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Symmetric periodic solutions of delay-coupled optoelectronic oscillators

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Abstract

Delay-coupled optoelectronic oscillators are considered. These structures are based on mutually coupled oscillators which oscillate at the same frequency. By taking the time delay as a bifurcation parameter, the stability of the zero equilibrium and the existence of Hopf bifurcations induced by delay are investigated, and then stability switches for the trivial solution are found. Conditions ensuring the stability and direction of the Hopf bifurcation are determined by applying the normal form theory and the center manifold theorem. Using the symmetric functional differential equation theories combined with the representation theory of Lie groups, the multiple Hopf bifurcations of the equilibrium are demonstrated. In particular, we find that the spatio-temporal patterns of bifurcating periodic oscillations will alternate according to the change of the propagation time delay in the coupling. The existence of multiple branches of bifurcating periodic solutions and their spatio-temporal patterns are obtained. Some numerical simulations are used to illustrate the effectiveness of the obtained results.

Keywords: optoelectronic oscillators; symmetric bifurcation; stability; spatio-temporal patterns; delay

1 Introduction

A system of coupled nonlinear oscillators is capable of displaying a rich dynamics and has applications in various areas of science and technology, such as physical, chemical, biological, and other systems [1–3]. When oscillators are coupled, non-negligible coupling delays naturally arise because signals in real systems inevitably propagate from one oscillator to the next with a finite propagation speed. These time delays can induce many new phenomena and complex dynamics. Examples include neuronal networks, biological oscillators, and physical models [4–6].

The coupled optoelectronic oscillator is a novel and unique device which simultaneously produces spectrally pure microwave reference signals as in a microwave oscillator [7], and short optical pulses, in a mode locked laser [8–10]. In [8], the authors response of two delay-coupled optoelectronic oscillators. Each oscillator operates under its own delayed feedback. They show that the system can display square-wave periodic solutions that can be synchronized in phase or out of phase depending on the ratio between self- and cross-delay times. Furthermore, they show that multiple periodic synchronized solutions can coexist for the same values of the fixed parameters. As a consequence, it is possible to

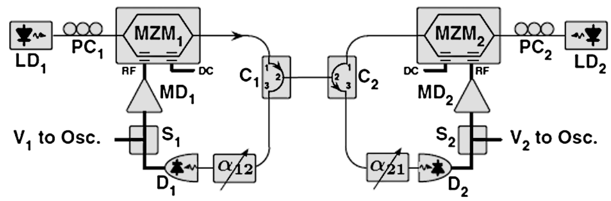


Figure 1 Schematic of the cross-coupled optoelectronic oscillators: LD, laser diodes; PC, polarization controllers; MZM, Mach-Zehnder modulators; C, optic circulators; α, adjustable optic attenuators; D, photodetectors; S, electronic splitters; MD, modulator drivers.

generate square-wave oscillations with different periods by just changing the initial conditions. Furthermore, in [9], the positive delayed feedback is considered, and they show that the scenario arising for positive feedback is much richer than with negative feedback. In [10], the authors model two non-identical delay-line optoelectronic oscillators mutually coupled through delayed cross-feedback. The system can generate multi-stable nanosecond periodic square-wave solutions which arise through a Hopf instability. They show that for suitable ratios between self- and cross-delay times, the two oscillators generate square waves with different amplitude but synchronized in phase, out of phase or with a dephasing of a quarter of the period.

In [11], two laser-pumped fiber-coupled Mach-Zehnder modulators (MZM) with a coupling scheme were experimentally set up, where the output intensity of one MZM is used to modulate the radio frequency (rf) input of the other MZM. In this paper we consider a system of two laser-pumped fiber-couple Mach-Zehnder modulators (MZM) as in [10], where the output intensity of one MZM is used to modulate the radio frequency (rf) input of the other MZM, as shown in Figure 1.

This system can be described by two coupled described by two coupled differential equations with nonlinearities $f(x) = \cos^2(m + d\tanh(x)) - \cos^2(m)$:

$$\begin{cases} \dot{x}_1 = -x_1(t) - y_1(t) + \gamma_{12} f(x_2(t-\tau)), \\ \dot{y}_1 = \varepsilon x_1(t), \\ \dot{x}_2 = -x_2(t) - y_2(t) + \gamma_{21} f(x_1(t-\tau)), \\ \dot{y}_2 = \varepsilon x_2(t), \end{cases} \tag{1.1}$$

where x_i $(i = 1, 2)$ account for the dimensionless and scaled variable corresponding to the measured output voltage, respectively; $\varepsilon > 0$ denotes the linear coefficient, and γ_{ij} $(i, j = 1, 2)$ the nonlinear coupled coefficient. The effective coupling strengths γ_{12} and γ_{21} take into account all losses and gains in the system and are directly proportional to α_{12} and α_{21}. In [11] it was shown in experiment and numerical simulations that this system has oscillatory solutions when the product of the two coupling strengths exceeds a critical value. Beyond the oscillation threshold, the oscillation amplitudes grow smoothly and the dependence of the amplitude on the coupling strengths can be described by a scaling law [11]. Beyond the oscillation threshold, the oscillation amplitudes grew smoothly and they found a scaling law that describes the dependence of the amplitude on the coupling strengths.

