{\Gamma(q-1)} \left(\int_{t'_{k+1}}^t \frac{ds}{s}\right)^{q-2} \\
&- \frac{1}{\Gamma(q)} \int_{t'_{k+1}}^t \left(\ln \frac{t}{s}\right)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s} \\
&- \sum_{\substack{1\le i\le (k+1)_0 \text{ and} \\ i\notin\{l_0+1,l_0+2,\ldots,(l+1)_0\}}} \lambda \Delta_i\bigl(u\bigl(t_i^-\bigr)\bigr)\biggl[\frac{u_1}{\Gamma(q)}\left(\int_a^t \frac{ds}{s}\right)^{q-1} + \frac{u_2}{\Gamma(q-1)}\left(\int_a^t \frac{ds}{s}\right)^{q-2} \\
&+ \frac{1}{\Gamma(q)} \int_a^t \left(\ln \frac{t}{s}\right)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s} - \frac{u_1 + \int_a^{t_i} f(s,u(s))\frac{ds}{s}}{\Gamma(q)} \left(\int_{t_i}^t \frac{ds}{s}\right)^{q-1} \\
&- \frac{u_1 \ln\frac{t_i}{a} + u_2 + \int_a^{t_i} (\ln\frac{t_i}{s}) f(s,u(s))\frac{ds}{s}}{\Gamma(q-1)} \left(\int_{t_i}^t \frac{ds}{s}\right)^{q-2} \\
&- \frac{1}{\Gamma(q)} \int_{t_i}^t \left(\ln \frac{t}{s}\right)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s}\biggr] \\
&- \sum_{\substack{1\le j\le (k+1)_1 \text{ and} \\ j\notin\{l_1+1,l_1+2,\ldots,(l+1)_1\}}} \hbar \bar{\Delta}_j\bigl(u\bigl(\bar{t}_j^-\bigr)\bigr)\biggl[\frac{u_1}{\Gamma(q)}\left(\int_a^t \frac{ds}{s}\right)^{q-1} + \frac{u_2}{\Gamma(q-1)}\left(\int_a^t \frac{ds}{s}\right)^{q-2} \\
&+ \frac{1}{\Gamma(q)} \int_a^t \left(\ln \frac{t}{s}\right)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s} - \frac{u_1 + \int_a^{\bar{t}_j} f(s,u(s))\frac{ds}{s}}{\Gamma(q)} \left(\int_{\bar{t}_j}^t \frac{ds}{s}\right)^{q-1} \\
&- \frac{u_1 \ln\frac{\bar{t}_j}{a} + u_2 + \int_a^{\bar{t}_j} (\ln\frac{\bar{t}_j}{s}) f(s,u(s))\frac{ds}{s}}{\Gamma(q-1)} \left(\int_{\bar{t}_j}^t \frac{ds}{s}\right)^{q-2} \\
&- \frac{1}{\Gamma(q)} \int_{\bar{t}_j}^t \left(\ln \frac{t}{s}\right)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s}\biggr] \quad \text{for } t\in\bigl(t'_{k+1},t'_{k+2}\bigr] \text{ and } 1\le l\le k+1. \qquad (3.46)
\end{aligned}
$$

