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# The Cauchy type problem for interval-valued fractional differential equations with the Riemann-Liouville gH-fractional derivative

Yonghong Shen*

*Correspondence:
shenyonghong2008@hotmail.com
School of Mathematics and Statistics, Tianshui Normal University, Tianshui, 741001, P.R. China
School of Mathematics and Statistics, Beijing Institute of Technology, Beijing, 100081, P.R. China

**Abstract**
In this paper, we establish the relationship between the Cauchy type problem for interval-valued fractional differential equations with the Riemann-Liouville gH-fractional derivative and the corresponding interval-valued integral equation. Moreover, we also consider the existence of the solutions to the interval-valued integral equation. Furthermore, we obtain the solutions to the Cauchy type problem under certain conditions.

**Keywords:** generalized Hukuhara difference; interval-valued function; interval-valued Riemann-Liouville fractional integral; Riemann-Liouville gH-fractional derivative; Cauchy type problem

## 1 Introduction

Fractional calculus can be regarded as a generalization of ordinary differentiation and integration to any real or complex order. In the past few decades, the subject has gained considerable popularity and importance due mainly to its demonstrated applications in many fields, such as rheology, viscoelasticity, electrochemistry, electromagnetism, diffusion processes, and so on. It does indeed provide several potentially useful tools for solving differential and integral equations, and various other problems involving special functions of mathematical physics. For more details, the reader can refer to several important monographs, such as Oldham and Spanier [1], Miller and Ross [2], Podlubny [3], Kilbas *et al.* [4], Laksmikantham *et al.* [5], *etc.*

In practice, many problems are often associated with different types of imprecision, for instance, randomness and uncertainty. Accordingly, it is necessary to take into account imprecision to study some dynamical systems. Interval numbers and fuzzy numbers are two important tools to deal with uncertainty problems. In 2010, Agarwal *et al.* [6] first introduced the concept of solution for fractional differential equations in the space of fuzzy numbers. In the following year, Arshad and Lupulescu [7] defined the concepts of fuzzy fractional integral and fuzzy fractional derivative by means of level sets of fuzzy numbers. Meantime, they also proved the existence and uniqueness to the initial value problem for fuzzy fractional differential equations. Hereafter, Allahviranloo *et al.* [8] introduced the notion of fuzzy Riemann-Liouville fractional derivative (or Riemann-Liouville H-derivative) based on the Hukuhara difference (or H-difference) of fuzzy numbers. In essence, this definition is based on the strongly generalized derivative ($G$-derivative) of fuzzy number-valued functions introduced by Bede and Gal [9]. Subsequently, Salahshour *et al.* [10] considered the solutions of fuzzy fractional differential equations under Riemann-Liouville H-derivative by using the fuzzy Laplace transform method. In the same year, they together with Baluanu [11] defined the concept of Caputo H-derivative in a similar way, and further studied the existence, uniqueness and approximate solutions of fuzzy fractional differential equations. Later, Malinowski [12] studied the existence and uniqueness of the solutions of two types of random fuzzy fractional integral equations. Meantime, the author established the boundedness of solutions and the insensitivity to small changes of parameters. Unlike previous methods, Takači *et al.* [13] analyzed fractional differential equations with fuzzy coefficients by Mikusiński fuzzy operators. Recently, Allahviranloo *et al.* [14] and Hoa [15] independently introduced the concept of Caputo gH-derivative by using the generalized Hukuhara difference (or gH-difference). In fact, the gH-difference is considered as an improvement of the H-difference of fuzzy numbers. But the gH-difference of two fuzzy numbers does not always exist. However, the gH-difference for interval numbers is well defined. Interval analysis emerged as a special case of set-valued analysis has a long history [16]. To a certain degree, interval analysis was introduced as an effective method to deal with interval uncertainty that appears in many practical problems. For this reason, it is very necessary to study interval-valued differential equations.

In a recent paper [17], the author introduced fractional calculus for interval-valued functions based on gH-difference of interval numbers. Based on these concepts, Lupulescu and Hoa [18] considered the solvability of the interval Abel integral equation. However, the purpose of the present paper is to establish the relationship between the Cauchy type problem for interval-valued fractional differential equations and the corresponding interval-valued integral equation. Furthermore, we shall characterize the solutions to the Cauchy type problem by the interval-valued integral equation under certain conditions.

## 2 Preliminaries

Let $\mathbb{N}$, $\mathbb{R}$, and $\mathcal{K}$ denote the set of all natural numbers, the set of all real numbers and the set of all nonempty compact convex subsets of the real line $\mathbb{R}$, respectively. Moreover, let $T = [a, b]$, $-\infty < a < b < \infty$, denote a finite interval on the real line $\mathbb{R}$.

