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A delayed Holling type III functional response predator-prey system with impulsive perturbation on the prey

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Abstract

A Holling type III functional response predator-prey system with constant gestation time delay and impulsive perturbation on the prey is investigated. The sufficient conditions for the global attractivity of a predator-extinction periodic solution are obtained by the theory of impulsive differential equations, *i.e.* the impulsive period is less than the critical value T_1^*. The conditions for the permanence of the system are investigated, *i.e.* the impulsive period is larger than the critical value T_2^*. Numerical examples show that the system has very complex dynamic behaviors, including (1) high-order periodic and quasi-periodic oscillations, (2) period-doubling and -halving bifurcations, and (3) chaos and attractor crises. Further, the importance of the impulsive period, the gestation time delay, and the impulsive perturbation proportionality constant are discussed. Finally, the impulsive control strategy and the biological implications of the results are discussed.

Keywords: predator-prey system; impulsive perturbation; time delay; extinction; permanence; chaos

1 Introduction

The time delay population dynamics system describes the current state of the population not only related to the current state but also related to the state of the population in the past. That is to say, the time delay effect is very important in population dynamics, which tends to destabilize the positive equilibria and cause a loss of stability, bifurcate into various periodic solutions, even make chaotic oscillations. Recently, there has been much work dealing with time delayed population systems (see [1–12]). For example, a stage-structured prey-predator system with time delay and Holling type-III functional response is considered by Wang *et al.* [4], the existence and properties of the Hopf bifurcations are established. A delayed eco-epidemiology model with Holling-III functional response was instigated by Zou *et al.* [5], and one found that time delay may lead to Hopf bifurcation under certain conditions. The Hopf bifurcation of a delayed predator-prey system with Holling type-III functional response also has been considered in [6–8]. The existence of positive periodic solutions of a delayed nonautonomous prey-predator system with Holling type-III functional response was considered in [9–11], by using the continuation theorem of coincidence degree theory. Therefore, time delays would make the prey-predator system subject to periodic oscillations via a local Hopf bifurcation, and destroy the stability of the system.

Recently, more and more authors have discussed the impulsive perturbation on prey-predator systems, since the system would be stabilized by impulsive effects, and would make the system subject to complex dynamical behaviors [13–21]. For example, a predator-prey model with impulsive effect and generalized Holling type III functional response was studied by Su *et al.* [15], and the sufficient conditions for the existence of a pest-eradication periodic solution and permanence of the system are obtained. The existence of positive periodic solutions of the nonautonomous prey-predator system with Holling type III functional response and impulsive perturbation is considered in [17, 18]. By using a continuation theorem of coincidence degree theory, the sufficient conditions for the existence of a positive periodic solution are obtained for the system.

Note that harmful delays would destroy the stability of the system via bifurcations and even lead the system to extinction. At this point, the impulsive control strategies can be considered, which can both improve the stability of the system and control the amplitude of the bifurcated periodic solution effectively. For example, a time delayed Holling type II functional response prey-predator system with impulsive perturbations is investigated by Jia *et al.* [22], and the problems of the predator-extinction periodic solution and the permanence of the system are investigated. So, how does the dynamical behavior go when the delayed system with impulsive effect? Especially, what would happen for the delayed predator-prey system with Holling type III functional response under impulsive perturbation?

Motivated by the aforementioned observations, we assume the predator needs a certain time to gestate the prey, and we consider the following delayed Holling type III functional response prey-predator system with impulsive perturbation on the prey:

$$\begin{cases} \left.\begin{aligned} x'(t) &= rx(t)\left(1-\frac{x(t)}{K}\right)-\frac{\alpha x^2(t)y(t)}{1+\beta x^2(t)},\\ y'(t) &= \frac{k\alpha x^2(t-\tau)y(t-\tau)}{1+\beta x^2(t-\tau)}-dy(t), \end{aligned}\right\} & t\neq nT,\\ \left.\begin{aligned} \Delta x(t) &= -px(t),\\ \Delta y(t) &= 0, \end{aligned}\right\} & t=nT, \end{cases} \tag{1}$$

where $x(t)$ and $y(t)$ are the prey and predator populations at time t, respectively; r, K, α, β, k, d are positive. r is the intrinsic rate of increase of the prey, K is the carrying capacity of the prey. α is the predation coefficient of the predator, which reflects the size of the predator's ability, and β is predation regulation factor (saturation factor) of the predator. d is the death rate of the predator, k $(0 < k < 1)$ is the rate of conversing prey into predator. $\Delta x(t) = x(t^+) - x(t)$, $\Delta y(t) = y(t^+) - y(t)$, T is the impulsive periodic. $n \in \mathbb{N}_+$, $\mathbb{N}_+ = \{1, 2, \ldots\}$, $p > 0$ is the proportionality constant which represents the rate of mortality due to the applied pesticide. The initial conditions for system (1) are

$$\big(\phi_1(s), \phi_2(s)\big) \in C_+ = C\big([-\tau, 0], \mathbb{R}^2_+\big), \quad \phi_i(0) > 0\ (i = 1, 2). \tag{2}$$

The paper is arranged as follows. Some notations and lemmas are given in the next section, and we consider the existence and global attraction of the predator-extinction periodic solutions of the system. The sufficient conditions for the permanence of the system are given by using the theory on impulsive and delay differential equation. Numerical examples are given to support the theoretical research, and some complex dynamic behaviors are shown. For example, we see period-halving and period-doubling bifurcations, periodic and high-order quasi-periodic oscillations, even chaotic oscillation. The importance of the impulsive period T, the gestation time delay τ, and the impulsive perturbation proportionality constant p are discussed. Finally, the impulsive control strategy and biological implications of the results are discussed.

2 Preliminaries

Let $\mathbb{R}_+ = [0,\infty)$, $\mathbb{R}^2_+ = \{x \in \mathbb{R}^2 | x \geq 0\}$. Denote by $f = (f_1, f_2)$ the map defined by the right hand of the first two equations of system (1), and let $\mathbb{N}$ be the set of all non-negative integers. Let $V : \mathbb{R}_+ \times \mathbb{R}^2_+ \to \mathbb{R}_+$, then V is said to belong to class V_0 if:

(1) V is continuous in $(t,x) \in (nT,(n+1)T] \times \mathbb{R}^2_+$ and for each $x \in \mathbb{R}^2_+$, $n \in \mathbb{N}$, $\lim_{(t,y)\to(nT^+,x)} V(t,y) = V(nT^+,x)$ exists.

(2) V is locally Lipschitzian in x.

Definition 2.1 Let $V \in V_0$, then for $(t,x) \in (nT,(n+1)T] \times \mathbb{R}^2_+$, the upper right derivative of $V(t,x)$ with respect to the impulsive differential system (1) is defined as

$$D^+V(t,x) = \lim_{h\to 0^+} \sup \frac{1}{h}\bigl[V\bigl(t+h, x+hf(t,x)\bigr) - V(t,x)\bigr].$$

Definition 2.2 System (1) is said to be permanent if there exist two positive constants m, M, and T_0 such that each positive solution $X(t) = (x(t), y(t))$ of the system (1) satisfies $m \leq x(t) \leq M$, $m \leq y(t) \leq M$ for all $t > T_0$.

The solution of system (1) is a piecewise continuous function $x : \mathbb{R}_+ \mapsto \mathbb{R}^2_+$, $x(t)$ is continuous on $(nT,(n+1)T]$, $n \in \mathbb{N}$, and $x(nT^+) = \lim_{t\to nT^+} x(t)$ exists, the smoothness properties of f guarantee the global existence and uniqueness of solutions of system (1), for details see [23, 24].