By (3.45) and (3.46) we have

$$
\begin{aligned}
&\hat{e}_{k+1}(t) \\
&\quad= \sum_{i=1}^{(k+1)_0} \frac{\Delta_i(u(t_i^-))}{\Gamma(q-1)}\biggl[\left(\int_{t_i}^t \frac{ds}{s}\right)^{q-2} - \left(\int_{t'_{k+1}}^t \frac{ds}{s}\right)^{q-2}\biggr] \\
&\qquad+ \sum_{j=1}^{(k+1)_1} \frac{\bar{\Delta}_j(u(\bar{t}_j^-))}{\Gamma(q)}\biggl[\left(\int_{\bar{t}_j}^t \frac{ds}{s}\right)^{q-1} - \left(\int_{t'_{k+1}}^t \frac{ds}{s}\right)^{q-1}\biggr] \\
&\qquad- \sum_{j=1}^{k_1} \bar{\Delta}_j\bigl(u\bigl(\bar{t}_j^-\bigr)\bigr)\ln\frac{t'_{k+1}}{\bar{t}_j}\left(\int_{t'_{k+1}}^t \frac{ds}{s}\right)^{q-2} + \frac{u_1}{\Gamma(q)}\left(\int_a^t \frac{ds}{s}\right)^{q-1} + \frac{u_2}{\Gamma(q-1)}\left(\int_a^t \frac{ds}{s}\right)^{q-2} \\
&\qquad+ \frac{1}{\Gamma(q)} \int_a^t \left(\ln \frac{t}{s}\right)^{q-1} f\bigl(s,u(s)\bigr)\frac{ds}{s} - \frac{u_1 + \int_a^{t'_{k+1}} f(s,u(s))\frac{ds}{s}}{\Gamma(q)} \left(\int_{t'_{k+1}}^t \frac{ds}{s}\right)^{q-1} \\
&\qquad- \frac{u_1 \int_a^{t'_{k+1}} \frac{ds}{s} + u_2 + \int_a^{t'_{k+1}} \ln \frac{t'_{k+1}}{s} f(s,u(s))\frac{ds}{s}}{\Gamma(q-1)} \left(\int_{t'_{k+1}}^t \frac{ds}{s}\right)^{q-2}
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{\Gamma(q)} \int_{t'_{k+1}}^{t} \left(\ln \frac{t}{s}\right)^{q-1} f\bigl(s, u(s)\bigr) \frac{ds}{s} \\
&- \sum_{i=1}^{(k+1)_0} \lambda \Delta_i\bigl(u\bigl(t_i^-\bigr)\bigr) \biggl[\frac{u_1}{\Gamma(q)} \left(\int_a^t \frac{ds}{s}\right)^{q-1} + \frac{u_2}{\Gamma(q-1)} \left(\int_a^t \frac{ds}{s}\right)^{q-2} \\
&+ \frac{1}{\Gamma(q)} \int_a^t \left(\ln \frac{t}{s}\right)^{q-1} f\bigl(s, u(s)\bigr) \frac{ds}{s} - \frac{u_1 + \int_a^{t_i} f(s, u(s)) \frac{ds}{s}}{\Gamma(q)} \left(\int_{t_i}^t \frac{ds}{s}\right)^{q-1} \\
&- \frac{u_1 \ln \frac{t_i}{a} + u_2 + \int_a^{t_i} (\ln \frac{t_i}{s}) f(s, u(s)) \frac{ds}{s}}{\Gamma(q-1)} \left(\int_{t_i}^t \frac{ds}{s}\right)^{q-2} \\
&- \frac{1}{\Gamma(q)} \int_{t_i}^t \left(\ln \frac{t}{s}\right)^{q-1} f\bigl(s, u(s)\bigr) \frac{ds}{s} \biggr] \\
&- \sum_{i=1}^{(k+1)_1} \hbar \bar{\Delta}_j\bigl(u\bigl(\bar{t}_j^-\bigr)\bigr) \biggl[\frac{u_1}{\Gamma(q)} \left(\int_a^t \frac{ds}{s}\right)^{q-1} + \frac{u_2}{\Gamma(q-1)} \left(\int_a^t \frac{ds}{s}\right)^{q-2} \\
&+ \frac{1}{\Gamma(q)} \int_a^t \left(\ln \frac{t}{s}\right)^{q-1} f\bigl(s, u(s)\bigr) \frac{ds}{s} - \frac{u_1 + \int_a^{\bar{t}_j} f(s, u(s)) \frac{ds}{s}}{\Gamma(q)} \left(\int_{\bar{t}_j}^t \frac{ds}{s}\right)^{q-1} \\
&- \frac{u_1 \ln \frac{\bar{t}_j}{a} + u_2 + \int_a^{\bar{t}_j} (\ln \frac{\bar{t}_j}{s}) f(s, u(s)) \frac{ds}{s}}{\Gamma(q-1)} \left(\int_{\bar{t}_j}^t \frac{ds}{s}\right)^{q-2} \\
&- \frac{1}{\Gamma(q)} \int_{\bar{t}_j}^t \left(\ln \frac{t}{s}\right)^{q-1} f\bigl(s, u(s)\bigr) \frac{ds}{s} \biggr] \quad \text{for } t \in \bigl(t'_{k+1}, t'_{k+2}\bigr]. \qquad (3.47)
\end{aligned}
$$