In this paper, considering the special case with $\gamma = \gamma_{12} = \gamma_{21}$, we attempt to analytically investigate how the time delay can affect its stability, the bifurcations of new solutions when stability is lost, and also spatio-temporal patterns of the bifurcating periodic oscillations. Spatio-temporal patterns reflect the relationship of the evolution of the dynamic behaviors of two sub-networks such as in-phase or anti-phase oscillation. The results show that the emerging oscillations can exhibit different spatio-temporal patterns sensitive to the delay. It has been shown that even small, compared to the oscillation period, delays may have a large impact on the dynamics of delay-coupled optoelectronic oscillators.

The plan for the article is as follows. In Section 2, we consider the linear stability of equation (1.1) and present some theorems about the region of stability of the trivial solution as a function of the physical parameters in the model. We find some new phenomena such as the stability switch for equation (1.1) which is not mentioned in [8–11]. Coupling can lead synchronization, phase trapping, phase locking, amplitude death, chaos, bifurcation of oscillators, and so on [12, 13]. Since two identical oscillators are coupled symmetrically, the most typical patterns of behavior are perfect synchrony or perfect antisynchrony (in which the oscillators are half a period out of phase with each other); see [14, 15]. In Section 3, we give the Z_2-equivariant property of equation (1.1) and the existence of multiple periodic solutions (synchronous (respectively, anti-phased). Then we investigate the direction and stability of the Hopf bifurcating periodic solutions in Section 4. In the final section, we present some numerical simulation to support our analytical results. The last section concludes the paper.

2 Stability and bifurcation analysis

It is clear that $(0,0,0,0)$ is an equilibrium point of equation (1.1). Linearizing system (1.1) at the trivial solution $(0,0,0,0)$ leads to the following linear system:

$$\begin{cases} \dot{x}_1 = -x_1(t) - y_1(t) + \gamma f'(0)x_2(t-\tau), \\ \dot{y}_1 = \varepsilon x_1(t), \\ \dot{x}_2 = -x_2(t) - y_2(t) + \gamma f'(0)x_1(t-\tau), \\ \dot{y}_2 = \varepsilon x_2(t). \end{cases} \tag{2.1}$$

The characteristic equation associated with the linearization of (2.1) is

$$\Delta = \left[\lambda^2 + \lambda + \varepsilon + a\lambda e^{-\lambda\tau}\right]\left[\lambda^2 + \lambda + \varepsilon - a\lambda e^{-\lambda\tau}\right] = \Delta_1\Delta_2 = 0, \tag{2.2}$$

where

$$\Delta_1 = \lambda^2 + \lambda + \varepsilon + a\lambda e^{-\lambda\tau},$$
$$\Delta_2 = \lambda^2 + \lambda + \varepsilon - a\lambda e^{-\lambda\tau},$$

and $a = \gamma f'(0)$. The distribution of zeros of equation (2.2) determines the dynamic properties of (1.1). In the following, the analysis on the distribution of the roots to equation (2.2) is based on the conclusion: the sum of the order of the zeros of equation (2.2) in an open right-plane can change only if a zero appears or crosses the imaginary axis as parameter is varied.

In the following, we assume that:

(H_1) either $1 - a^2 > 2\varepsilon$ or $0 < 1 - a^2 < 4\varepsilon$;
(H_2) $a > 1$.

Consider equation $\Delta_1 = 0$. Let $i\omega$ ($\omega > 0$) be a root of $\Delta_1 = 0$, then plugging $i\omega$ into $\Delta_1 = 0$ to get $-\omega^2 + i\omega + \varepsilon + ia\omega e^{-i\omega\tau} = 0$. Solving ω we have: if (H_1) holds, then $\Delta_1 = 0$ has no purely imaginary roots; if (H_2) holds, then $\Delta_1 = 0$ has a pair of purely imaginary roots $\pm i\omega^+$ ($\pm i\omega^-$, respectively) at $\tau = \tau_j^{1+}$ ($\tau = \tau_j^{1-}$, respectively), where

$$\begin{cases} \tau_j^{1+} = \frac{1}{\omega^+}(2j\pi + \pi - \arccos\frac{1}{a}), \\ \tau_j^{1-} = \frac{1}{\omega^-}(2j\pi + \pi + \arccos\frac{1}{a}), \quad j = 0, 1, 2, \ldots, \end{cases} \tag{2.3}$$

$$\omega^{\pm} = \sqrt{\frac{-1 + a^2 + 2\varepsilon \pm \sqrt{(-1 + a^2 + 2\varepsilon)^2 - 4\varepsilon^2}}{2}}. \tag{2.4}$$