For $A = [a^-, a^+]$, $B = [b^-, b^+] \in \mathcal{K}$, $\lambda \in \mathbb{R}$, the Minkowski addition $A + B$ and scalar multiplication $\lambda \cdot A$ (or $\lambda A$) can be defined by

$$A + B = \left[a^-, a^+\right] + \left[b^-, b^+\right] = \left[a^- + b^-, a^+ + b^+\right]$$

and

$$\lambda \cdot A = \lambda \cdot \left[a^-, a^+\right] = \left[\min\left\{\lambda a^-, \lambda a^+\right\}, \max\left\{\lambda a^-, \lambda a^+\right\}\right],$$

respectively. Then the opposite $-A := (-1) \cdot A = [-a^+, -a^-]$. However, in general, $A + (-A) \neq \{0\}$, which implies that the opposite of $A$ is not the inverse of $A$ with respect to the Minkowski addition, unless $A$ is a singleton.

Let $A, B \in \mathcal{K}$. If there exists $C \in \mathcal{K}$ such that $A = B + C$, then $C$ is called the Hukuhara difference (H-difference for short) of $A$ and $B$, and it is denoted by $C := A \ominus B$. Note that the H-difference is unique, but it does not always exist. A necessary condition for $A \ominus B$ to exist is that $A$ contains a translation of $B$, *i.e.*, there exists an element $c$ such that $\{c\} + B \subseteq A$. To overcome this shortcoming, a generalized Hukuhara difference (gH-difference for short) is introduced by Stefanini [19].

**Definition 2.1** The *generalized Hukuhara difference* (or gH-difference) of two intervals $A$ and $B$ is defined as follows:

$$A \ominus_g B = \begin{cases} \text{(i) } A = B + C \quad \Leftrightarrow \quad A \ominus B = C, \\ \text{or (ii) } B = A + (-C) \quad \Leftrightarrow \quad B \ominus A = -C. \end{cases}$$

For $A = [a^-, a^+]$, $B = [b^-, b^+] \in \mathcal{K}$, it is easy to verify that the following equalities hold:

$$\begin{aligned} A \ominus_g B &= \left[a^-, a^+\right] \ominus_g \left[b^-, b^+\right] \\ &= \left[\min\left\{a^- - b^-, a^+ - b^+\right\}, \max\left\{a^- - b^-, a^+ - b^+\right\}\right] \\ &= \begin{cases} [a^- - b^-, a^+ - b^+], & w(A) \geq w(B), \\ [a^+ - b^+, a^- - b^-], & w(A) < w(B), \end{cases} \end{aligned}$$

where $w(\cdot)$ denotes the width of the interval, that is, $w(A) = a^+ - a^-$.

Now we define a functional $\|\cdot\| : \mathcal{K} \to [0, \infty)$ by $\|A\| = \max\{|a^-|, |a^+|\}$ for every $A = [a^-, a^+] \in \mathcal{K}$. It can easily be shown that $\|\cdot\|$ is a norm on $\mathcal{K}$, and thus the quadruple $(\mathcal{K}, +, \cdot, \|\cdot\|)$ is a normed quasilinear space [20].

Given two intervals $A = [a^-, a^+]$, $B = [b^-, b^+] \in \mathcal{K}$, the Hausdorff-Pompeiu metric between $A$ and $B$ is defined by $H(A, B) = \max\{|a^- - b^-|, |a^+ - b^+|\}$. It is well known that $(\mathcal{K}, H)$ is a complete and separable metric space. Furthermore, the following relationships exist between the Hausdorff-Pompeiu metric $H$ and the norm $\|\cdot\|$:

$$\|A\| = H\big(A, \{0\}\big), \qquad H(A, B) = \|A \ominus_g B\|.$$

Throughout this paper, we denote by $AC([a, b], \mathcal{K})$ the set of all absolutely continuous interval-valued functions from $[a, b]$ to $\mathcal{K}$. For $1 \leq p \leq \infty$, we denote by $L^p([a, b], \mathcal{K})$ the set of all interval-valued functions $F : [a, b] \to \mathcal{K}$ such that $\|F(t)\| \in L^p[a, b]$. The space $L^p([a, b], \mathcal{K})$ is a complete metric space with respect to the metric $\mathcal{H}_p$ defined by $\mathcal{H}_p(F, G) := \|F \ominus_g G\|_p$, where

$$\|F\|_p := \begin{cases} \int_a^b (\|F(t)\|^p)^{\frac{1}{p}} \, dt, & 1 \leq p < \infty, \\ \operatorname{ess\,sup}_{t \in [a,b]} \|F(t)\|, & p = \infty. \end{cases}$$