Thus,

$$
\begin{aligned}
u(t) &= \hat{u}(t) + \hat{e}_{k+1}(t) \\
&= \frac{u_1}{\Gamma(q)} \left(\int_a^t \frac{ds}{s}\right)^{q-1} + \frac{u_2}{\Gamma(q-1)} \left(\int_a^t \frac{ds}{s}\right)^{q-2} + \frac{1}{\Gamma(q)} \int_a^t \left(\ln \frac{t}{s}\right)^{q-1} f\bigl(s, u(s)\bigr) \frac{ds}{s} \\
&\quad + \sum_{i=1}^{(k+1)_0} \frac{\Delta_i(u(t_i^-))}{\Gamma(q-1)} \left(\int_{t_i}^t \frac{ds}{s}\right)^{q-2} + \sum_{j=1}^{(k+1)_1} \frac{\bar{\Delta}_j(u(\bar{t}_j^-))}{\Gamma(q)} \left(\int_{\bar{t}_j}^t \frac{ds}{s}\right)^{q-1} \\
&\quad - \sum_{i=1}^{(k+1)_0} \lambda \Delta_i\bigl(u\bigl(t_i^-\bigr)\bigr) \biggl[\frac{u_1}{\Gamma(q)} \left(\int_a^t \frac{ds}{s}\right)^{q-1} + \frac{u_2}{\Gamma(q-1)} \left(\int_a^t \frac{ds}{s}\right)^{q-2} \\
&\quad + \frac{1}{\Gamma(q)} \int_a^t \left(\ln \frac{t}{s}\right)^{q-1} f\bigl(s, u(s)\bigr) \frac{ds}{s} - \frac{u_1 + \int_a^{t_i} f(s, u(s)) \frac{ds}{s}}{\Gamma(q)} \left(\int_{t_i}^t \frac{ds}{s}\right)^{q-1} \\
&\quad - \frac{u_1 \ln \frac{t_i}{a} + u_2 + \int_a^{t_i} (\ln \frac{t_i}{s}) f(s, u(s)) \frac{ds}{s}}{\Gamma(q-1)} \left(\int_{t_i}^t \frac{ds}{s}\right)^{q-2} \\
&\quad - \frac{1}{\Gamma(q)} \int_{t_i}^t \left(\ln \frac{t}{s}\right)^{q-1} f\bigl(s, u(s)\bigr) \frac{ds}{s} \biggr] \\
&\quad - \sum_{i=1}^{(k+1)_1} \hbar \bar{\Delta}_j\bigl(u\bigl(\bar{t}_j^-\bigr)\bigr) \biggl[\frac{u_1}{\Gamma(q)} \left(\int_a^t \frac{ds}{s}\right)^{q-1} + \frac{u_2}{\Gamma(q-1)} \left(\int_a^t \frac{ds}{s}\right)^{q-2} \\
&\quad + \frac{1}{\Gamma(q)} \int_a^t \left(\ln \frac{t}{s}\right)^{q-1} f\bigl(s, u(s)\bigr) \frac{ds}{s} - \frac{u_1 + \int_a^{\bar{t}_j} f(s, u(s)) \frac{ds}{s}}{\Gamma(q)} \left(\int_{\bar{t}_j}^t \frac{ds}{s}\right)^{q-1}
\end{aligned}
$$