Similarly, under the condition (H_2), we have: $\Delta_2 = 0$ has a pair of purely imaginary roots $\pm i\omega^+$ ($\pm i\omega^-$, respectively) at $\tau = \tau_2^{2+}$ ($\tau = \tau_j^{2-}$, respectively), where

$$\begin{cases} \tau_j^{2+} = \frac{1}{\omega^+}(2k\pi + 2\pi - \arccos\frac{1}{a}), \\ \tau_j^{2-} = \frac{1}{\omega^-}(2j\pi + \arccos\frac{1}{a}), \quad j = 0, 1, 2, \ldots, \end{cases} \tag{2.5}$$

and $\omega^{\pm}$ meets equation (2.4).

Rewrite equation (2.3) and equation (2.5):

$$\begin{cases} \tau_j^{+} = \frac{1}{\omega^+}((j+1)\pi - \arccos\frac{1}{a}), \\ \tau_j^{-} = \frac{1}{\omega^-}(j\pi + \arccos\frac{1}{a}), \quad j = 0, 1, 2, \ldots. \end{cases} \tag{2.6}$$

Then we have the following lemma.

Lemma 2.1

(1) *If* (H_1) *holds, then equation* (2.2) *has no purely imaginary roots.*

(2) *If* (H_2) *holds, then equation* (2.2) *has a pair of purely imaginary roots* $\pm i\omega^+$ ($\pm i\omega^-$, *respectively*) *at* $\tau = \tau_j^+$ ($\tau = \tau_j^-$, *respectively*). *Furthermore, the transversality condition is satisfied at* $\tau_j^{\pm}$ ($j = 0, 1, 2, \ldots$):

$$\begin{cases} \frac{d\,\mathrm{Re}(\lambda(\tau))}{d\tau}\big|_{\tau=\tau_j^+, \omega=\omega^+} > 0, \\ \frac{d\,\mathrm{Re}(\lambda(\tau))}{d\tau}\big|_{\tau=\tau_j^-, \omega=\omega^-} < 0. \end{cases}$$

When $\tau = 0$, equation (2.2) has at least one root with positive real part. Since we are concerned about the stability of equation (1.1), we prefer to investigate the relations of τ_0^+ and τ_0^-. From equation (2.6), we have $\tau_0^+ > \tau_0^-$ if and only if $\frac{\omega^- \pi}{\omega^+ + \omega^-} > \arccos\frac{1}{a}$.

Applying the above conclusions and bifurcation theorems for functional differential equations, we have the following results presenting the stability and existence bifurcations to the symmetric system (1.1).

Theorem 2.1

(1) *If either* $1 - a^2 > 2\varepsilon$ *or* $0 < 1 - a^2 < 4\varepsilon$, *then the zero solution of equation* (1.1) *is unstable for all* $\tau \geq 0$.

(2) *If* $a > 1$ *and* $\frac{\omega^- \pi}{\omega^+ + \omega^-} > \arccos \frac{1}{a}$, *then* $\tau_0^- < \tau_0^+$, *and there exists a positive integer* j *such that* $\tau_{j-1}^- < \tau_{j-1}^+ < \tau_j^+ < \tau_j^-$. *Then the zero solution of equation* (1.1) *is asymptotically stable for* $\tau \in (\tau_0^-, \tau_0^+) \cup (\tau_1^-, \tau_1^+) \cup \cdots \cup (\tau_{j-1}^-, \tau_{j-1}^+)$ *and unstable for* $\tau \in [0, \tau_0^-) \cup (\tau_{j-1}^-, +\infty)$.

(3) *If* $a > 1$, *then for any* $k, l \in \{0, 1, 2, \ldots\}$, *the equation* (1.1) *undergoes a Hopf bifurcation at* τ_k^- *or* τ_l^+ *for* $\tau_k^- \neq \tau_l^+$.

Remark 2.1 Theorems 2.1 shows that under the conditions $a > 1$, and $\frac{\omega^- \pi}{\omega^+ + \omega^-} > \arccos \frac{1}{a}$, there are j switches from stability to instability to stability.