Lemma 2.1 *Let $X(t)$ be a solution of system* (1) *with $X(0^+) \geq 0$, then $X(t) \geq 0$ for all $t \geq 0$ and further $X(t) > 0$ for all $t \geq 0$ if $X(0^+) > 0$.*

Lemma 2.2 [23] *Suppose $V \in V_0$. Assume that*

$$\begin{cases} D^+V(t,x) \leq g(t,V(t,x)), & t \neq nT, \\ V(t,x(t^+)) \leq \psi_n(V(t,x)), & t = nT, \end{cases} \tag{3}$$

where $g : \mathbb{R}_+ \times \mathbb{R}_+ \mapsto \mathbb{R}$ is continuous in $(nT,(n+1)T] \times \mathbb{R}_+$ and for $u \in \mathbb{R}_+$, $n \in \mathbb{N}$, $\lim_{(t,y)\to(nT^+,u)} = g(nT^+,u)$ exists, $\psi_n : \mathbb{R}_+ \to \mathbb{R}_+$ is non-decreasing. Let $r(t)$ be the maximal solution of the scalar impulsive differential equation

$$\begin{cases} u'(t) = g(t,u(t)), & t \neq nT, \\ u(t^+) = \psi_n(u(t)), & t = nT, \\ u(0^+) = u_0, \end{cases} \tag{4}$$

existing on $[0,\infty)$. Then $V(0^+,x_0) \leq u_0$ implies that $V(t,x(t)) \leq r(t)$, $t \geq 0$ where $X(t)$ is any solution of system (1).

We consider the following subsystem of system (1):

$$\begin{cases} x'(t) = rx(t)(1 - \frac{x(t)}{K}), & t \neq nT, \\ \Delta x(t) = -px(t), & t = nT. \end{cases} \tag{5}$$

Clearly, if $(1-p)e^{rT} > 1$,

$$x^*(t) = \frac{x_0^*}{(1 - \frac{x_0^*}{K})e^{-r(t-nT)} + \frac{x_0^*}{K}}, \quad t \in \bigl(nT, (n+1)T\bigr], \qquad x_0^* = \frac{K[(1-p)e^{rT} - 1]}{e^{rT} - 1},$$

is a globally asymptotically stable positive periodic solution of system (5) [22].

Lemma 2.3 *If* $(1-p)e^{rT} > 1$, *system* (1) *has a predator-extinction periodic solution* $X(t) = (x^*(t), 0)$ *for* $t \in (nT, (n+1)T]$, *and for any solution* $X(t) = (x(t), y(t))$ *of system* (1), *we have* $x(t) \to x^*(t)$ *as* $t \to +\infty$.

Lemma 2.4 *Consider the following delay differential equation* [25]:

$$x'(t) = ax(t-\tau) - bx(t),$$

where a, b, τ *are all positive constants and* $x(t) > 0$ *for* $t \in [-\tau, 0]$.

(i) *If* $a < b$, *then* $\lim_{t\to+\infty} x(t) = 0$.

(ii) *If* $a > b$, *then* $\lim_{t\to+\infty} x(t) = +\infty$.

3 Extinction and permanence

Denote

$$X_1^* = \frac{K[(1-p)e^{rT} - 1]}{(1-p)(e^{rT} - 1)}, \qquad \Re_1 = \frac{k\alpha X_1^{*2}}{d(1 + \beta X_1^{*2})}.$$

Theorem 3.1 *If* $\Re_1 < 1$ *and* $(1-p)e^{rT} > 1$, *then the predator-extinction periodic solution* $X(t) = (x^*(t), 0)$ *of system* (1) *is globally attractive*.

Proof Let $X(t) = (x(t), y(t))$ be any solution of system (1) with initial conditions (2). By the second equation of system (1), we get

$$y'(t) \le \frac{k\alpha}{\beta} y(t-\tau) - dy(t).$$

Consider the following delayed comparison equation:

$$z'(t) = \frac{k\alpha}{\beta} z(t-\tau) - dz(t).$$

If $k\alpha < d\beta$, then $\Re_1 < 1$. According to Lemma 2.4, we have $\lim_{t\to+\infty} z(t) = 0$. Note that $y(t) = z(t) = \phi_2(t) > 0$ for all $t \in [-\tau, 0]$, then we have $y(t) \to 0$ as $t \to +\infty$. We assume that $k\alpha > d\beta$ in the rest of this paper. $\Re_1 < 1$ and we note that $\frac{k\alpha z^2}{1+\beta z^2}$ is a monotonically increasing function about variable z. Therefore, there exists a sufficiently small positive constant ε such that

$$\frac{k\alpha(X_1^*+\varepsilon)^2}{1+\beta(X_1^*+\varepsilon)^2}-d<0.$$

By the first and third equations of system (1), we have

$$\begin{cases} x'(t)\le rx(t)(1-\frac{x(t)}{K}), & t\neq nT,\\ \Delta x(t)=-px(t), & t=nT, \end{cases}$$

then we consider the following comparison system:

$$\begin{cases} z'(t)=rz(t)(1-\frac{z(t)}{K}), & t\neq nT,\\ \Delta z(t)=-pz(t), & t=nT, \end{cases} \tag{6}$$

with $z(0^+)=z(0)=x(0)$. Recalling system (5), we obtain a unique globally asymptotically stable positive T-periodic solution of system (6), where

$$z^*(t)=\frac{K}{[\frac{e^{rT}-1}{(1-p)e^{rT}-1}-1]e^{-r(t-nT)}+1},\quad t\in\big(nT,(n+1)T\big].$$

Then, there exists an arbitrarily small positive constant ε and $n_1\in\mathbb{N}$ such that

$$x(t)\le z^*(t)+\varepsilon\le\frac{K}{[\frac{e^{rT}-1}{(1-p)e^{rT}-1}-1]e^{-rT}+1}+\varepsilon=X_1^*+\varepsilon\triangleq\eta \tag{7}$$

for all $t\ge n_1T$. From (7) and the second equation of system (1), we have

$$y'(t)\le\frac{k\alpha\eta^2}{1+\beta\eta^2}y(t-\tau)-dy(t)$$

for $t\ge n_1T+\tau$. Consider the following delayed comparison equation:

$$z'(t)=\frac{k\alpha\eta^2}{1+\beta\eta^2}z(t-\tau)-dz(t).$$

According to Lemma 2.4, we get $\lim_{t\to+\infty}z(t)=0$. Note that $y(t)=z(t)=\phi_2(t)>0$ for all $t\in[-\tau,0]$, then we have $y(t)\to 0$ as $t\to+\infty$.

Without loss of generality, we may suppose that $0<y(t)<\varepsilon$ and $r_1=r-\alpha K\varepsilon>0$, $K_1=r_1K/r$ for ε small enough and $t\ge 0$. By the first equation of system (1), we obtain

$$x'(t)\ge r_1x(t)\left(1-\frac{x(t)}{K_1}\right).$$

Then we have $z_1^*(t)\to x^*(t)$ as $\varepsilon\to 0$ $(t\to+\infty)$, where $z_1^*(t)$ is the unique periodic solution of the following comparison system:

$$\begin{cases} z_1'(t)=r_1z_1(t)(1-\frac{z_1(t)}{K_1}), & t\neq nT,\\ \Delta z_1(t)=-pz_1(t), & t=nT, \end{cases} \tag{8}$$

with initial condition $z_1(0^+) = x(0^+)$. From (8), we get

$$z_1^*(t) = \frac{z_1^*}{(1 - \frac{z_1^*}{K_1})e^{-r_1(t-nT)} + \frac{z_1^*}{K_1}}, \quad t \in \bigl(nT, (n+1)T\bigr],$$

where

$$z_1^* = \frac{K_1[(1-p)e^{r_1 T} - 1]}{e^{r_1 T} - 1}.$$

Therefore, for any $\varepsilon_1 > 0$ there exists a $T_0 > 0$ such that

$$x(t) > z_1^*(t) - \varepsilon_1 \tag{9}$$

for $t > T_0$.