$$- \frac{u_1 \ln \frac{\bar{t}_j}{a} + u_2 + \int_a^{\bar{t}_j} (\ln \frac{\bar{t}_j}{s}) f(s,u(s)) \frac{ds}{s}}{\Gamma(q-1)} \left(\int_{\bar{t}_j}^t \frac{ds}{s} \right)^{q-2}$$

$$- \frac{1}{\Gamma(q)} \int_{\bar{t}_j}^t \left(\ln \frac{t}{s} \right)^{q-1} f\big(s,u(s)\big) \frac{ds}{s} \Bigg] \quad \text{for } t \in \big(t'_{k+1}, t'_{k+2}\big].$$

So, system (1.1) is equivalent to the integral equation (3.36). The proof is now completed. □

Corollary 3.5 *Let $q \in (1,2)$, and let λ, $\hbar$ be two constants. System* (1.2) *is equivalent to the fractional integral equation*

$$u(t) = \begin{cases} \frac{u_1}{\Gamma(q)} (\int_a^t \frac{ds}{s})^{q-1} + \frac{u_2}{\Gamma(q-1)} (\int_a^t \frac{ds}{s})^{q-2} + \frac{1}{\Gamma(q)} \int_a^t (\ln \frac{t}{s})^{q-1} f(s,u(s)) \frac{ds}{s} \\ \quad \textit{for } t \in (a,t_1], \\ \frac{u_1}{\Gamma(q)} (\int_a^t \frac{ds}{s})^{q-1} + \frac{u_2}{\Gamma(q-1)} (\int_a^t \frac{ds}{s})^{q-2} + \frac{1}{\Gamma(q)} \int_a^t (\ln \frac{t}{s})^{q-1} f(s,u(s)) \frac{ds}{s} \\ \quad + \sum_{i=1}^k [\frac{\Delta_i(u(t_i^-))}{\Gamma(q-1)} (\int_{t_i}^t \frac{ds}{s})^{q-2} + \frac{\bar{\Delta}_i(u(t_i^-))}{\Gamma(q)} (\int_{t_i}^t \frac{ds}{s})^{q-1}] \\ \quad - \sum_{i=1}^k [\lambda \Delta_i(u(t_i^-)) + \hbar \bar{\Delta}_i(u(t_i^-))] [\frac{u_1}{\Gamma(q)} (\int_a^t \frac{ds}{s})^{q-1} + \frac{u_2}{\Gamma(q-1)} (\int_a^t \frac{ds}{s})^{q-2} \\ \quad + \frac{1}{\Gamma(q)} \int_a^t (\ln \frac{t}{s})^{q-1} f(s,u(s)) \frac{ds}{s} \\ \quad - \frac{u_1 + \int_a^{t_i} f(s,u(s)) \frac{ds}{s}}{\Gamma(q)} (\int_{t_i}^t \frac{ds}{s})^{q-1} - \frac{u_1 \ln \frac{t_i}{a} + u_2 + \int_a^{t_i} (\ln \frac{t_i}{s}) f(s,u(s)) \frac{ds}{s}}{\Gamma(q-1)} (\int_{t_i}^t \frac{ds}{s})^{q-2} \\ \quad - \frac{1}{\Gamma(q)} \int_{t_i}^t (\ln \frac{t}{s})^{q-1} f(s,u(s)) \frac{ds}{s}] \\ \quad \textit{for } t \in J_k, k = 1,2,\ldots,m \end{cases} \tag{3.48}$$

provided that the integral in (3.48) *exists.*

4 Example

In this section, we give an example to illustrate the usefulness of our results.