3 Existence of multiple periodic solutions

In the following, we consider the symmetric properties of equation (1.1). Using the theories of functional differential equations, (1.1) can be written as

$$\begin{aligned} &\dot{x}(t) = Lx_t + Fx_t, \\ &L\phi = \begin{pmatrix} A_1 & O \\ O & A_1 \end{pmatrix} \phi(0) + \begin{pmatrix} O & A_2 \\ A_2 & O \end{pmatrix} \phi(-\tau), \end{aligned} \tag{3.1}$$

where $x_t = x(t + \theta)$ for $-\tau \leq \theta \leq 0$,

$$A_1 = \begin{pmatrix} -1 & -1 \\ \varepsilon & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} \gamma f'(0) & 0 \\ 0 & 0 \end{pmatrix},$$

and

$$F\phi = -\begin{pmatrix} \frac{\gamma f''(0)}{2} \phi_3^2(-\tau) + \frac{\gamma f'''(0)}{6} \phi_3^3(-\tau) \\ 0 \\ \frac{\gamma f''(0)}{2} \phi_1^2(-\tau) + \frac{\gamma f'''(0)}{6} \phi_1^3(-\tau) \\ 0 \end{pmatrix}$$

for $\phi = (\phi_1, \phi_2, \phi_3, \phi_4) \in C([-\tau, 0], R^4)$.

It is clear that the system (3.1) is Z_2-equivariant with

$$(\rho U)_r = U_{r+1} (\mathrm{mod}\, 2)$$

for any U_r in R^2. It is very interesting to consider the spatio-temporal patterns of bifurcating periodic solutions. For this purpose, we give the concepts of some spatio-temporal symmetric periodic solutions. Assume that the state $(u_1(t), v_1(t), u_2(t), v_2(t))$ can possess two different types of symmetry: spatial and temporal.

The oscillators $(u_1(t), v_1(t))$ and $(u_2(t), v_2(t))$ are synchronized if the state takes the form

$$\bigl(u(t), v(t), u(t), v(t)\bigr)$$

for all times t. On the other hand, the oscillator $(u_1(t), v_1(t))$ is half a period out of phase with the (anti-synchronous) oscillator $(u_2(t), v_2(t))$ meaning the state takes the form

$$\left(u(t), v(t), u\left(t + \frac{T}{2}\right), v\left(t + \frac{T}{2}\right)\right).$$

Now, we explore the possible (spatial) symmetry of the system (3.1). Consider the action of $Z_2 \times S^1$ on $([-\tau, 0], R^4)$ with

$$(r,\theta)x(t) = rx(t+\theta), \quad (r,\theta) \in Z_2 \times S^1,$$

where S^1 is the temporal. Let $T = \frac{2\pi}{\omega^+}$ or $T = \frac{2\pi}{\omega^-}$, and denote by P_T the Banach space of all continuous T-periodic functions $x(t)$. Denoting by SP_T the subspace of P_T consisting of all T-periodic solution of system (3.1) with $\tau = \tau_k^{\pm}$, then, for each subgroup $\Sigma \subset Z_2 \times S^1$,

$$Fix(\Sigma, SP_T) = \bigl\{x \in SP_T, (r,\theta)x = x, \text{ for all } (r,\theta) \in \Sigma\bigr\}$$

is a subspace.

Theorem 3.1 *The trivial solution of system* (3.1) *undergoes a Hopf bifurcation giving rise to one branch of synchronous* (*respectively, anti-phased*) *periodic solutions.*

Proof Let $\pm i\omega^+$ (respectively, $\pm i\omega^-$) satisfy equation (2.4). The corresponding eigenvectors of $\Delta(\lambda)$ can be chosen as

$$q_1(\theta) = \bigl(\nu_1(\theta)^T, \nu_1(\theta)^T\bigr)^T,$$

where $\nu_1(\theta)$ satisfies $(A_1 + e^{-i\omega^{\pm}\tau_j^{2\pm}} A_2)\nu_1(\theta) = i\omega^{\pm}\nu_1(\theta)$;

$$q_2(\theta) = \bigl(\nu_2(\theta)^T, -\nu_2(\theta)^T\bigr)^T,$$

and $\nu_2(\theta)$ satisfies $(A_1 - e^{-i\omega^{\pm}\tau_j^{1\pm}} A_2)\nu_2(\theta) = i\omega^{\pm}\nu_2(\theta)$.