By the first equation of system (1), we get

$$x(t) \le rx(t)\left(1 - \frac{x(t)}{K}\right).$$

Consider the following comparison system:

$$\begin{cases} z_2'(t) = rz_2(t)(1 - \frac{z_2(t)}{K}), & t \neq nT, \\ \Delta z_2(t) = -pz_2(t), & t = nT, \end{cases} \tag{10}$$

with initial condition $z_2(0^+) = x(0^+)$. Then, we have

$$x(t) < z_2^*(t) + \varepsilon_1 \tag{11}$$

as $t \to +\infty$ and $z_2^*(t) = x^*(t)$, where $z_2^*(t)$ is the unique positive periodic solution of (10). Let $\varepsilon \to 0$, and by (9) and (11), we have

$$x^*(t) - \varepsilon_1 < x(t) < x^*(t) + \varepsilon_1$$

for sufficiently large t. This implies $x(t) \to x^*(t)$ as $t \to +\infty$. □

Theorem 3.2 *There exists a constant $Y_0 = M_1/d - kK > 0$, such that $x(t) \le K$ and $y(t) \le Y_0$ for any solution $X(t) = (x(t), y(t))$ of system* (1) *with all t large enough.*

Proof Let $V(t) = kx(t) + y(t+\tau)$. Note that $V \in V_0$. We get $V'(t)$ by calculating the upper right derivative of $V(t)$ along a solution of system (1):

$$V'(t) = krx(t)\left(1 - \frac{x(t)}{K}\right) - dy(t+\tau) = -dV(t) + kx(t)\left[d + r\left(1 - \frac{x(t)}{K}\right)\right].$$

Let $M_1 = \max\{Kkd, \frac{K(r+d)^2}{4}\}$, then we have

$$\frac{\mathrm{d}V(t)}{\mathrm{d}t} \le -dV(t) + M_1.$$

Consider the following differential inequalities:

$$\begin{cases} V'(t) \le -dV(t) + M_1, & t \ne nT, \\ V(t^+) \le V(t), & t = nT, \end{cases}$$

according to Lemma 2.2, we have

$$V(t) \le \left(V\left(0^+\right) - \frac{M_1}{d} \right) e^{-dt} + \frac{M_1}{d}.$$

Therefore $\lim_{t\to+\infty} V(t) \le M_1/d \triangleq M$, then $V(t)$ is ultimately bounded. Therefore, any positive solution of system (1) is uniformly ultimately bounded. □

Denote

$$\Re_2 = \frac{k\alpha x_0^{*2}}{d(1 + \beta x_0^{*2})}.$$

Theorem 3.3 *If* $\Re_2 > 1$ *and* $r > \alpha KY_0$, *then system* (1) *is uniformly persistent.*

Proof Suppose that $X(t) = (x(t), y(t))$ is each positive solution of system (1) with initial conditions (2). Rewrite the second equation of system (1) as follows:

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = \left(\frac{k\alpha x^2(t)}{1 + \beta x^2(t)} - d \right) y(t) - k\alpha \frac{\mathrm{d}}{\mathrm{d}t} \int_{t-\tau}^{t} \frac{x^2(\theta)}{1 + \beta x^2(\theta)} y(\theta)\,\mathrm{d}\theta.$$

Define

$$V(t) = y(t) + k\alpha \int_{t-\tau}^{t} \frac{x^2(\theta)}{1 + \beta x^2(\theta)} y(\theta)\,\mathrm{d}\theta.$$

Calculating $V'(t)$ along the solution of system (1), we have

$$V'(t) = d\left(\frac{k\alpha}{d} \frac{x^2(t)}{1 + \beta x^2(t)} - 1 \right) y(t). \tag{12}$$

Since $\Re_2 > 1$, there exist two positive constants m_2^* and ε_1 small enough such that

$$\frac{k\alpha}{d} \frac{\rho^2}{1 + \beta\rho^2} > 1, \tag{13}$$

where

$$\rho = \frac{K_2[(1-p)e^{r_2 T} - 1]}{e^{r_2 T} - 1} - \varepsilon_1 > 0, \qquad r_2 = r - \alpha K m_2^* > 0,$$
$$K_2 = \frac{Kr_2}{r} > 0, \qquad 0 < m_2^* < \frac{rT + \ln(1-p)}{\alpha KT}.$$

We claim that: for any positive constant t_0, the inequality $y(t) < m_2^*$ cannot hold for all $t \ge t_0$. Otherwise, we can choose a positive constant t_0 such that $y(t) < m_2^*$ for all $t \ge t_0$.

By the first and third equations of system (1), we obtain

$$\begin{cases} x'(t) \geq r_2 x(t)(1 - \frac{x(t)}{K_2}), & t \neq nT, \\ \Delta x(t) = -px(t), & t = nT. \end{cases} \tag{14}$$

Then, we have $z_2^*(t) \leq x(t)$, where $z_2^*(t)$ is an unique positive solution of the following comparison system:

$$\begin{cases} z_2'(t) = r_2 z_2(t)(1 - \frac{z_2(t)}{K_2}), & t \neq nT, \\ \Delta z_2(t) = -pz_2(t), & t = nT, \end{cases} \tag{15}$$

with initial condition $z_2(0^+) = x(0^+)$. From (15), we obtain

$$z_2^*(t) = \frac{z_2^*}{(1 - \frac{z_2^*}{K_2})e^{-r_2(t-nT)} + \frac{z_2^*}{K_2}},$$

for $t \in (nT, (n+1)T]$, where

$$z_2^* = \frac{K_2[(1-p)e^{r_2 T} - 1]}{e^{r_2 T} - 1}.$$

Then, for any $\varepsilon_1 > 0$ there exists a $T_1 > 0$ such that

$$x(t) > z_2^*(t) - \varepsilon_1 \geq z_2^* - \varepsilon_1 \triangleq \rho, \tag{16}$$

for $t > T_1$.

When $t \geq T_1$, from (12) and (16) we get

$$V'(t) > d\left(\frac{k\alpha}{d} \frac{\rho^2}{1 + \beta\rho^2} - 1\right) y(t). \tag{17}$$

Let $y_1 = \min\{y(t) | T_1 \leq t \leq T_1 + \tau\}$. We show that $y(t) \geq y_1$ for all $t \geq T_1$. Otherwise, there exists a nonnegative constant T_2 such that

$$y(t) \geq y_1 \quad (T_1 \leq t \leq T_1 + T_2 + \tau), \qquad y(T_1 + T_2 + \tau) = y_1, \qquad y'(T_1 + T_2 + \tau) \leq 0.$$

Thus, by the second equation of system (1), (13), and (17), we obtain

$$y'(T_1 + T_2 + \tau) > d\left(\frac{k\alpha}{d} \frac{\rho^2}{1 + \beta\rho^2} - 1\right) y_1 > 0,$$

which is a contradiction. Hence, we get $y(t) > y_1 > 0$ for all $t \geq T_1$. From (17), we have

$$V'(t) > d\left(\frac{k\alpha}{d} \frac{\rho^2}{1 + \beta\rho^2} - 1\right) y(t) > 0,$$

which implies $V(t) \to +\infty$ as $t \to +\infty$. This is a contradiction to $V(t) \leq M$. Then, for any given positive constant t_0, the inequality $y(t) < m_2^*$ cannot hold for all $t \geq t_0$.

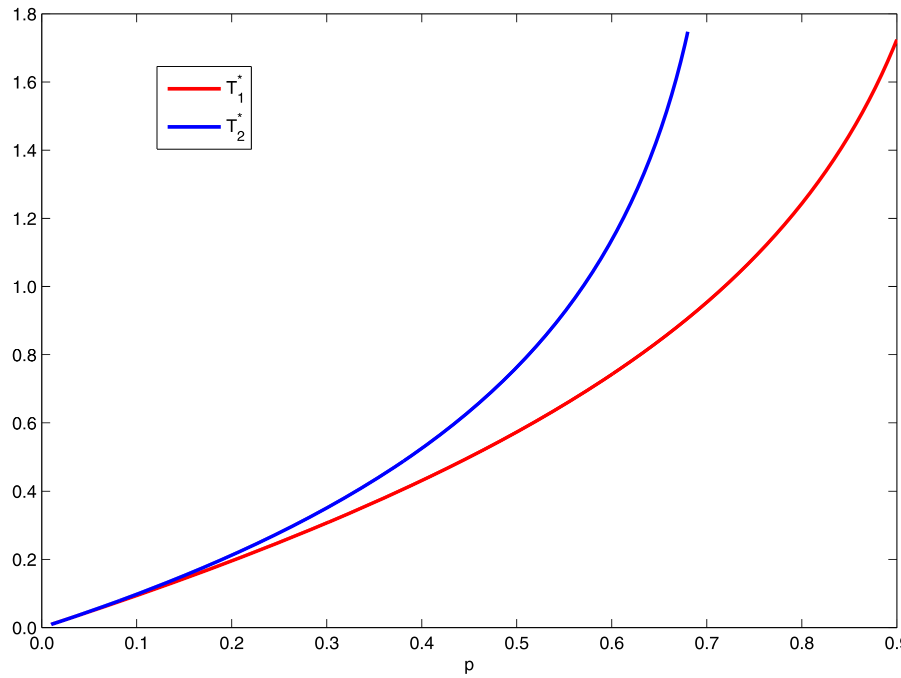

Figure 1 The two thresholds T_1^* and T_2^* for system (18).

