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# Impulsive functional delay differential inclusions of fractional order at variable times

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**Abstract**
We are concerned with some sufficient conditions for the existence of solutions of a class of initial value problems for impulsive fractional differential inclusions with functional delay at variable moments.

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**Keywords:** Caputo fractional derivative; existence and uniqueness; functional differential inclusions; impulsive differential inclusions; variable times

## 1 Introduction

This work considers the existence of solutions to the following initial value problem (IVP) for a class of impulsive retarded fractional differential inclusions at variable times:

$$^{C}D^{\alpha}\left[^{C}D^{\beta}x(t)-g(t,x_t)\right]\in F(t,x_t),\quad t\in J, t\neq\tau_k\bigl(x(t)\bigr),\tag{1}$$

$$x\bigl(t^{+}\bigr)=I_k\bigl(x(t)\bigr),\quad t=\tau_k\bigl(x(t)\bigr),\tag{2}$$

$$^{C}D^{\beta}x\bigl(t^{+}\bigr)=I_k^{*}\bigl(x(t)\bigr),\quad t=\tau_k\bigl(x(t)\bigr),\tag{3}$$

$$x(t)=\phi(t),\quad t\in[-r,0],\tag{4}$$

$$^{C}D^{\beta}x(0)=\mu\in R,\tag{5}$$

where $^{C}D^{\alpha}$ and $^{C}D^{\beta}$ are Caputo fractional derivatives, $0<\alpha,\beta\le 1, 1<\alpha+\beta<2, J=[0,T]$, $0<r<\infty$, $\mathcal{D}=\{\psi:[-r,0]\to R$ is continuous everywhere except for a finite number of points $s$ at which $\psi(s^-)$ and $\psi(s^+)$ exist and $\psi(s^-)=\psi(s)\}$, and $\phi\in\mathcal{D}$, $F:J\times\mathcal{D}\to\mathcal{P}(R)$ is compact convex valued multivalued map ($\mathcal{P}(R)$ is the family of all nonempty subsets of $R$), $g:J\times\mathcal{D}\to R$, $I_k,I_k^{*}:R\to R$, $\tau_k:R\to R$, $k=1,2,\dots,p$ are given functions satisfying some conditions to be specified later. For any function $x$ defined on $[-r,T]$ and any $t\in J$ we denote by $x_t$ the element of $\mathcal{D}$ defined by $x_t=x(t+\theta)$, $\theta\in[-r,0]$. Here $x_t(\cdot)$ represents the history of the state from time $t-r$ up to the present time $t$.

The subject of impulsive fractional differential equations and inclusions has generated a good deal of interest among a good many researchers due to fact that fractional calculus and impulsive theory arise in mathematical modeling of some certain problems in science and engineering [1–7]. We refer the interested reader to [8–18] and [19–24] for some recent works on fractional differential equations and inclusions and for those on impulsive ones, respectively.

Furthermore, several authors investigate the existence of solutions of functional (neutral or retarded) differential equations and inclusions of fractional order [25–27] and impulsive functional fractional differential inclusions with fixed moments [28–30]. However, to the best of our knowledge, impulsive retarded fractional differential inclusions with variable moments have not been considered yet.

Therefore, inspired by mentioned works above as well as the study [31] including the following problem:

$$\begin{aligned}
&\frac{d}{dt}\big[y'(t)-g(t,y_t)\big]\in F\big(t,y(t)\big), \quad \text{a.e. } t\in[0,T], t\neq\tau_k\big(y(t)\big),\\
&y\big(t^+\big)=I_k\big(y(t)\big), \quad t=\tau_k\big(y(t)\big),\\
&y'\big(t^+\big)=\overline{I}_k\big(y(t)\big), \quad t=\tau_k\big(y(t)\big),\\
&y(t)=\phi(t), \quad t\in[-r,0], \qquad y'(0)=\eta,
\end{aligned}$$

we deal with the existence of an initial value problem for impulsive retarded functional fractional differential inclusions with variable times (1)-(5) in view of fixed point theorem for multivalued maps.

The present paper is organized as follows: We will briefly give some fundamentals and preliminary results on fractional calculus and multivalued maps in Section 2. We will establish some existence results of the IVP (1)-(5) by making use of the nonlinear alternative of Leray-Schauder type for multivalued maps in Section 3.

## 2 Preliminaries

In this section, let us introduce some notations, definitions, and preliminary facts to be used throughout this study.

By $C(J,R)$, $C([-r,0],R)$, and $C([-r,T],R)$ we denote the Banach space of all continuous functions from $J$ into $R$ with the norm

$$\|x\|_C:=\sup\big\{\big|x(t)\big|:t\in J\big\},$$

the Banach space of all continuous functions from $[-r,0]$ into $R$ with the norm

$$\|x\|_{\mathcal{D}}:=\sup\big\{\big|\phi(\theta)\big|:\theta\in[-r,0]\big\}$$

and the Banach space of all continuous functions from $[-r,T]$ into $R$ with the norm

$$\|x\|:=\|x\|_C+\|x\|_{\mathcal{D}},$$

respectively. Let us denote the Banach space of all continuous $\beta$-differentiable functions from $[-r,T]$ into $R$ by $C^\beta([-r,T],R)$ with the norm

$$\|x\|_\beta:=\max\big\{\|x\|,\big\|{}^{\mathrm{C}}D^\beta x\big\|\big\},$$

where $C^\beta([-r,T],R)=\{x\in C([-r,T],R):{}^{\mathrm{C}}D^\beta x(t)$ exists and ${}^{\mathrm{C}}D^\beta x(t)\in C([-r,T],R)\}$.