The isotropic subgroup of $Z_2 \times S^1$ is $z_2(\rho)$, the center space associated to eigenvalues $\pm i\omega^{\pm}$, which implies that it is spanned by $q_1(\theta)$ and $\bar{q}_1(\theta)$, and the bifurcated periodic solutions are synchronous, taking the form

$$\bigl(u(t), v(t), u(t), v(t)\bigr).$$

Similarly, $Z_2 \times S^1$ has another isotropic subgroup $z_2(\rho, \pi)$, the center space associated to eigenvalues $\pm i\omega^{\pm}$ is spanned by $q_2(\theta)$, $\bar{q}_2(\theta)$, which implies that the bifurcated periodic solutions are anti-phased, *i.e.*, taking the form

$$\left(u(t), v(t), u\left(t + \frac{T}{2}\right), v\left(t + \frac{T}{2}\right)\right),$$

where T is a period. □

4 Direction and stability of the Hopf bifurcation

In this section, we shall study the direction, stability, and the period of the bifurcating periodic solutions. We first focus on the case $\Delta_1 = 0$, because the other case can be dealt with analogously. We re-scale the time by $t \mapsto t/\tau$, to normalize the delay of the system

(3.1) and introduce the new parameter $\nu = \tau - \tau_k^{1\pm}$ such that $\nu = 0$ is a Hopf bifurcation value. Then equation (3.1) can be written as

$$\begin{aligned} &\dot{u}(t) = L_\nu u_t + F u_t, \\ &L_\nu \phi = \tau \left[\begin{pmatrix} A_1 & O \\ O & A_1 \end{pmatrix} \phi(0) + \begin{pmatrix} O & A_2 \\ A_2 & O \end{pmatrix} \phi(-1) \right], \end{aligned} \tag{4.1}$$

where $u_t = x(t+\theta)$ for $-1 \le \theta \le 0$,

$$A_1 = \begin{pmatrix} -1 & -1 \\ -\varepsilon & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} \gamma f'(0) & 0 \\ 0 & 0 \end{pmatrix},$$

and

$$F\phi = -\tau \begin{pmatrix} \frac{\gamma f''(0)}{2}\phi_3^2(-1) + \frac{\gamma f'''(0)}{6}\phi_3^3(-1) \\ 0 \\ \frac{\gamma f''(0)}{2}\phi_1^2(-1) + \frac{\gamma f'''(0)}{6}\phi_1^3(-1) \\ 0 \end{pmatrix}$$

for $\phi = (\phi_1, \phi_2, \phi_3, \phi_4) \in C([-1,0], R^4)$.

By the Riesz representation theorem, there exists a function $\eta(\theta, \mu)$ $(0 \le \theta \le 1)$, whose elements are of bounded variation such that

$$L_\nu \phi = \int_{-1}^{0} d\eta(\theta, \nu)\phi(\theta), \quad \phi \in C.$$

In fact, we choose

$$\eta(\theta, \nu) = \left(\tau_j^{1\pm} + \nu\right) A\delta(\theta) + \left(\tau_j^{1\pm} + \nu\right) B\delta(\theta + 1),$$

where δ is determined by

$$\delta(\theta) = \begin{cases} 1, & \theta = 0, \\ 0, & \theta \neq 0. \end{cases}$$

For $\phi \in C^1([-1,0], R^4)$, define

$$A(\nu)\phi = \begin{cases} d\phi(\theta)/d\theta, & \theta \in [-1, 0), \\ \int_{-1}^{0} d\eta(t, \nu)\phi(t), & \theta = 0, \end{cases}$$

and

$$R(\nu)\varphi = \begin{cases} 0, & \theta \in [-1, 0), \\ F(\nu, \varphi), & \theta = 0. \end{cases}$$

Then system (4.1) is equivalent to the following operator equation:

$$\dot{u_t} = A(\nu)u_t + R(\nu)u_t, \tag{4.2}$$

where $u_t = u(t+\theta)$, $\theta \in [-1,0]$. For $\psi \in C^1([0,1],(R^4)^*)$, define

$$A^*\psi(s) = \begin{cases} -d\psi(s)/ds, & s \in (0,1], \\ \int_{-1}^0 d\eta^T(s,\nu)\phi(-s), & s = 0, \end{cases}$$

and a bilinear form

$$\langle \psi(s), \phi(\theta) \rangle = \bar{\psi}(0)\phi(0) - \int_{\theta=1}^{0}\int_{\xi=0}^{\theta} \bar{\psi}(\xi-\theta)\,d\eta(\theta)\phi(\xi)\,d\xi,$$

where $\eta(\theta) = \eta(\theta,0)$, then $A(0)$ and A^* are adjoint operators. From Section 3, $q_1(\theta)$ and $q_1^*(s)$ are the eigenvectors of $A(0)$ and A^* corresponding to $i\tau_j^{1\pm}\omega^{\pm}$ and $-i\tau_j^{1\pm}\omega^{\pm}$, where $q_1(\theta) = (-\varepsilon, i\omega^{\pm}, -\varepsilon, i\omega^{\pm})^T$, $q^*(s) = \bar{D}(i\omega^{\pm}, 1, i\omega^{\pm}, 1)$, and $D = \tau_j^{1\pm}e^{i\omega^{\pm}\tau_j^{1\pm}}$. Then $\langle q^*, q\rangle = 1$ and $\langle q^*, \bar{q}\rangle = 0$.