On the one hand, if $y(t) \geq m_2^*$ holds true for all sufficiently large t, then our aim is reached. On the other hand, suppose $y(t)$ is oscillatory about m_2^*. Let $m_2 = \min\{m_2^*/2, m_2^* \exp(-d\tau)\}$ and we will prove that $y(t) \geq m_2$. There exist two positive constants $\bar{t}$ and ω such that $y(\bar{t}) = y(\bar{t} + \omega) = m_2^*$ and $y(t) < m_2^*$ for $t \in (\bar{t}, \bar{t} + \omega)$. The inequality $x(t) > \rho$ holds true for $t \in (\bar{t}, \bar{t} + \omega)$ when $\bar{t}$ is large enough.

Since there is no impulsive effect on $y(t)$, $y(t)$ is uniformly continuous. Then, there exists a constant T_3 (with $0 < T_3 < \tau$ and T_3 is dependent of the choice of $\bar{t}$) such that $y(t) > m_2^*/2$ for all $t \in [\bar{t}, \bar{t} + T_3]$.

If $\omega \leq T_3$, our aim is reached. If $T_3 < \omega \leq \tau$, by the second equation of system (1) we get $y'(t) \geq -dy(t)$ for $t \in (\bar{t}, \bar{t} + \omega]$. Then we get $y(t) \geq m_2^* \exp(-d\tau)$ for $\bar{t} < t \leq \bar{t} + \omega \leq \bar{t} + \tau$ since $y(t) = m_2^*$. Therefore, $y(t) \geq m_2$ for $t \in (\bar{t}, \bar{t} + \omega]$.

If $\omega \geq \tau$, from the second equation of system (1), then we get $y(t) \geq m_2$ for $t \in (\bar{t}, \bar{t} + \tau]$. Thus, we have $y(t) \geq m_2$ for $t \in [\bar{t} + \tau, \bar{t} + \omega]$. According to the above proof. Since the interval $[\bar{t}, \bar{t} + \omega]$ is arbitrarily chosen, we have $y(t) \geq m_2$ for sufficiently large t. From our arguments above, the choice of m_2 is independent of the positive solution of system (1) which shows that $y(t) \geq m_2$ holds for sufficiently large t.

By Theorem 3.2, we obtain $y(t) \leq Y_0$ for sufficiently large t. Therefore, by the first equation of system (1), we get

$$x'(t) \geq r_3 x(t)\left(1 - \frac{x(t)}{K_3}\right)$$

for sufficiently large t, where $r_3 = r - \alpha K Y_0$ and $K_3 = K r_3 / r$. Therefore, we have $x(t) \geq z_3^*(t)$, where $z_3^*(t)$ is a unique globally asymptotically stable positive periodic solution of

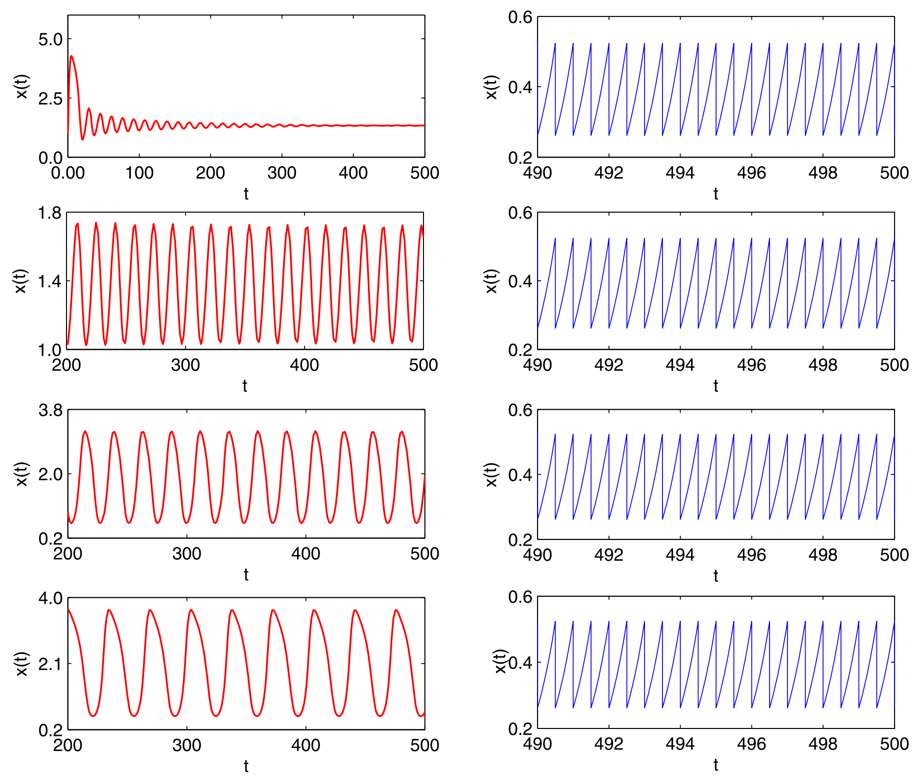

Figure 2 Time-series of prey population $x(t)$ of system (18) without (left) and with (right) impulsive period $T = 0.5 < T_1^*$ when the time delay $\tau = 0.5, 0.7, 3, 6$, respectively.

the following comparison system:

$$\begin{cases} z_3'(t) = r_3 z_3(t)(1 - \frac{z_3(t)}{K_3}), & t \neq nT, \\ \Delta z_3(t) = -p z_3(t), & t = nT, \end{cases}$$

with initial condition $z_3(0^+) = x(0^+)$. Similarly, we can choose a $\varepsilon > 0$ small enough such that

$$x(t) > z_3^*(t) - \varepsilon \geq \frac{K_3[(1-p)e^{r_3 T} - 1]}{e^{r_3 T} - 1} - \varepsilon \triangleq m_1$$

holds for sufficiently large t. □

Remark 1 Let $(1-p)e^{rT} > 1$ and

$$F_1(f_1(p,T)) = \frac{k\alpha f_1^2(p,T)}{d(1 + f_1^2(p,T))} - 1, \qquad f_1(p,T) = \frac{K[(1-p)e^{rT} - 1]}{(1-p)(e^{rT} - 1)}.$$

Note that $F_1(f_1(p,T))$ is a monotonically increasing function with respect to $f_1(p,T)$, and $f_1(p,T)$ is a monotonically increasing function with respect to T ($T > -\ln(1-p)/r$) and a monotonically decreasing function with respect to p ($0 < p < 1 - \exp(-rT)$), since

$$f_{1p}'(p,T) = \frac{-K}{(e^{rT} - 1)(1-p)^2} < 0, \qquad f_{1T}'(p,T) = \frac{rpKe^{rT}}{(1-p)(e^{rT} - 1)^2} > 0.$$

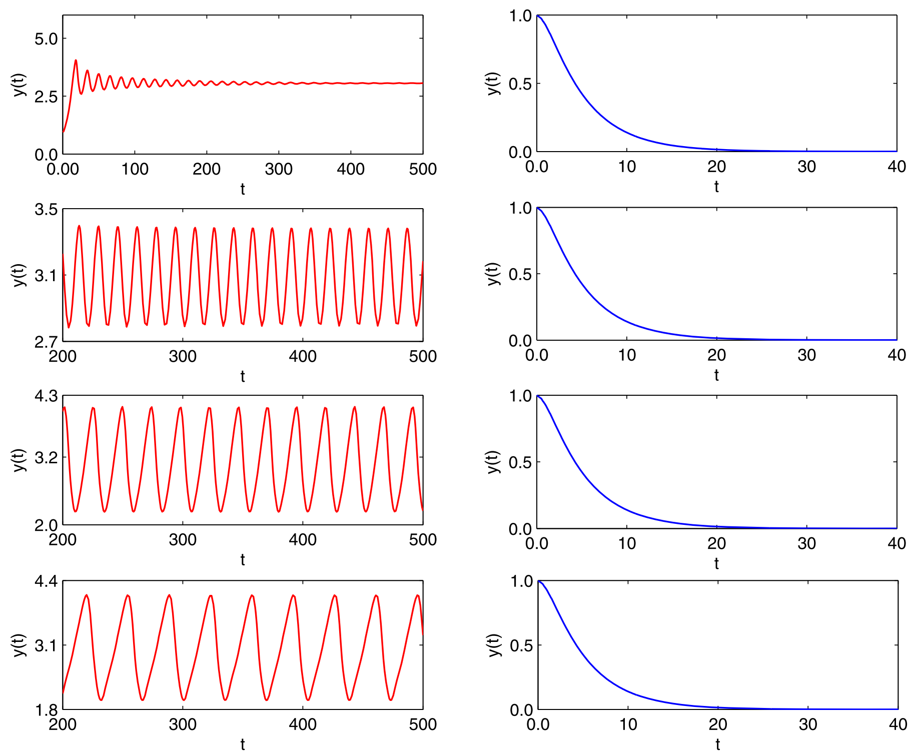

Figure 3 Time-series of predator population *y*(*t*) of system (18) without (left) and with (right) impulsive period $T = 0.5 < T_1^*$ when time delay $\tau = 0.5, 0.7, 3, 6$, respectively.