In addition, in order to define the solutions of the problem (1)-(5) we will consider the piecewise continuous spaces:

$\Omega = \{x : [-r, T] \to R$ : there exists $0 = t_0 < t_1 < t_2 < \cdots < t_p < t_{p+1} = T$ such that $t_k = \tau_k(x(t_k))$ and $x_{k+1} \in C((t_k, t_{k+1}], R)$, $k = 0, 1, 2, \dots, p$; also, there exist $x(t_k^+)$ and $x(t_k^-)$ with $x(t_k^-) = x(t_k)$ for $k = 1, 2, \dots, p$, and $x(t) = \phi(t)$, $t \le t_0\}$

and

$\Omega^\beta = \{x \in \Omega, {}^{\mathrm{C}}D^\beta x(t) \in C((t_k, t_{k+1}], R)$ : there exist ${}^{\mathrm{C}}D^\beta x(t_k^+)$ and ${}^{\mathrm{C}}D^\beta x(t_k^-)$ with ${}^{\mathrm{C}}D^\beta x(t_k^-) = {}^{\mathrm{C}}D^\beta x(t_k)$, $1 \le k \le p$, $0 < \beta \le 1\}$, where $x_{k+1}$ is the restriction of $x$ over $(t_k, t_{k+1}]$ and denoted by $x_{k+1} := x|_{(t_k, t_{k+1}]}$, $k = 0, 1, 2, \dots, p$.

The spaces $\Omega$ and $\Omega^\beta$ form Banach spaces with the norms

$$\|x\|_\Omega := \max\bigl\{\|x_{k+1}\|, k = 0, 1, \dots, p+1\bigr\} + \|x\|_{\mathcal{D}}$$

and

$$\|x\|_{\Omega^\beta} := \max\bigl\{\|x\|_\Omega, \bigl\|{}^{\mathrm{C}}D^\beta x\bigr\|_\Omega\bigr\},$$

respectively.

Let $L^1(J, R)$ denote the Banach space of measurable functions $x : J \to R$ which are Lebesgue integrable with the norm

$$\|x\|_{L^1} = \int_0^T \bigl|x(t)\bigr|\,dt \quad \text{for all } x \in L^1(J, R).$$

**Definition 1** ([1, 2]) The fractional (arbitrary) order integral of the function $h \in L^1(J, R)$ of order $q \in R_+$ is defined by

$$I_{0^+}^q h(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} h(s)\,ds,$$

where $\Gamma(\cdot)$ is the Euler gamma function.

**Definition 2** ([1, 2]) For a function $h$ given on the interval $J$, the Caputo fractional derivative of order $q > 0$ is defined by

$${}^{\mathrm{C}}D_{0^+}^q h(t) = \int_0^t \frac{(t-s)^{n-q-1}}{\Gamma(n-q)} h^{(n)}(s)\,ds, \quad n = [q] + 1,$$

where the function $h(t)$ has absolutely continuous derivatives up to order $(n-1)$.

Now, we focus on some fundamental facts of multivalued maps. See Gorniewicz [32], Aubin and Frankowska [33], Deimling [34], and Hu and Papageorgiou [35].

For a Banach space $(X, \|\cdot\|)$, let us denote:

$\mathcal{P}(X) = \{Y \subseteq X : Y \neq \emptyset\}$,
$\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : Y$ is closed$\}$,
$\mathcal{P}_b(X) = \{Y \in \mathcal{P}(X) : Y$ is bounded$\}$,
$\mathcal{P}_{cv}(X) = \{Y \in \mathcal{P}(X) : Y$ is convex$\}$,
$\mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X) : Y$ is compact$\}$,
$\mathcal{P}_{cv,cp}(X) = \mathcal{P}_{cv}(X) \cap \mathcal{P}_{cp}(X)$.

A multivalued map $G : X \to \mathcal{P}(X)$ has convex (closed) values if $G(x)$ is convex (closed) for all $x \in X$. $G$ is bounded on bounded sets if $G(B) = \bigcup_{x\in B} G(x)$ is bounded in $X$ for all $B \in \mathcal{P}_b(X)$ (*i.e.* $\sup_{x\in B}\{\sup\{\|y\| : y \in G(x)\}\} < \infty$).

A multivalued map $G : [0,1] \to \mathcal{P}_{cl}(X)$ is said to be measurable if for every $x \in X$, the function $Y : [0,1] \to X$ defined by $Y(t)$: $\operatorname{dist}(x, G(t)) = \inf\{\|x - z\| : z \in G(t)\}$ is Lebesgue measurable.