Let u_t be the solution of equation (4.2) when $\nu = 0$. $z(t) = \langle q^*, u_t\rangle$, $W(t,\theta) = u_t(\theta) - 2\operatorname{Re}\{z(t)q(\theta)\}$. On the center manifold C_0 we have $W(t,\theta) = W(z(t),\bar{z}(t),\theta)$, where

$$W(z,\bar{z},\theta) = W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\bar{z} + W_{02}(\theta)\frac{z^2}{2} + W_{30}\frac{z^3}{6} + \cdots$$

z and $\bar{z}$ are local coordinates for the center manifold C_0 in the direction of q^* and $\bar{q}^*$. Note that W is real if u_t is real. We only consider real solutions. For the solution $u_t \in C_0$ of equation (4.2), since $\nu = 0$, we have

$$\begin{aligned} z'(t) &= \mathrm{i}\omega^{\pm}z + \bigl\langle q^*(\theta), F\bigl(W + 2\operatorname{Re}\bigl\{z(t)q(\theta)\bigr\}\bigr)\bigr\rangle \\ &\stackrel{\text{def}}{=} \mathrm{i}\omega^{\pm}z + \bar{q}^*(0)F_0(z,\bar{z}). \end{aligned}$$

We rewrite this equation as

$$z'(t) = \mathrm{i}\omega^{\pm}z(t) + g(z,\bar{z}) \tag{4.3}$$

with

$$\begin{aligned} g(z,\bar{z}) &= \bar{q}^*(0)\tilde{F}\bigl(W(z,\bar{z},0) + 2\operatorname{Re}\bigl\{z(t)q(0)\bigr\}\bigr) \\ &= g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + g_{21}\frac{z^2\bar{z}}{2} + \cdots. \end{aligned} \tag{4.4}$$

It follows from (4.3) and (4.4) that

$$W' = u_t' - z'q - \bar{z}'\bar{q} = \begin{cases} AW - 2\operatorname{Re}\{\bar{q}^*(0)Fq(\theta)\}, & \theta \in [-1,0), \\ AW - 2\operatorname{Re}\{\bar{q}^*(0)Fq(\theta)\} + F, & \theta = 0. \end{cases}$$

Comparing of coefficients we have

$$\begin{aligned} g_{20} &= -\frac{\gamma f''(0)\bar{D}}{2}i\omega^{\pm}\varepsilon^2 e^{-2i\omega^{\pm}\tau_k^{1\pm}}, \\ g_{11} &= -\gamma f''(0)\bar{D}i\omega^{\pm}\varepsilon^2, \end{aligned}$$

$$g_{02} = -\frac{\gamma f''(0)\bar{D}}{2} i\omega^{\pm}\varepsilon^2 e^{2i\omega^{\pm}\tau_k^{1\pm}},$$
$$g_{21} = \frac{\gamma f'''(0)\bar{D}}{6}\big[i\omega^{\pm}\varepsilon\big(W_{11}^{(3)}(-1) + W_{11}^{(1)}(-1)\big)e^{-2i\omega^{\pm}\tau_k^{1\pm}} + i\omega^{\pm}\varepsilon^3 e^{-i\omega^{\pm}\tau_k^{1\pm}}\big],$$

where

$$W_{11}(\theta) = \frac{g_{11}}{\omega^{\pm}} q(0)e^{i\omega^{\pm}\theta} - \frac{\bar{g}_{11}}{i\omega^{\pm}}\bar{q}(0)e^{-i\omega^{\pm}\theta} + E_2.$$

Moreover, E_2 satisfies the following equations:

$$\begin{pmatrix} -1 & -1 & \gamma f'(0) & 0 \\ \varepsilon & 0 & 0 & 0 \\ \gamma f'(0) & 0 & -1 & -1 \\ 0 & 0 & -\varepsilon & 0 \end{pmatrix} E_2 = \gamma f''(0)\bar{D}\begin{pmatrix} \varepsilon^2 \\ 0 \\ \varepsilon^2 \\ 0 \end{pmatrix}.$$

Then we can compute the following quantities:

$$\begin{aligned} c_1(0) &= \frac{i}{2\omega^{\pm}}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2\right) + \frac{g_{21}}{2}, \\ \nu_2 &= -\frac{\operatorname{Re} c_1(0)}{\operatorname{Re}\lambda'(\tau_k^{1\pm})}, \\ \beta_2 &= 2\operatorname{Re} c_1(0). \end{aligned} \tag{4.5}$$

Theorem 4.1 *Assume that Δ_1 crosses critical values* 0. *In* (4.5), *the sign of ν_2 determines the direction of the Hopf bifurcation*: *if $\nu_2 > 0$ ($\nu_2 < 0$), then the Hopf bifurcation is supercritical* (*subcritical*) *and the bifurcating periodic solution exists for $\tau > \tau_j^{1\pm}$ ($\tau < \tau_j^{1\pm}$). β_2 determines the stability of the bifurcating periodic solution*: *the bifurcating periodic solution is stable* (*unstable*) *if $\beta_2 < 0$ ($\beta_2 > 0$).*

We also can focus on the case $\Delta_2 = 0$. Since the conclusions are the same, we shall omit it.