So, there exists a unique T_1^* such that $F_1(f_1(p, T_1^*)) = 0$ if we fix p, similarly, there exists a unique p_1^* such that $F_1(f_1(p_1^*, T)) = 0$ if we fix T. Therefore, the condition $\Re_1 < 1$ is equivalent to $T < T_1^*$ (or $p > p_1^*$), where $T_1^*(p_1^*)$ is the unique solution of $F_1(f_1(p, T)) = 0$.

Similarly, let

$$F_2(f_2(p, T)) = \frac{k\alpha f_2^2(p, T)}{d(1 + f_2^2(p, T))} - 1, \qquad f_2(p, T) = \frac{K[(1-p)e^{rT} - 1]}{e^{rT} - 1}.$$

Note that $F_2(f_2(p, T))$ is a monotonically increasing function with respect to $f_2(p, T)$, and $f_2(p, T)$ is a monotonically increasing function with respect to T ($T > -\ln(1-p)/r$) and a monotonically decreasing function with respect to p ($0 < p < 1 - \exp(-rT)$), since

$$f'_{2p}(p, T) = -Ke^{rT} < 0, \qquad f'_{2T}(p, T) = \frac{rpKe^{rT}}{(e^{rT} - 1)^2} > 0.$$

So, there exists a unique T_2^* such that $F_2(f_2(p, T_2^*)) = 0$ if we fix p, similarly, there exists a unique p_2^* such that $F_2(f_2(p_2^*, T)) = 0$ if we fix T. Therefore, the condition $\Re_2 > 1$ is equivalent to $T > T_2^*$ (or $p < p_2^*$), where $T_2^*(p_2^*)$ is the unique solution of $F_2(f_2(p, T)) = 0$.

Remark 2 Note that $f_1(p, T) > f_2(p, T)$, so $T_1^* < T_2^*$ for $F_1(f_1(p, T_1^*)) = 0$ and $F_2(f_2(p, T_2^*)) = 0$ with respect to the same value of the parameter p. Therefore, if $T < T_1^*$ the predator-extinction periodic solution is globally attractive and if $T > T_2^*$ the system has permanence. If system (1) is without time delay, according to [14], we know that there would be

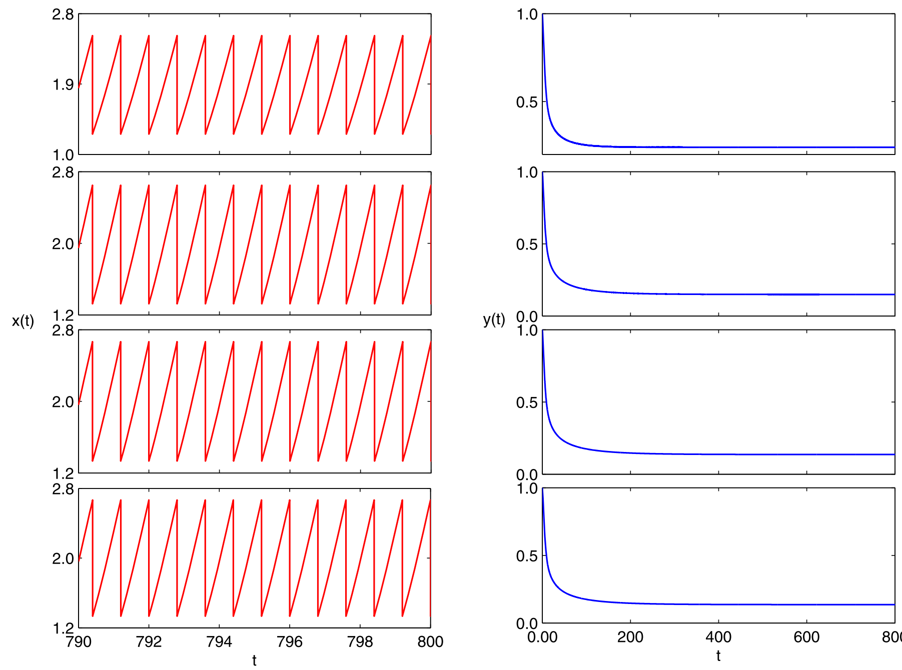

Figure 4 Time-series of prey (left) and predator (right) population $y(t)$ of system (18) with impulsive period $T = 0.8 > T_2^*$ when the time delay $\tau = 0.5, 0.7, 3, 6$, respectively.

a threshold $T_{\max}$. If $T < T_{\max}$, then the prey- (or predator)-eradication periodic solution is locally asymptotically stable; if $T > T_{\max}$ the system is permanent. But we get two thresholds T_1^* and T_2^*, and there is no information as regards the system when $T_1^* < T < T_2^*$. This is essentially different when system (1) is with or without time delay.

4 Numerical analysis

Numerical experiments are carried out to integrate the system by using the DDE23 algorithm method in MATLAB.

4.1 Example 1

We consider the following delayed Holling type-III response prey-predator system with impulsive perturbation:

$$\begin{cases} \left.\begin{aligned} x'(t) &= 1.5x(t)\left(1 - \frac{x(t)}{5}\right) - \frac{0.75x^2(t)y(t)}{1+x^2(t)}, \\ y'(t) &= \frac{3x^2(t-\tau)y(t-\tau)}{8[1+x^2(t-\tau)]} - 0.24y(t), \end{aligned}\right\} & t \neq nT, \\ \left.\begin{aligned} \Delta x(t) &= -px(t), \\ \Delta y(t) &= 0, \end{aligned}\right\} & t = nT, \end{cases} \tag{18}$$

where $r = 1.5$, $K = 5$, $\alpha = 0.75$, $\beta = 1$, $k = 0.5$, $d = 0.24$, with initial conditions $(\phi_1(t), \phi_2(t)) = (1,1)$, $t \in [-\tau, 0]$. From Remark 1, we can get thresholds T_1^* and T_2^* when varying the values of parameter p from 0.01 to 0.9. From Figure 1, we see that T_1^* gets more and more close to T_2^* when p gets more and more close to zero.

According to [4, 6], there would be a time delay critical value τ_0, and there is a Hopf bifurcation when the time delay τ crosses the critical value τ_0 when system (18) is without

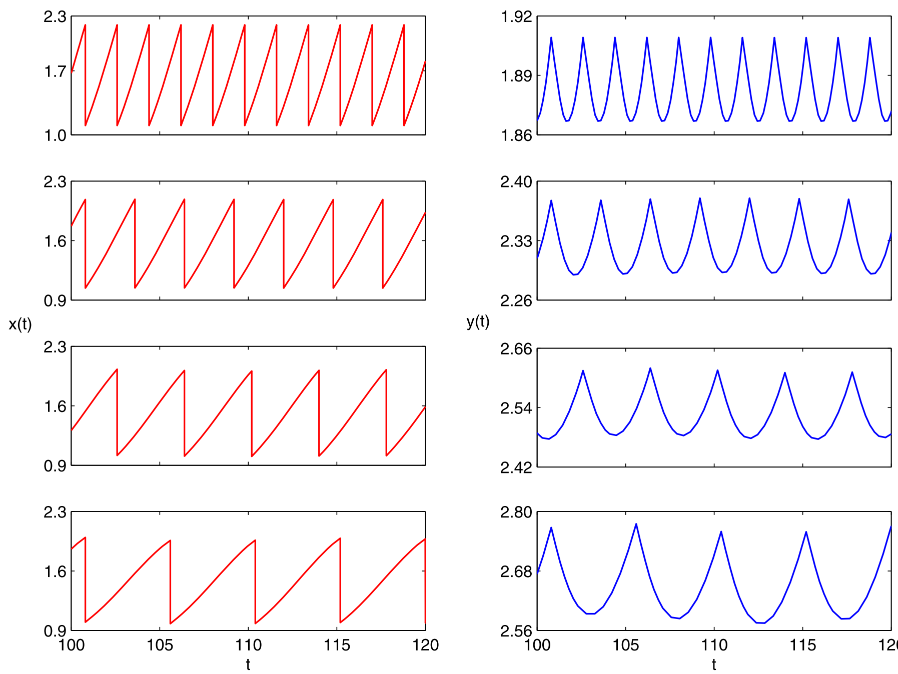

Figure 5 Time-series of prey (left) and predator (right) population $y(t)$ of system (18) with time delay $\tau = 0.5$ when the impulsive period $T = 1.8, 2.8, 3.8, 4.8$, respectively.