A multivalued map $F : J \times \mathcal{D} \to \mathcal{P}(R)$ is said to be $L^1$-Caratheodory if

(i) $t \to F(t,u)$ is measurable for each $u \in \mathcal{D}$,

(ii) $u \to F(t,u)$ is upper semi-continuous for almost all $t \in J$,

(iii) for each $q > 0$, there exists $\phi_q \in L^1(J, R_+)$ such that

$$\|F(t,u)\| = \sup\{|v| : v \in F(t,u)\} \le \phi_q(t)$$

for all $\|u\|_{\mathcal{D}} \le q$ and for almost all $t \in J$.

For a function $u \in \Omega^\beta$, we define the set

$$S_{F,u} = \{v \in L^1(J,R) : v(t) \in F(t,u) \text{ for a.e. } t \in J\},$$

which is known as the set of selection functions of $F$.

The next lemmas and proposition play a pivotal role in the subsequent results.

**Lemma 1** ([36]) *Let $F : J \times \mathcal{D} \to \mathcal{P}_{cv,cp}(R)$ be $L^1$-Caratheodory multivalued map with $S_{F,x} \neq \emptyset$ and let $\mathcal{L}$ be a linear continuous mapping from $L^1(J,R_+)$ to $C(J,R)$, then the operator*

$$\begin{aligned} &\mathcal{L} \circ S_F : C(J,R) \to \mathcal{P}_{cp,c}(C(J,R)) \\ &\qquad x \mapsto (\mathcal{L} \circ S_F)(x) := \mathcal{L}(S_{F,x}) \end{aligned}$$

*is a closed graph operator in $C(J,R) \times C(J,R)$.*

**Proposition 1** ([32]) *Assume $\varphi : X \to Y$ is a multivalued map such that $\varphi(X) \subset K$ and the graph $\Gamma_\varphi$ of $\varphi$ is closed, where $K$ is a compact set. Then $\varphi$ is upper semi-continuous.*

**Lemma 2** ([37]) *Let $X$ be a Banach space with $C \subset X$ convex. Assume that $U$ is a nonempty open subset of $C$ with $0 \in U$ and let $G : \overline{U} \to \mathcal{P}_{cp,cv}(C)$ be an upper semi-continuous and compact map. Then either,*

(a) *$G$ has a fixed point in $U$, or*

(b) *there exist $u \in \partial U$ and $\lambda \in (0,1)$ with $u \in \lambda G(u)$.*

## 3 Existence of solutions

**Definition 3** A function $x \in \Omega^\beta$ is said to be a solution of (1)-(5) if there exists a function $v(t) \in S_{F,x}$ for which the equation ${}^{C}D^\alpha[{}^{C}D^\beta x(t) - g(t,x_t)] = v(t)$ holds for a.e. $t \in J$, $t \neq \tau_k(x(t))$, $k = 1,2,\ldots,p$, where the conditions $x(t^+) = I_k(x(t))$, ${}^{C}D^\beta x(t^+) = I_k^*(x(t))$, $t = \tau_k(x(t))$, $k = 1,2,\ldots,p$, and $x(t) = \phi(t)$, ${}^{C}D^\beta x(0) = \mu \in R$, $t \in [-r,0]$, $0 < r < \infty$ are satisfied for $x$.

**Lemma 3** *The function $x(t) \in C^\beta([-r,T],R)$ is a solution of the problem*

$${}^{C}D^\alpha\left[{}^{C}D^\beta x(t) - g(t,x_t)\right] = v(t), \quad t \in J,$$

$$x(t) = \phi(t), \quad t \in [-r, 0],$$
$$^{C}D^{\beta}x(0) = \mu \in R,$$

*if and only if* $x(t)$ *satisfies the following integral equation*:

$$x(t) = \begin{cases} \phi(t), & t \in [-r, 0], \\ \phi(0) + [\mu - g(0, \phi)] \frac{t^{\beta}}{\Gamma(\beta+1)} & \\ \quad + \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} g(s, x_s)\, ds + \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} v(s)\, ds, & t \in J, \end{cases}$$

*where* $\alpha$, $\beta$, $J$ *are stated as above.*

From now on, for the sake of convenience, we assume that

$$P(t) = \phi(0) + \bigl[\mu - g(0, \phi)\bigr] \frac{t^{\beta}}{\Gamma(\beta + 1)} + \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} g(s, x_s)\, ds$$

and

$$P_{t_k}^{(k)}(t) = I_k\bigl(x_k(t_k)\bigr) + \bigl(I_k^*\bigl(x_k(t_k)\bigr) - g(t_k, x_{t_k})\bigr) \frac{(t - t_k)^{\beta}}{\Gamma(\beta + 1)} + \int_{t_k}^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} g(s, x_s)\, ds,$$

where $k = 1, 2, \ldots, p$.