Example 1 Let us consider the general solution of the impulsive fractional system

$$\begin{cases} {}_H D_{1^+}^{\frac{3}{2}} u(t) = \ln t, \quad t \in (1,3] \text{ and } t \neq 2, \\ \Delta({}_H \mathcal{J}_{1^+}^{\frac{1}{2}} u)|_{t=2} = {}_H \mathcal{J}_{1^+}^{\frac{1}{2}} u(2^+) - {}_H \mathcal{J}_{1^+}^{\frac{1}{2}} u(2^-) = \delta, \\ \Delta({}_H D_{1^+}^{\frac{1}{2}} u)|_{t=2} = {}_H D_{1^+}^{\frac{1}{2}} u(2^+) - {}_H D_{1^+}^{\frac{1}{2}} u(2^-) = \bar{\delta}, \\ {}_H \mathcal{J}_{1^+}^{\frac{1}{2}} u(1^+) = u_2, \qquad {}_H D_{1^+}^{\frac{1}{2}} u(1^+) = u_1. \end{cases} \tag{4.1}$$

By the Theorem 3.4 the general solution is obtained by

$$u(t) = \begin{cases} \frac{u_1}{\Gamma(\frac{3}{2})} (\int_1^t \frac{ds}{s})^{\frac{3}{2}-1} + \frac{u_2}{\Gamma(\frac{3}{2}-1)} (\int_1^t \frac{ds}{s})^{\frac{3}{2}-2} + \frac{1}{\Gamma(\frac{3}{2})} \int_1^t (\ln \frac{t}{s})^{\frac{3}{2}-1} \ln s \frac{ds}{s} \\ \quad \text{for } t \in (1,2], \\ \frac{u_1}{\Gamma(\frac{3}{2})} (\int_1^t \frac{ds}{s})^{\frac{3}{2}-1} + \frac{u_2}{\Gamma(\frac{3}{2}-1)} (\int_1^t \frac{ds}{s})^{\frac{3}{2}-2} + \frac{1}{\Gamma(\frac{3}{2})} \int_1^t (\ln \frac{t}{s})^{\frac{3}{2}-1} \ln s \frac{ds}{s} \\ \quad + \frac{\delta}{\Gamma(\frac{3}{2}-1)} (\int_2^t \frac{ds}{s})^{\frac{3}{2}-2} + \frac{\bar{\delta}}{\Gamma(\frac{3}{2})} (\int_2^t \frac{ds}{s})^{\frac{3}{2}-1} \\ \quad - (\lambda\delta + \hbar\bar{\delta}) [\frac{u_1}{\Gamma(\frac{3}{2})} (\int_1^t \frac{ds}{s})^{\frac{3}{2}-1} + \frac{u_2}{\Gamma(\frac{3}{2}-1)} (\int_1^t \frac{ds}{s})^{\frac{3}{2}-2} \\ \quad + \frac{1}{\Gamma(\frac{3}{2})} \int_1^t (\ln \frac{t}{s})^{\frac{3}{2}-1} \ln s \frac{ds}{s} - \frac{u_1 + \int_1^2 \ln s \frac{ds}{s}}{\Gamma(\frac{3}{2})} (\int_2^t \frac{ds}{s})^{\frac{3}{2}-1} \\ \quad - \frac{u_1 \ln 2 + u_2 + \int_1^2 (\ln \frac{2}{s}) \ln s \frac{ds}{s}}{\Gamma(\frac{3}{2}-1)} (\int_2^t \frac{ds}{s})^{\frac{3}{2}-2} - \frac{1}{\Gamma(\frac{3}{2})} \int_2^t (\ln \frac{t}{s})^{\frac{3}{2}-1} \ln s \frac{ds}{s}] \\ \quad \text{for } t \in (2,3]. \end{cases} \tag{4.2}$$

Here λ, $\hbar$ are two constants. Next, we will verify that Eq. (4.2) satisfies all conditions of system (4.1).