impulsive perturbation (*i.e.* $p = 0$). From the time-series of the prey population (Figure 2 (left)) and the predator population (Figure 3 (left)) with time delay $\tau = 0.5, 0.7, 3, 6$, respectively, we know that the time delay critical value τ_0 is between 0.5 and 0.7. The prey and predator populations are locally stable when the time delay $\tau \leq 0.5$ and coexist with sable cycles when $\tau \geq 0.7$.

From Theorem 3.1, Theorem 3.3, and Remark 1, one knows that if $T < T_1^*$ the predator-extinction periodic solution is globally attractive and if $T > T_2^*$ the system has permanence. From Figure 1, we see $T_1^* \approx 0.5734$ and $T_2^* \approx 0.7634$ when $p = 0.5$. Then, the predator-extinction periodic solution is globally attractive when $T = 0.5 < T_1^*$ (see Figure 2 (right) and Figure 3 (right)) where the time delay $\tau = 0.5, 0.7, 3, 6$, respectively. The system is permanent when $T = 0.8 > T_2^*$ where the time delay $\tau = 0.5, 0.7, 3, 6$, respectively (see Figure 4). Furthermore, when the time delay $\tau = 0.5 < \tau_0$ the predator and prey populations coexist with sable cycles, where the impulsive period $T = 1.8, 2.8, 3.8, 4.8$, respectively (see Figure 5). That is to say, an impulsive effect would destabilize the system under some conditions. If we increasing the time delay from 2 to 4.8, that large time delay could stabilize the system (see Figure 6 and Figure 7).

4.2 Example 2

We consider the following delayed Holling type-III response predator-prey system with impulsive perturbation:

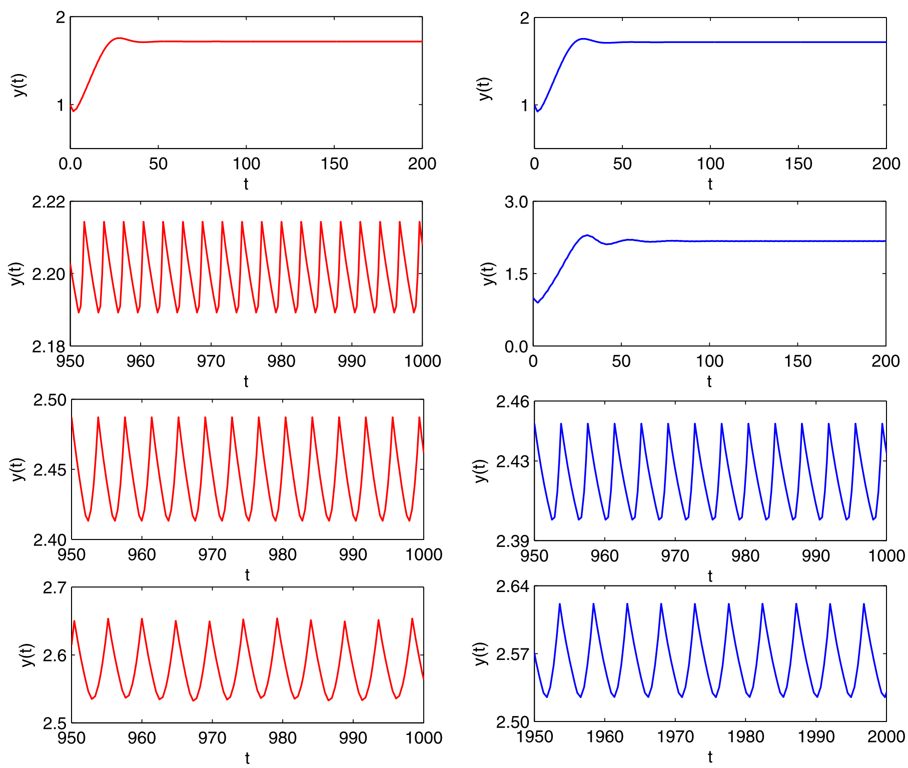

Figure 6 Time-series of prey-predator population $y(t)$ of system (18) with time delay $\tau = 2$ (left) and $\tau = 2.5$ (right) when the impulsive period $T = 1.8, 2.8, 3.8, 4.8$, respectively.

$$\begin{cases} \left.\begin{aligned} x'(t) &= 2.2x(t)\left(1 - \frac{x(t)}{7}\right) - \frac{0.75x^2(t)y(t)}{1+0.9x^2(t)}, \\ y'(t) &= \frac{0.525x^2(t-\tau)y(t-\tau)}{1+0.9x^2(t-\tau)} - 0.34y(t), \end{aligned}\right\} \quad t \neq nT, \\ \left.\begin{aligned} \Delta x(t) &= -px(t), \\ \Delta y(t) &= 0, \end{aligned}\right\} \quad t = nT, \end{cases} \tag{19}$$

where $r = 2.2$, $K = 7$, $\alpha = 0.75$, $\beta = 0.9$, $k = 0.7$, $d = 0.34$, with initial condition $(\phi_1(s), \phi_2(s)) = (1,1)$, $s \in [-\tau, 0]$. First of all, we let $p = 0.3$, $\tau = 3$, and we consider the effect of the impulsive period T on system (19). The bifurcation diagrams of the impulsive period T over $[0.5, 8.5]$ and $[6.8, 8.5]$, show that system (19) has complex dynamics (see Figure 8 and Figure 9), including high-order periodic and quasi-periodic oscillating, period-doubling and period-halving bifurcations, and chaos and attractor crises. For example, there exist $5T$, $4T$, $3T$ periodic solutions when $T = 3.5, 4.5, 6.5$, respectively (see Figure 10). When T increases from 6.8 to 7.1, there is an attractor crisis leading to a chaotic solution (see Figure 11). These results imply that an impulsive effect could destroy the stability of the system and lead to multiple attractors, bifurcations, even chaos oscillations, which makes the dynamical behaviors very complex.

Second, we let $p = 0.3$, $T = 7$ and consider the effect of the time delay τ on system (19). The bifurcation diagrams of time delay τ over $[0.2, 7.0]$ and $[2.5, 3.1]$, show that system (19) has very complex dynamic behaviors (see Figure 12 and Figure 13). These results imply that the time delay would make the dynamical behaviors more complex. For example, there is a cascade of period-doubling bifurcations leading to chaos (see Figure 14).