**Theorem 1** *Suppose that the following conditions are satisfied*:

(A1) *There exist a continuous non-decreasing function* $\psi : [0, \infty) \to (0, \infty)$ *and a function* $m(t) \geq 0$, $\forall t \in J$ *with* $m^0 = \sup\{|m(t)| : t \in J\}$ *such that* $|F(t, u)| \leq m(t)\psi(\|u\|_{\mathcal{D}})$ *for* $\forall t \in J$, $\forall u \in \mathcal{D}$, *where the function* $F : J \times \mathcal{D} \to \mathcal{P}_{cv,cp}(R)$ *is* $L^1$*-Caratheodory.*

(A2) *The function* $g : J \times \mathcal{D} \to R$ *is continuous such that* $|g(t, u)| \leq c_1\|u\|_{\mathcal{D}} + c_2$ *for* $\forall t \in J$, $\forall u \in \mathcal{D}$ *and constants* $c_1, c_2 \geq 0$.

(A3) *The functions* $I_k, I_k^* : R \to R$, $k = 1, 2, \ldots, p$ *are continuous.*

(A4) *There exists a number* $\kappa > 0$ *such that*

$$\min\left\{ \frac{(1 - \frac{2c_1 T^{\beta}}{\Gamma(\beta+1)})\kappa}{|\phi(0)| + (|\mu| + 2c_2)\frac{T^{\beta}}{\Gamma(\beta+1)} + \frac{m^0 T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\psi(\kappa)}, \frac{(1 - \frac{2c_1 T^{\beta}}{\Gamma(\beta+1)})\kappa}{|I_k(\kappa)| + (|I_k^*(\kappa)| + 2c_2)\frac{T^{\beta}}{\Gamma(\beta+1)} + \frac{m^0 T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\psi(\kappa)} \right\} > 1.$$

(A5) *There exist functions* $\tau_k \in C^1(R, R)$ *for* $k = 1, 2, \ldots, p$ *such that* $0 < \tau_1(x) < \tau_2(x) < \cdots < \tau_p(x) < T$ *for* $\forall x \in R$.

(A6) *For all* $\forall x \in R$, $\tau_k(I_k(x)) \leq \tau_k(x) < \tau_{k+1}(I_k(x))$, $k = 1, 2, \ldots, p$.

(A7) *Let* $x \in \Omega$, *then for* $\forall t \in J$, *for every constant* $\zeta \in J$, *and for all* $x_t \in \mathcal{D}$ *we have*

$$\left\langle \tau_k'\bigl(x(t)\bigr), \frac{d}{dt} P_{\zeta}^{(k)}(t) + I_{\zeta}^{\alpha+\beta-1} v(t) \right\rangle \neq 1$$

*for* $k = 1, 2, \ldots, p$ *and for all* $v(t) \in S_{F,x}$.

*Then the IVP* (1)-(5) *has at least one solution on* $J$.

*Proof* The proof will be given in several steps for convenience.

*Step* 1: Consider the following problem:

$$ {}^{\mathrm{C}}D^{\alpha}\big[{}^{\mathrm{C}}D^{\beta}x(t)-g(t,x_t)\big]\in F(t,x_t),\quad t\in J, \tag{6}$$

$$ x(t)=\phi(t),\quad t\in[-r,0], \tag{7}$$

$$ {}^{\mathrm{C}}D^{\beta}x(0)=\mu\in R, \tag{8}$$

where $0<\alpha,\beta\leq 1$, $1<\alpha+\beta<2$, $J=[0,T]$, $0<r<\infty$.

Let us transform the problem (6)-(8) into a fixed point problem. By using Lemma 3 we consider the operator $\mathcal{N}:C^{\beta}([-r,T],R)\to\mathcal{P}(C^{\beta}([-r,T],R))$ defined by $\mathcal{N}(x)=\{h\in C^{\beta}([-r,T],R)\}$ where, for $v(t)\in S_{F,x}$,

$$ h(t)=\begin{cases}\phi(t), & t\in[-r,0],\\ P(t)+\int_0^t\frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}v(s)\,ds, & t\in J.\end{cases}$$

It is obvious that the set of fixed points of the operator $\mathcal{N}$ is solution to the problem (6)-(8). In this position, we shall use the nonlinear alternative of Leray-Schauder type in order to show that the operator $\mathcal{N}$ has fixed points. Then let us try to satisfy the conditions of the nonlinear alternative of Leray-Schauder type (Lemma 2).

First, we show that $\mathcal{N}(x)$ is convex for each $x\in C^{\beta}([-r,T],R)$. To do this, let $h_1$ and $h_2$ belong to $\mathcal{N}(x)$ with $v_1,v_2\in S_{F,x}$ such that

$$ h_i(t)=P(t)+\int_0^t\frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}v_i(s)\,ds,\quad i=1,2,$$

then, for each $t\in J$, we have

$$ \big[dh_1(t)+(1-d)h_2(t)\big]=P(t)+\int_0^t\frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\big[dv_1(s)+(1-d)v_2(s)\big]\,ds,$$

where $0\leq d\leq 1$.

Since $S_{F,x}$ is convex (*i.e.* $dv_1(s)+(1-d)v_2(s)\in S_{F,x}$ for $v_1,v_2\in S_{F,x}$ and $0\leq d\leq 1$) then $dh_1(t)+(1-d)h_2(t)\in\mathcal{N}(x)$.

Next, we need to show that $\mathcal{N}$ is a compact multivalued map.