5 Numerical simulations

In this section, we use some numerical simulations to illustrate the analytical results we obtained in the previous sections.

Let $m = \frac{\pi}{4}$, $d = 1$. Then $f(x) = \cos^2(\frac{\pi}{4} + \tanh(x)) - \cos^2(m)$.

For parameters $\gamma = 1.155$, $\varepsilon = 1$, using the algorithm in Section 3, we obtain $\omega^+ = 1.33$, $\omega^- = 0.75$. Then $\tau_0^- = 0.696 < \tau_0^+ = 1.97 < \tau_1^+ = 4.33 < \tau_1^- = 4.87$. From Theorem 4.1, the zero solution is asymptotically stable with $\tau_0^- < \tau < \tau_0^+$. When τ meets $\tau_j^{1\pm}$ or $\tau_j^{2\pm}$, the anti-phased or synchronous periodic solutions will appear.

Figures 2-3 show that a branch of anti-phased periodic solutions is bifurcated from the trivial solution with $\tau \le \tau_0^- = 0.696$.

Figures 4-5 show that the zero solution is asymptotically stable when $\tau_0^- = 0.696 < \tau < \tau_0^+ = 1.97$.

Figures 6-7 show a branch of synchronous periodic solutions is bifurcated from the trivial solution when $\tau_0^+ = 1.97 \le \tau \le \tau_1^+ = 4.33$. Figure 8 shows the anti-phased periodic solutions appear again from the synchronous periodic solutions.

Figure 2 The bifurcation diagram for equation (2.1). The zero solution is unstable for all $\tau \geq 0$ when $(a, \varepsilon) \in D_1$. When $(a, \varepsilon) \in D_2$, the zero solution is unstable for all $\tau \geq 0$ and system undergoes a Hopf bifurcation at $\tau = \tau_j^{\pm}$ $(j = 0, 1, 2, \ldots)$. When $(a, \varepsilon) \in D_3$, $\tau_0^+ < \tau_0^- < \tau_1^+$ and the zero solution is stable.

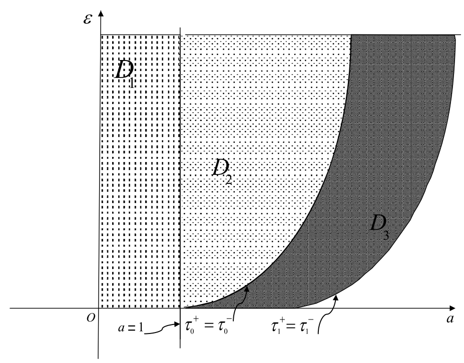


Figure 3 A branch of the anti-phased periodic solution is bifurcated from the trivial solution with $\tau = 0.5$ and initial condition $(-2, 1.3, -1.4, 0.2)$.

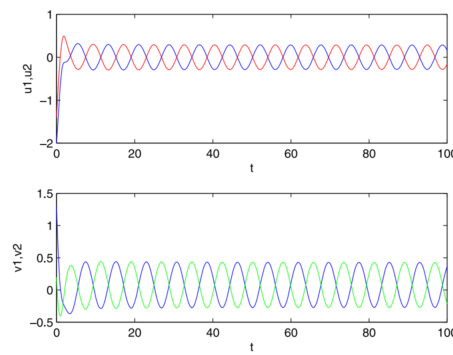


Figure 4 A branch of the anti-phased periodic solution is bifurcated from the trivial solution with $\tau = 0.697$ and initial condition $(-2, 1.3, -1.4, 0.2)$.

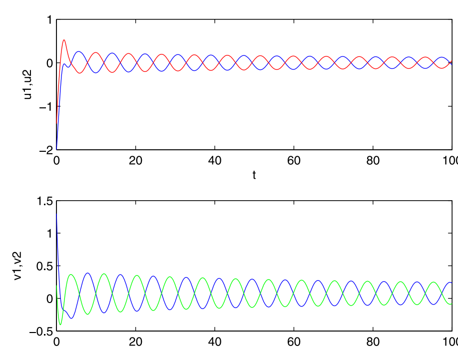


Figure 5 The zero solution is asymptotically stable with $\tau = 1.2$ and initial condition $(-2, 1.3, -1.4, 0.2)$.