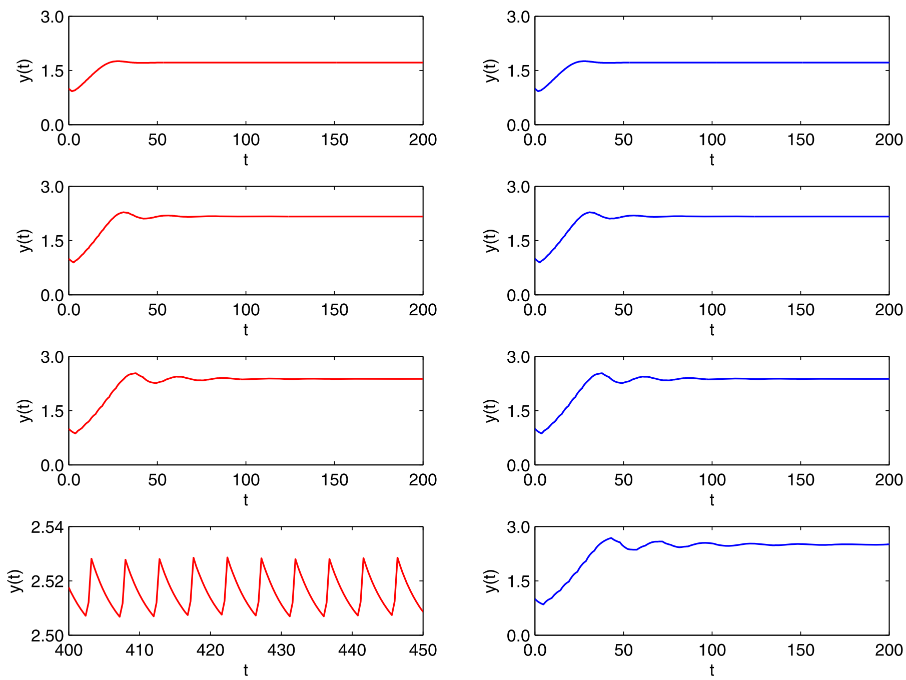

Figure 7 **Time-series of prey predator population $y(t)$ of system (18) with time delay $\tau = 4$ (left) and $\tau = 4.8$ (right) when the impulsive period $T = 1.8, 2.8, 3.8, 4.8$, respectively.**

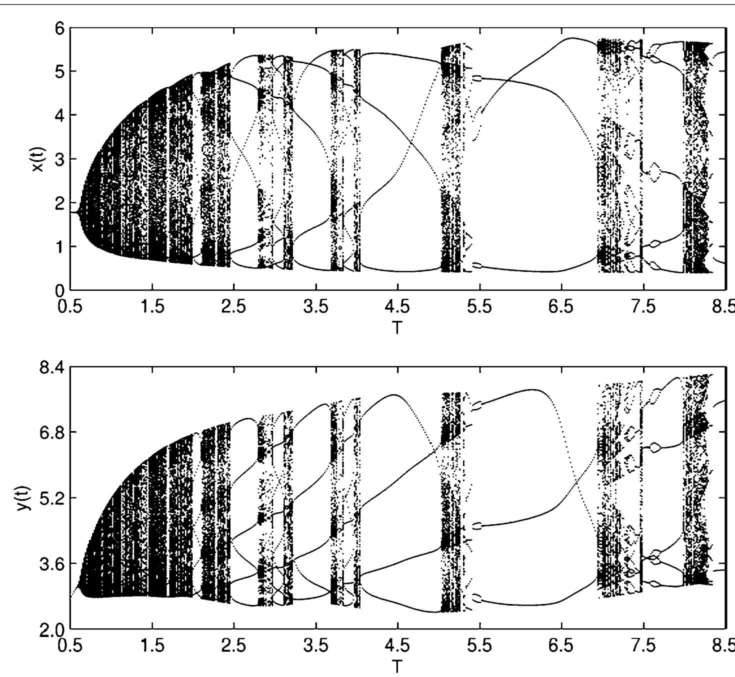

Figure 8 **Bifurcation diagrams of system (19): prey population $x(t)$ (top) and predator population $y(t)$ (bottom) when $p = 0.3$, $\tau = 3$ with impulsive period T over [0.5, 8.5].**

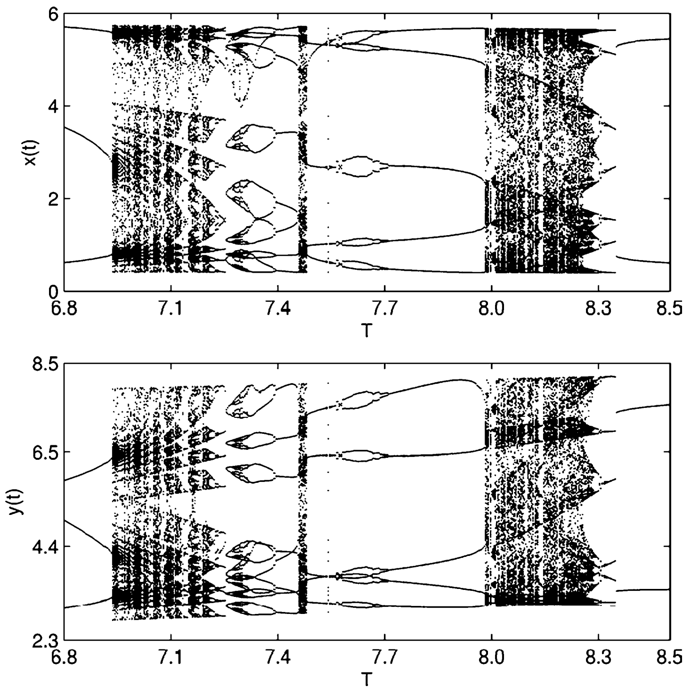

Figure 9 Bifurcation diagrams of system (19): prey population $x(t)$ (top) and predator population $y(t)$ (bottom) when $p = 0.3$, $\tau = 3$ with impulsive period T over [6.9, 7.5].

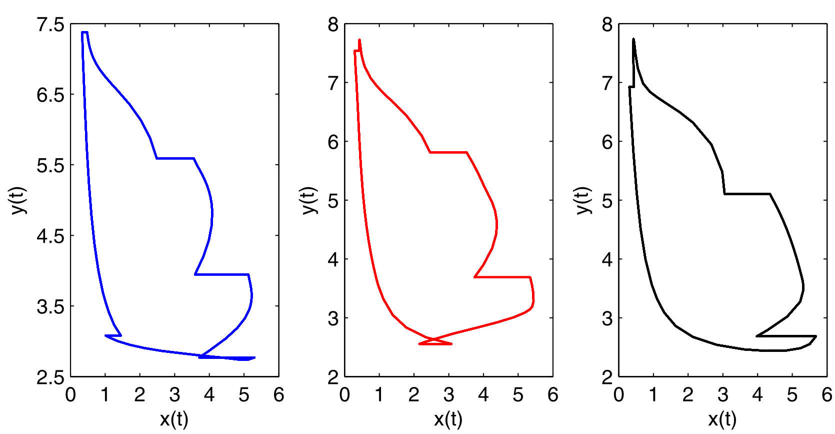

Figure 10 Phase portrait of the system (19). $5T$, $4T$, $3T$ periodic solution when $T = 3.5, 4.5, 6.5$, respectively.

Finally, we let $\tau = 7$, $T = 7$, and we investigate the impulsive perturbation proportionality constant p on system (19). The bifurcation diagrams of impulsive perturbation proportionality constant p over $[0.2, 0.74]$ show that system (19) has complex dynamics (see Figure 15). These results imply that the impulsive perturbation proportionality constant p would make the dynamical behaviors more complex, too. From the bifurcation diagrams (Figures 8, 9, 12, 13 and 15), we know that the parameters impulsive period T, time delay τ, and impulsive perturbation proportionality constant p would be important factors to affect the dynamic behaviors of the system, and make the system subject to complex dynamical behaviors.

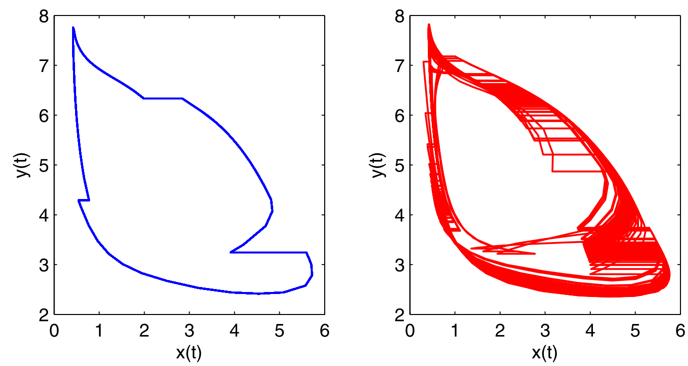

Figure 11 Crisis are shown: from left to right there is a crisis that the chaos suddenly appears when $T = 6.93, 6.94$, respectively.