(i) ($\mathcal{N}$ maps bounded sets into bounded sets in $C^{\beta}([-r,T],R)$.)

Actually, it is enough to show that there exists a constant $l>0$ such that we have $\|\mathcal{N}x\|\leq l$ for each $x\in B_r=\{x(t)\in C^{\beta}([-r,T],R):\|x\|_{\beta}\leq r\}$ for any $r>0$. Let $x\in B_r$ and $h\in\mathcal{N}(x)$ with $v\in S_{F,x}$, then for each $t\in J$ we obtain

$$ \begin{aligned}\big|\mathcal{N}(x)(t)\big| &\leq \big|P(t)\big|+\int_0^t\frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\big|v(s)\big|\,ds\\ &\leq\|\phi\|_{\mathcal{D}}+\big(|\mu|+c_1\|\phi\|_{\mathcal{D}}+c_2\big)\frac{t^{\beta}}{\Gamma(\beta+1)}\\ &\quad+\big(c_1\|x_t\|_{\mathcal{D}}+c_2\big)\int_0^t\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\,ds+\int_0^t\frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}m(s)\psi\big(\|x_s\|_{\mathcal{D}}\big)\,ds\end{aligned}$$

$$\leq r + \left(|\mu| + 2c_1 r + 2c_2\right)\frac{t^{\beta}}{\Gamma(\beta+1)} + m^0\psi(r)\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} := l,$$

$$\left\|\mathcal{N}(x)(t)\right\|_{\beta} \leq l,$$

which implies that the operator $\mathcal{N}$ is uniformly bounded.

(ii) ($\mathcal{N}$ maps bounded sets into equicontinuous sets of $C^{\beta}([-r,T],R)$.)

Assume that $\theta_1,\theta_2 \in J$, $\theta_1 < \theta_2$, and $B_r$ is a bounded set as in (i). Let $x \in B_r$ and $h \in \mathcal{N}(x)$ with $v \in S_{F,x}$, then for each $t \in J$ we have

$$\begin{aligned}
\left|\mathcal{N}(x)(\theta_2) - \mathcal{N}(x)(\theta_1)\right| \leq{}& \left|\mu - g(0,\phi)\right|\frac{\theta_2^{\beta} - \theta_1^{\beta}}{\Gamma(\beta+1)} \\
&+ \int_0^{\theta_1}\frac{[(\theta_2-s)^{\beta-1} - (\theta_1-s)^{\beta-1}]}{\Gamma(\beta)}\left|g(t,x_s)\right|ds \\
&+ \int_{\theta_1}^{\theta_2}\frac{(\theta_2-s)^{\beta-1}}{\Gamma(\beta)}\left|g(t,x_s)\right|ds \\
&+ \int_0^{\theta_1}\frac{[(\theta_2-s)^{\alpha+\beta-1} - (\theta_1-s)^{\alpha+\beta-1}]}{\Gamma(\alpha+\beta)}m(s)\psi\left(\|x_s\|_{\mathcal{D}}\right)ds \\
&+ \int_{\theta_1}^{\theta_2}\frac{(\theta_2-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}m(s)\psi\left(\|x_s\|_{\mathcal{D}}\right)ds, \\
\left\|\mathcal{N}(x)(\theta_2) - \mathcal{N}(x)(\theta_1)\right\|_{\beta} \leq{}& \left|\mu - g(0,\phi)\right|\frac{\theta_2^{\beta} - \theta_1^{\beta}}{\Gamma(\beta+1)} \\
&+ \frac{c_1 r + c_2}{\Gamma(\beta+1)}\left|2(\theta_2-\theta_1)^{\alpha} + \theta_1^{\alpha} - \theta_2^{\alpha}\right| \\
&+ \frac{m^0\psi(r)}{\Gamma(\alpha+\beta+1)}\left|2(\theta_2-\theta_1)^{\alpha+\beta} + \theta_1^{\alpha+\beta} - \theta_2^{\alpha+\beta}\right|,
\end{aligned}$$

implying that $\mathcal{N}$ is equicontinuous on $J$ since the right-hand side of the inequality tends to zero as $\theta_1 \to \theta_2$. Thus, as a consequence of (i) and (ii) together with the Arzela-Ascoli theorem, the operator $\mathcal{N} : C^{\beta}([-r,T],R) \to \mathcal{P}(C^{\beta}([-r,T],R))$ is a compact multivalued map.

Now, let us show that $\mathcal{N}$ has a closed graph. Let $x_n \to x_*$, $h_n \to h_*$, and $h_n \in \mathcal{N}(x_n)$ with $v_n \in S_{F,x_n}$ such that, for each $t \in J$,

$$h_n(t) = P_n(t) + \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}v_n(s)\,ds.$$

Then we have to show that there exists $v_* \in S_{F,x_*}$ in order to prove that $h_* \in \mathcal{N}(x_*)$ such that, for each $t \in J$,

$$h_*(t) = P_{n_*}(t) + \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}v_*(s)\,ds. \tag{9}$$

Let us consider the continuous linear operator $\mathcal{L} : L^1(J,R_+) \to C(J,R)$,

$$v \to (\mathcal{L}v)(t) = \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}v(s)\,ds.$$

Obviously, by the continuity of $g$, we have

$$\left\| h_n(t) - P_n(t) - \left(h_*(t) - P_{n_*}(t)\right)\right\| \to 0$$

as $n \to \infty$.