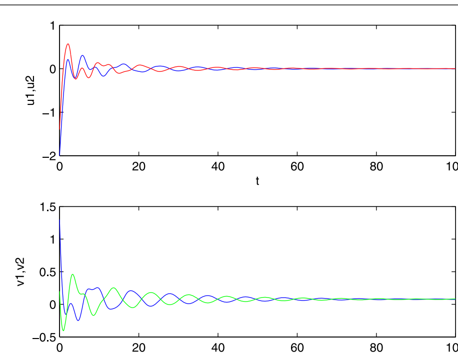


Figure 6 A branch of the synchronous periodic solution is bifurcated from the trivial solution with $\tau = 1.97$ and initial condition $(-2, 1.3, -1.4, 0.2)$.

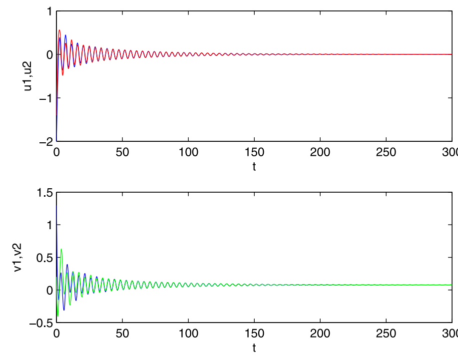


Figure 7 The zero solution is asymptotically stable with $\tau = 1.8$ and initial condition $(-2, 1.3, -1.4, 0.2)$.

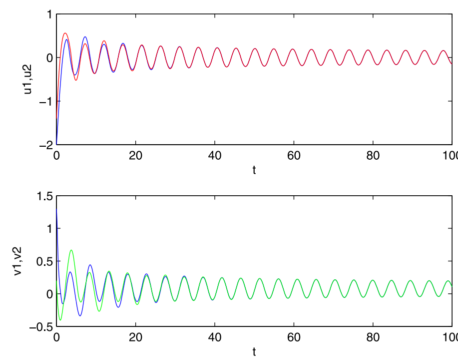


Figure 8 A branch of the synchronous periodic solution is bifurcated from the trivial solution with $\tau = 4.33$ and initial condition $(-2, 1.3, -1.4, 0.2)$.

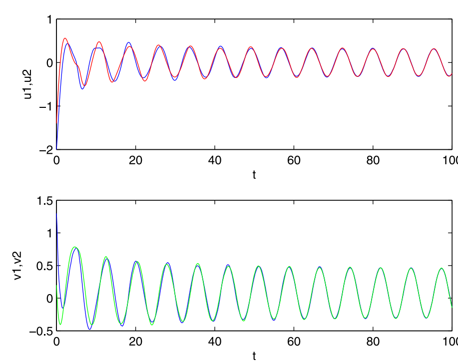
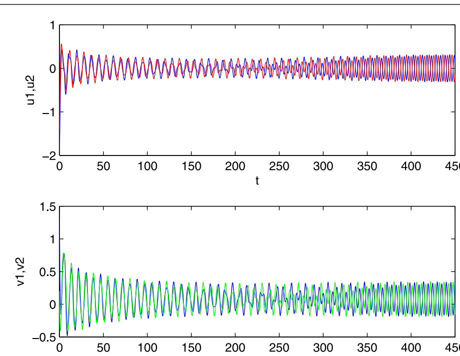


Figure 9 A anti-phased periodic solution appears again from synchronous periodic solutions with $\tau = 4.87$ and initial condition $(-2, 1.3, -1.4, 0.2)$.



6 Conclusion

In this paper, a delay-coupled optoelectronic oscillators described by (1.1) is considered. The effect of the time delay on the linear stability of the system is investigated. Based on the standard Hopf bifurcation theory, we find that as the time delay increases and crosses through the critical values, there exist stability switches in a certain region of the plane of the linear coefficient ε and the coupling strength γ and a branch of periodic solutions bifurcate from the zero equilibrium. By means of the general symmetric local Hopf bifurcation theorem, we not only investigated the effect of a delay of the signal transmission on the pattern formation of model (1.1), but we also obtain some important results about the spontaneous bifurcation of multiple branches of periodic solutions and their spatio-temporal patterns. A remarkable finding is that the spatio-temporal patterns between the two output voltage depend not only on the critical value τ of the coupling time delay, but also on the parameter region where the bifurcation occurs. There are two types of spatio-temporal patterns: one is in phase and the other is in antiphase.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors have made equal contributions to each part of this paper. All the authors read and approved the final manuscript.

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