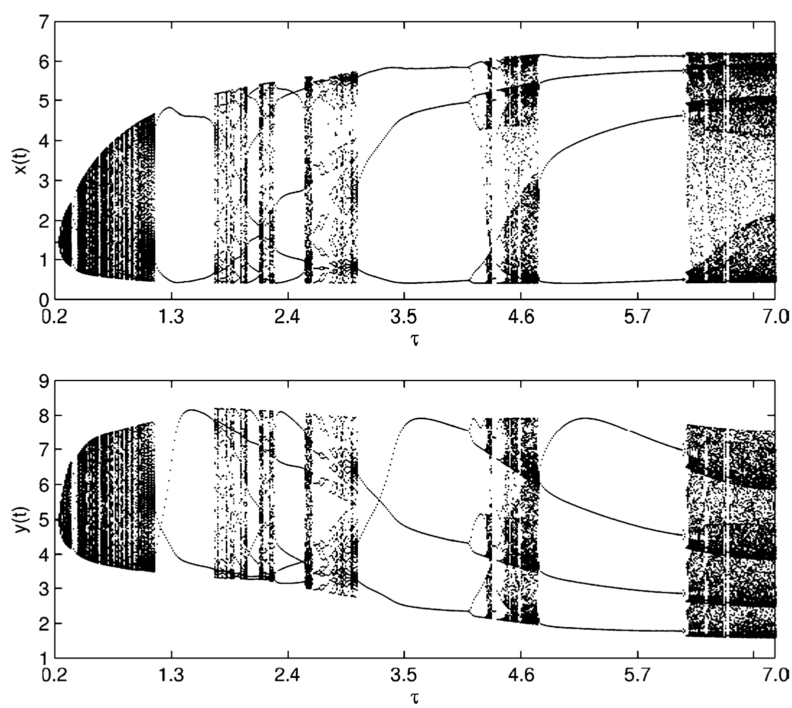

Figure 12 Bifurcation diagrams of system (19): prey population $x(t)$ (top) and predator population $y(t)$ (bottom) when $p = 0.3$, $T = 7$ with time delay τ over [0.2, 7.0].

5 Conclusion

In this paper, we have investigated a prey-predator system with constant gestation time delay and impulsive perturbation on the prey in detail. We have shown that there exists a globally attractive predator-extinction periodic solution when the impulsive period T is less than the critical value T_1^*. The system is permanent when the impulsive period T is larger than the critical value T_2^*. Therefore, we get two thresholds T_1^* and T_2^*, and there is no information as regards the system (1) when $T_1^* < T < T_2^*$. If the system (1) is without time delay, then $T_1^* = T_{\max} = T_2^*$ would be the unique threshold. This is essentially different when system (1) is with or without time delay.

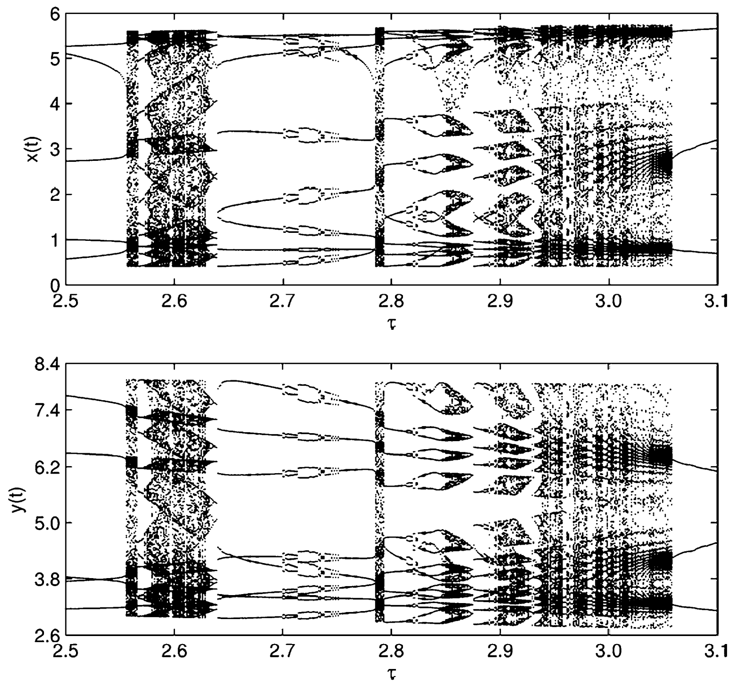

Figure 13 Bifurcation diagrams of system (19): prey population $x(t)$ (top) and predator population $y(t)$ (bottom) when $p = 0.3$, $T = 7$ with time delay τ over $[2.5, 3.1]$.

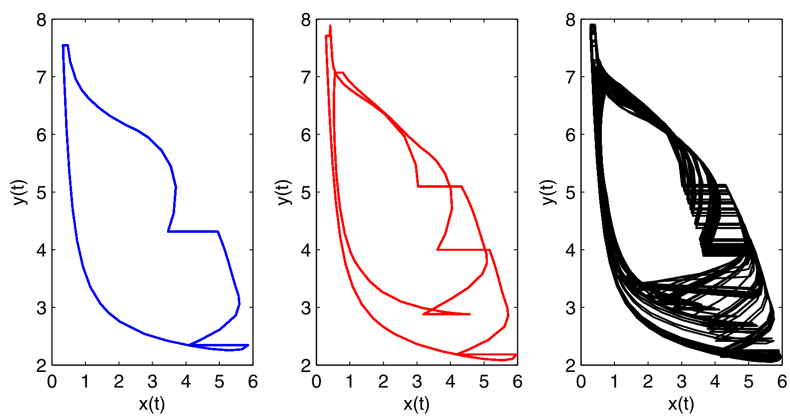

Figure 14 Period-doubling bifurcation leads to chaos. From left to right: phase portrait of T-periodic, $2T$-periodic solutions and chaos when $\tau = 4.1, 4.2, 4.3$, respectively.

Numerical examples show that system (1) have various kinds of periodic oscillations, including high-order periodic and quasi-periodic oscillations, chaotic oscillations. These results imply that the parameters of impulsive period T, time delay τ, and impulsive perturbation proportionality constant p would be important factors to affect the dynamic behaviors of the system (1), and make the system (1) subject to complex dynamical behaviors. That large time delay could stabilize the system, an impulsive effect could destabilize the system. Therefore, the dynamical behaviors would be more complex when the system is subject to both time delay and an impulsive effect.

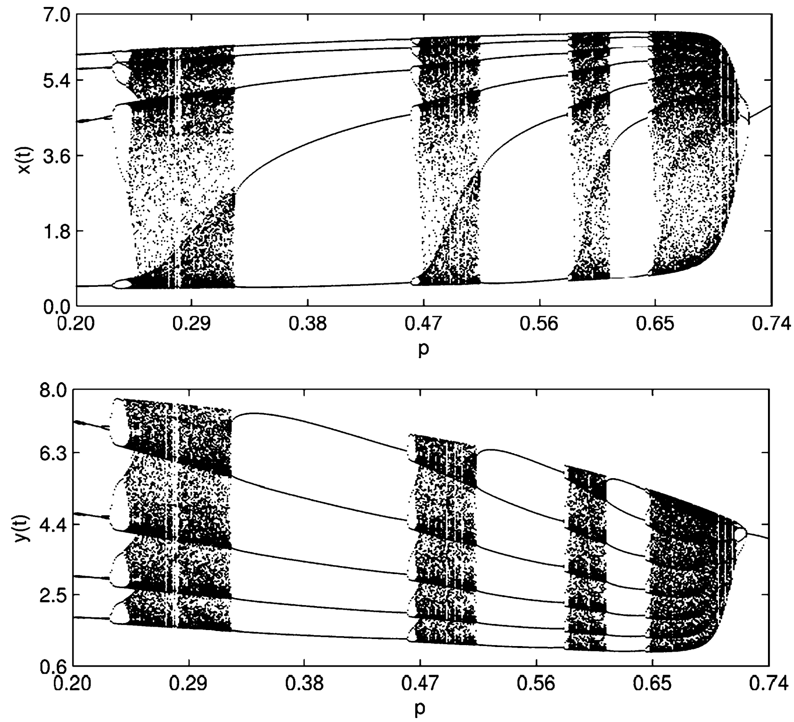

Figure 15 Bifurcation diagrams of system (19): prey population $x(t)$ (top) and predator population $y(t)$ (bottom) when $\tau = 7$, $T = 7$ with parameter p over $[0.2, 0.74]$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
SL carried out the main part of this manuscript. WL participated in the discussion and gave the examples. All authors read and approved the final manuscript.

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