It results from Lemma 1 that $\mathcal{L} \circ S_F$ is a closed graph operator. What is more, since $(h_n(t) - P_n(t)) \in \mathcal{L}(S_{F,x_n})$ and $x_n \to x_*$, Lemma 1 implies that equation (9) holds for some $v_* \in S_{F,x_*}$.

Thus, by Proposition 1, $\mathcal{N}$ is an upper semi-continuous compact map with convex closed values.

Finally, it remains to discuss *a priori* bounds on solutions. Let $x$ be a possible solution of the problem (1)-(5). Then there exists $v \in L^1(J,R_+)$ with $v \in S_{F,x}$ such that, for each $t \in J$, we have

$$\begin{aligned} \left|(x)(t)\right| &\le \left|P(t)\right| + \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\left|v(s)\right|\,ds \\ &\le \left|\phi(0)\right| + \left(\mu + c_1\|\phi\|_{\mathcal{D}} + c_2\right)\frac{t^\beta}{\Gamma(\beta+1)} \\ &\quad + \left(c_1\|x_t\|_{\mathcal{D}} + c_2\right)\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\,ds + \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} m(s)\psi\left(\|x_s\|_{\mathcal{D}}\right)ds, \\ \|x\|_\beta &\le \left|\phi(0)\right| + \left(|\mu| + 2c_1\|x\|_\beta + 2c_2\right)\frac{T^\beta}{\Gamma(\beta+1)} + m^0\psi\left(\|x\|_\beta\right)\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}. \end{aligned} \tag{10}$$

Thus we get

$$\frac{\left(1 - \frac{2c_1T^\beta}{\Gamma(\beta+1)}\right)\|x\|_\beta}{|\phi(0)| + (|\mu| + 2c_2)\frac{T^\beta}{\Gamma(\beta+1)} + \frac{m^0T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\psi(\|x\|_\beta)} \le 1. \tag{11}$$

In view of (A4), there exists $\kappa$ such that $\|x\|_\beta \neq \kappa$. Then let us set

$$U = \left\{x \in C^\beta\left([-r,T],R\right) : \|x\|_\beta < \kappa\right\}.$$

We note that the operator $\mathcal{N} : \overline{U} \to \mathcal{P}(C^\beta([-r,T],R))$ is also an upper semi-continuous and compact multivalued map. Accordingly, the choice of $U$ shows that there is no $x \in \partial U$ such that $x \in \lambda\mathcal{N}(x)$ for some $\lambda \in (0,1)$. Consequently, thanks to the nonlinear alternative of Leray-Schauder type (Lemma 2), we conclude that $\mathcal{N}$ has a fixed point $x \in \overline{U}$ which is a solution of the problem (1)-(5). Denote this solution by $x_1$.

Now, we shall discuss at which discontinuity moment the solution $x(t)$ beats. Let us define the following function which is able to make the discussion easier:

$$r_{k,1}(t) = \tau_k\left(x_1(t)\right) - t, \quad t \ge 0.$$

From (A5) we have

$$r_{k,1}(0) = \tau_k\left(x_1(0)\right) \neq 0, \quad k = 1, 2, \ldots, p.$$

If $r_{k,1}(t) \neq 0$, that is, $\tau_k(x_1(t)) \neq t$ on $J$ for $k = 1, 2, \dots, p$, then $x_1(t)$ is a solution of both (6)-(8) and (1)-(5).

Now, we consider the case when

$$r_{1,1}(t) = 0, \quad \textit{i.e. } t = \tau_1\bigl(x_1(t)\bigr) \text{ for some } t \in J.$$

Since $r_{1,1}$ is continuous and $r_{1,1}(0) \neq 0$ by (A5), there exists $t_1 > 0$ such that

$$r_{1,1}(t_1) = 0 \quad \text{and} \quad r_{1,1}(t) \neq 0 \quad \text{for all } t \in [0, t_1).$$

Thus by (A5) we have

$$r_{k,1}(t) \neq 0 \quad \text{for all } t \in [0, t_1) \text{ and } k = 1, 2, \dots, p.$$

Hence, we have established the discontinuity point $t_1$ where the solution $x(t)$ beats.