Taking the Hadamard fractional derivative of the both sides of Eq. (4.2), we have

(i) for $t \in (1,2]$,

$$\begin{aligned}
&{}_HD_{1^+}^{\frac{3}{2}}u(t)\\
&= \frac{1}{\Gamma(2-\frac{3}{2})}\left(t\frac{d}{dt}\right)^2\int_1^t\left(\ln\frac{t}{\eta}\right)^{2-\frac{3}{2}-1}\left[\frac{u_1}{\Gamma(\frac{3}{2})}\left(\int_1^\eta\frac{ds}{s}\right)^{\frac{3}{2}-1}+\frac{u_2}{\Gamma(\frac{3}{2}-1)}\left(\int_1^\eta\frac{ds}{s}\right)^{\frac{3}{2}-2}\right.\\
&\quad\left.+\frac{1}{\Gamma(\frac{3}{2})}\int_1^\eta\left(\ln\frac{\eta}{s}\right)^{\frac{3}{2}-1}\ln s\frac{ds}{s}\right]\frac{d\eta}{\eta}\\
&= \frac{1}{\Gamma(\frac{1}{2})\Gamma(\frac{3}{2})}\left(t\frac{d}{dt}\right)^2\int_1^t\left(\ln\frac{t}{\eta}\right)^{\frac{1}{2}-1}\left[\int_1^\eta\left(\ln\frac{\eta}{s}\right)^{\frac{3}{2}-1}\ln s\frac{ds}{s}\right]\frac{d\eta}{\eta}=\ln t,
\end{aligned}$$

(ii) for $t \in (2,3]$,

$$\begin{aligned}
{}_HD_{1^+}^{\frac{3}{2}}u(t) &= \frac{1}{\Gamma(2-\frac{3}{2})}\left(t\frac{d}{dt}\right)^2\int_1^t\left(\ln\frac{t}{\eta}\right)^{2-\frac{3}{2}-1}\left\{\frac{u_1}{\Gamma(\frac{3}{2})}\left(\int_1^\eta\frac{ds}{s}\right)^{\frac{3}{2}-1}\right.\\
&\quad+\frac{u_2}{\Gamma(\frac{3}{2}-1)}\left(\int_1^\eta\frac{ds}{s}\right)^{\frac{3}{2}-2}+\frac{1}{\Gamma(\frac{3}{2})}\int_1^\eta\left(\ln\frac{\eta}{s}\right)^{\frac{3}{2}-1}\ln s\frac{ds}{s}\\
&\quad+\frac{\delta}{\Gamma(\frac{3}{2}-1)}\left(\int_2^\eta\frac{ds}{s}\right)^{\frac{3}{2}-2}+\frac{\bar{\delta}}{\Gamma(\frac{3}{2})}\left(\int_2^\eta\frac{ds}{s}\right)^{\frac{3}{2}-1}\\
&\quad-(\lambda\delta+\hbar\bar{\delta})\left[\frac{u_1}{\Gamma(\frac{3}{2})}\left(\int_1^\eta\frac{ds}{s}\right)^{\frac{3}{2}-1}+\frac{u_2}{\Gamma(\frac{3}{2}-1)}\left(\int_1^\eta\frac{ds}{s}\right)^{\frac{3}{2}-2}\right.\\
&\quad+\frac{1}{\Gamma(\frac{3}{2})}\int_1^\eta\left(\ln\frac{\eta}{s}\right)^{\frac{3}{2}-1}\ln s\frac{ds}{s}-\frac{u_1+\int_1^2\ln s\frac{ds}{s}}{\Gamma(\frac{3}{2})}\left(\int_2^\eta\frac{ds}{s}\right)^{\frac{3}{2}-1}\\
&\quad-\frac{u_1\ln 2+u_2+\int_1^2(\ln\frac{2}{s})\ln s\frac{ds}{s}}{\Gamma(\frac{3}{2}-1)}\left(\int_2^\eta\frac{ds}{s}\right)^{\frac{3}{2}-2}\\
&\quad\left.\left.-\frac{1}{\Gamma(\frac{3}{2})}\int_2^\eta\left(\ln\frac{\eta}{s}\right)^{\frac{3}{2}-1}\ln s\frac{ds}{s}\right]\right\}\frac{d\eta}{\eta}\\
&=\left\{\ln t|_{t\geq 1}+\frac{1}{\Gamma(\frac{1}{2})}\left(t\frac{d}{dt}\right)^2\int_2^t\left(\ln\frac{t}{\eta}\right)^{\frac{1}{2}-1}\left[\frac{\delta}{\Gamma(\frac{1}{2})}\left(\int_2^\eta\frac{ds}{s}\right)^{\frac{3}{2}-2}\right.\right.\\
&\quad\left.+\frac{\bar{\delta}}{\Gamma(\frac{3}{2})}\left(\int_2^\eta\frac{ds}{s}\right)^{\frac{3}{2}-1}\right]\frac{d\eta}{\eta}\\
&\quad-(\lambda\delta+\hbar\bar{\delta})\left(\ln t|_{t\geq 1}-\frac{1}{\Gamma(\frac{1}{2})}\left(t\frac{d}{dt}\right)^2\int_2^t\left(\ln\frac{t}{\eta}\right)^{\frac{1}{2}-1}\right.\\
&\quad\times\left[\frac{u_1+\int_1^2\ln s\frac{ds}{s}}{\Gamma(\frac{3}{2})}\left(\int_2^\eta\frac{ds}{s}\right)^{\frac{3}{2}-1}+\frac{u_1\ln 2+u_2+\int_1^2(\ln\frac{2}{s})\ln s\frac{ds}{s}}{\Gamma(\frac{1}{2})}\left(\int_2^\eta\frac{ds}{s}\right)^{\frac{3}{2}-2}\right.\\
&\quad\left.\left.\left.+\frac{1}{\Gamma(\frac{3}{2})}\int_2^\eta\left(\ln\frac{\eta}{s}\right)^{\frac{3}{2}-1}\ln s\frac{ds}{s}\right]\frac{d\eta}{\eta}\right)\right\}_{t\in(2,3]}
\end{aligned}$$