*Step* 2: Consider the following problem:

$$^{\mathrm{C}}D^{\alpha}\bigl[^{\mathrm{C}}D^{\beta}x(t) - g(t, x_t)\bigr] \in F(t, x_t), \quad t \in [t_1, T], \tag{12}$$

$$x\bigl(t_1^+\bigr) = I_1\bigl(x_1(t_1)\bigr), \tag{13}$$

$$^{\mathrm{C}}D^{\beta}x\bigl(t_1^+\bigr) = I_1^*\bigl(x_1(t_1)\bigr), \tag{14}$$

$$x(t) = x_1(t), \quad t \in [t_1 - r, t_1]. \tag{15}$$

Let us transform the problem (12)-(15) into a fixed point problem by considering the operator $\mathcal{N}_1 : C^{\beta}([t_1 - r, T], R) \to \mathcal{P}(C^{\beta}([t_1 - r, T], R))$ defined by $\mathcal{N}_1(x) = \{h \in C^{\beta}([t_1 - r, T], R)\}$ where, for $v(t) \in S_{F,x}$,

$$h(t) = \begin{cases} x_1(t), & t \in [t_1 - r, t_1], \\ P_{t_1}^{(1)}(t) + \int_{t_1}^{t} \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} v(s)\,ds, & t \in [t_1, T]. \end{cases} \tag{16}$$

In the sense of Step 1, $\mathcal{N}_1$ is an upper semi-continuous compact map with convex closed values. Then, for the discussion of *a priori* bounds on solutions as in (10) and (11), taking into account (16) and assumptions (A1)-(A4) we have

$$\frac{(1 - \frac{2c_1 T^{\beta}}{\Gamma(\beta+1)})\|x\|_{\beta}}{|I_1(x_1(t_1))| + (I_1^*(x_1(t_1)) + 2c_2)\frac{T^{\beta}}{\Gamma(\beta+1)} + \frac{m^0 T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\psi(\|x\|_{\beta})} \leq 1.$$

As a consequence of Lemma 2 as in Step 1, the choice of

$$U = \bigl\{x \in C^{\beta}\bigl([t_1 - r, T], R\bigr) : \|x\|_{\beta} < \kappa\bigr\}$$

results in the operator $\mathcal{N}_1 : \overline{U} \to \mathcal{P}(C^{\beta}([t_1 - r, T], R))$ to have a fixed point $x \in \overline{U}$, which is a solution of the problem (12)-(15) on $[t_1, T]$. Denote this solution by $x_2$.

Now, we shall discuss at which discontinuity moment after $t_1$ the solution $x(t)$ beats. Let us define the following function:

$$r_{k,2}(t) = \tau_k\bigl(x_2(t)\bigr) - t, \quad t \geq t_1. \tag{17}$$

If $r_{k,2}(t) \neq 0$, that is, $\tau_k(x_2(t)) \neq t$ on $(t_1, T]$ for $k = 1, 2, \ldots, p$, then $x_2(t)$ is a solution of (12)-(15). That is,

$$x(t) = \begin{cases} x_1(t), & t \in [t_0, t_1], \\ x_2(t), & t \in (t_1, T], \end{cases}$$

is a solution of (1)-(5).

Now, we consider the case when

$$r_{2,2}(t) = 0, \quad \textit{i.e.}\ t = \tau_2\bigl(x_2(t)\bigr) \text{ for some } t \in (t_1, T].$$

From (A6) we have

$$\begin{aligned} r_{2,2}\bigl(t_1^+\bigr) &= \tau_2\bigl(x_2\bigl(t_1^+\bigr)\bigr) - t_1 \\ &= \tau_2\bigl(I_1\bigl(x_1(t_1)\bigr)\bigr) - t_1 \\ &> \tau_1\bigl(x_1(t_1)\bigr) - t_1 \\ &= r_{1,1}(t_1) = 0. \end{aligned}$$

Since $r_{2,2}$ is continuous, there exists $t_2 > t_1$ such that

$$r_{2,2}(t_2) = 0 \quad \text{and} \quad r_{2,2}(t) \neq 0 \quad \text{for all } t \in (t_1, t_2).$$

Thus by (A5) we have

$$r_{k,2}(t) \neq 0 \quad \text{for all } t \in (t_1, t_2) \text{ and } k = 2, 3, \ldots, p.$$

Also, let us show that there does not exist any $\xi \in (t_1, t_2)$ such that $r_{1,2}(\xi) = 0$. Suppose now that there exists $\xi \in (t_1, t_2)$ such that $r_{1,2}(\xi) = 0$. By (A6) it follows that

$$\begin{aligned} r_{1,2}\bigl(t_1^+\bigr) &= \tau_1\bigl(x_2\bigl(t_1^+\bigr)\bigr) - t_1 \\ &= \tau_1\bigl(I_1\bigl(x_1(t_1)\bigr)\bigr) - t_1 \\ &\leq \tau_1\bigl(x_1(t_1)\bigr) - t_1 \\ &= r_{1,1}(t_1) = 0. \end{aligned}$$

And from (A5) we have

$$\begin{aligned} r_{1,2}(t_2) &= \tau_1\bigl(x_2(t_2)\bigr) - t_2 \\ &< \tau_2\bigl(x_2(t_2)\bigr) - t_2 \\ &= r_{2,2}(t_2) = 0. \end{aligned}$$

Since $r_{1,2}(t_1^+) \leq 0$, $r_{1,2}(t_2) < 0$, and $r_{1,2}(\xi) = 0$ for some $\xi \in (t_1, t_2)$, the function $r_{1,2}$ gets a nonnegative maximum at some point $\eta \in (t_1, t_2)$. On the other hand, in view of equation (12), since the function $x_2(t)$ holds for

$$^{C}D^{\alpha}\bigl[{}^{C}D^{\beta}x_2(t) - g(t, x_{2t})\bigr] \in F(t, x_{2t}), \quad \text{a.e. } t \in (t_1, T) \tag{18}$$

subject to conditions (13)-(15), there exists $\nu(\cdot) \in L^1((t_1, T))$ with $\nu(t) \in F(t, x_{2t})$, a.e. $t \in (t_1, T)$ such that

$$ {}^{\mathrm{C}}D^{\alpha}\left[{}^{\mathrm{C}}D^{\beta}x_2(t) - g(t, x_{2t})\right] = \nu(t). $$

Subsequently, from (18) and Lemma 3 the equalities

$$ \begin{aligned} x_2(t) &= I_1\bigl(x_1(t_1)\bigr) + \bigl(I_1^*\bigl(x_1(t_1)\bigr) - g(t_1, x_{t_1})\bigr)\frac{(t-t_1)^{\beta}}{\Gamma(\beta+1)} \\ &\quad + \int_{t_1}^{t} \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} g(s, x_s)\,ds + \int_{t_1}^{t} \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \nu(s)\,ds \end{aligned} $$

and

$$ \begin{aligned} x_2'(t) &= \bigl(I_1^*\bigl(x_1(t_1)\bigr) - g(t_1, x_{t_1})\bigr)\frac{(t-t_1)^{\beta-1}}{\Gamma(\beta)} \\ &\quad + \int_{t_1}^{t} \frac{(t-s)^{\beta-2}}{\Gamma(\beta-1)} g(s, x_s)\,ds + \int_{t_1}^{t} \frac{(t-s)^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)} \nu(s)\,ds \\ &= \frac{d}{dt}P_{t_1}^{(k)}(t) + I_{t_1}^{\alpha+\beta-1}\nu(t) \end{aligned} \tag{19} $$

are derived. Thus, in view of (17) and (19), and for some point $\eta \in (t_1, t_2]$, we obtain

$$ r_{1,2}'(\eta) = \tau_1'\bigl(x_2(\eta)\bigr)x_2'(\eta) - 1 = 0, $$

that is,

$$ \left\langle \tau_1'\bigl(x_2(\eta)\bigr), \frac{d}{dt}P_{t_1}^{(k)}(t) + I_{t_1}^{\alpha+\beta-1}\nu(t) \right\rangle = 1. $$

But this contradicts (A7).

Hence, we have established a second discontinuity point $t_2 > t_1$ where the solution $x(t)$ beats in such a way that $r_{2,2}(t_2) = 0$ and $r_{k,2}(t) \neq 0$ for all $t \in (t_1, t_2)$ and $k = 1, 2, 3, \ldots, p$.

*Step* 3: Let us continue the process as in Steps 1 and 2 by taking into account that $x_p := x|_{(t_{p-1}, T]}$ is a solution of the following problem:

$$ \begin{aligned} &{}^{\mathrm{C}}D^{\alpha}\left[{}^{\mathrm{C}}D^{\beta}x(t) - g(t, x_t)\right] \in F(t, x_t), \quad t \in [t_{p-1}, T], \\ &x\bigl(t_{p-1}^+\bigr) = I_{p-1}\bigl(x_{p-1}(t_{p-1})\bigr), \\ &{}^{\mathrm{C}}D^{\beta}x\bigl(t_{p-1}^+\bigr) = I_{p-1}^*\bigl(x_{p-1}(t_{p-1})\bigr), \\ &x(t) = x_{p-1}(t), \quad t \in [t_{p-1} - r, t_{p-1}], \end{aligned} $$

by considering the operator $\mathcal{N}_{p-1} : C^{\beta}([t_{p-1} - r, T], R) \to \mathcal{P}(C^{\beta}([t_{p-1} - r, T], R))$ defined by $\mathcal{N}_{p-1}(x) = \{h \in C^{\beta}([t_{p-1} - r, T], R)\}$ where, for $\nu(t) \in S_{F,x}$,

$$ h(t) = \begin{cases} x_{p-1}(t), & t \in [t_{p-1} - r, t_{p-1}], \\ P_{t_{p-1}}^{(p-1)}(t) + \int_{t_{p-1}}^{t} \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \nu(s)\,ds, & t \in [t_{p-1}, T]. \end{cases} $$

At the end, as in the previous steps, we establish a $p$th discontinuity point $t_p > t_{p-1} > \cdots > t_2 > t_1$ where the solution $x(t)$ beats in such a way that $r_{p,p}(t_p) = 0$ and $r_{p,p}(t) \neq 0$ for all $t \in (t_{p-1}, t_p)$. Then the solution $x$ of the problem (1)-(5) is defined by

$$x(t) = \begin{cases} x_1(t), & \text{if } t \in [t_0, t_1], \\ x_2(t), & \text{if } t \in (t_1, t_2], \\ \ldots, & \\ x_p(t), & \text{if } t \in (t_{p-1}, t_p], \\ x_{p+1}(t), & \text{if } t \in (t_p, T]. \end{cases} \qquad \square$$

**Competing interests**
The author declares that there are no competing interests.

**Author's contributions**
The author read and approved the final manuscript.

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