$$= \left\{\ln t|_{t\geq 1} - (\lambda\delta + \hbar\bar{\delta})[\ln t|_{t\geq 1} - \ln t|_{t\geq 2}]\right\}_{t\in(2,3]}$$
$$= \ln t|_{t\in(2,3]}.$$

So, Eq. (4.2) satisfies the Hadamard fractional derivative condition of system (4.1).

By Definition 2.1 we obtain

$${}_H\mathcal{J}_{1^+}^{\frac{1}{2}}u(t) = \begin{cases} u_1\ln t + u_2 + \int_1^t \ln\frac{t}{s}\ln s\frac{ds}{s} & \text{for } t\in[1,2], \\ u_1\ln t + u_2 + \int_1^t \ln\frac{t}{s}\ln s\frac{ds}{s} + \delta + \bar{\delta}(\ln t - \ln 2) & \text{for } t\in(2,3], \end{cases}$$

and

$${}_HD_{1^+}^{\frac{1}{2}}u(t) = \begin{cases} u_1 + \int_1^t \ln s\frac{ds}{s} & \text{for } t\in[1,2], \\ u_1 + \int_1^t \ln s\frac{ds}{s} + \bar{\delta} & \text{for } t\in(2,3]. \end{cases}$$

Therefore,

$${}_H\mathcal{J}_{1^+}^{\frac{1}{2}}u(2^+) - {}_H\mathcal{J}_{1^+}^{\frac{1}{2}}u(2^-) = \delta \quad \text{and} \quad {}_HD_{1^+}^{\frac{1}{2}}u(2^+) - {}_HD_{1^+}^{\frac{1}{2}}u(2^-) = \bar{\delta}.$$

That is, Eq. (4.2) satisfies the impulsive condition in system (4.1).

Finally, it is obvious that Eq. (4.2) satisfies the three implicit conditions (i)-(iii). So, Eq. (4.2) is the general solution of system (4.1).

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally in this article. They read and approved the final manuscript.

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