In particular, when $p = 1$, $L^1([a, b], \mathcal{K}) = L([a, b], \mathcal{K})$. Moreover, for $F, G \in L([a, b], \mathcal{K})$, we can obtain

$$\mathcal{H}_1(F, G) = \|F \ominus_g G\|_1 = \int_a^b \big\|F(t) \ominus_g G(t)\big\| \, dt = \int_a^b H\big(F(t), G(t)\big) \, dt.$$

Let $F : T \to \mathcal{K}$ be an interval-valued function. We say that $F$ is $w$-increasing ($w$-decreasing) on $T$ if $w(F(t))$ is increasing (decreasing) on $T$. Especially, we call $F$ is $w$-monotonic on $T$ if $w(F(t))$ is increasing or decreasing on $T$.

**Definition 2.2** (See Stefanini [21]) Let $F : T \to \mathcal{K}$ be an interval-valued function and let $t_0 \in T$ such that $t_0 + h \in T$. If the limit

$$\lim_{h\to 0} \frac{F(t_0 + h) \ominus_g F(t_0)}{h}$$

exists, written as $F'(t_0)$, we say that $F$ is *generalized Hukuhara differentiable* (gH-differentiable for short) at $t_0$. Meantime, $F'(t_0)$ is referred to as the *generalized Hukuhara derivative* (or gH-derivative) at $t_0$. At the endpoints of the interval $T$, we consider only the one sided gH-derivatives.

**Definition 2.3** (See Allahviranloo *et al.* [8], Lupulescu [17]) Let $F = [f^-, f^+] \in L([a,b], \mathcal{K})$ and $\alpha > 0$. The *interval-valued Riemann-Liouville fractional integral* of order $\alpha$ is defined as follows:

$$\mathcal{J}^{\alpha}_{a+} F(t) = \bigl[I^{\alpha}_{a+} f^-(t), I^{\alpha}_{a+} f^+(t)\bigr], \quad t > a,$$

where

$$I^{\alpha}_{a+} f^{\mp}(t) := \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f^{\mp}(s)\, ds$$

denote the classical Riemann-Liouville fractional integrals of orders $\alpha$ of the real valued functions $f^-(t)$ and $f^+(t)$, respectively. Here, $\Gamma(\cdot)$ stands for the Gamma function.

**Definition 2.4** (See Allahviranloo *et al.* [8], Lupulescu [17]) Let $F = [f^-, f^+] \in L([a,b], \mathcal{K})$ and $\alpha \in [0,1]$. Define the interval-valued function $F_{1-\alpha} : T \to \mathcal{K}$ by

$$F_{1-\alpha}(t) = \mathcal{J}^{1-\alpha}_{a+} F(t) := \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-s)^{-\alpha} F(s)\, ds, \quad t \in [a,b].$$

If $F_{1-\alpha}(t)$ is gH-differentiable for almost everywhere (*i.e.*, a.e.) $t \in [a,b]$, then the gH-derivative $F'_{1-\alpha}(t)$ is called the *interval-valued Riemann-Liouville gH-fractional derivative* of order $\alpha$ and it is denoted by

$$\mathcal{D}^{\alpha}_{a+} F(t) = F'_{1-\alpha}(t) = \bigl(\mathcal{J}^{1-\alpha}_{a+} F\bigr)'(t)$$

for a.e. $t \in [a,b]$.

In particular, when $\alpha = 0$ and $\alpha = 1$, we have $\mathcal{D}^0_{a+} F(t) = F(t)$, $\mathcal{D}^1_{a+} F(t) = F'(t)$.

**Lemma 2.1** (See Markov [22]) *Let $f : T \to \mathbb{R}$ be a differentiable real valued function and let $C \in \mathcal{K}$. Then the interval-valued function $f \cdot C : T \to \mathcal{K}$ is gH-differentiable and $(f(t) \cdot C)' = f'(t) \cdot C$.*

**Lemma 2.2** (See Lupulescu [17]) *The interval-valued Riemann-Liouville fractional integration operator $\mathcal{J}^{\alpha}_{a+}$ with $\alpha > 0$ is a bounded operator from $L^p([a,b],\mathcal{K})$ into $L^p([a,b],\mathcal{K})$, $1 \le p \le \infty$. More precisely, we have*

$$\left\| \mathcal{J}^{\alpha}_{a+} F \right\|_p \le \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)} \|F\|_p.$$

Based on Lemma 1.1 in [4], we can obtain the following characterization of the space $AC([a,b],\mathcal{K})$.

**Lemma 2.3** *The space $AC([a,b],\mathcal{K})$ consists of those and only those interval-valued functions $F = [f^-, f^+]$ that can be represented in the form*

$$F(t) = \int_a^t \Phi(s)\,ds + C,$$

*where $\Phi = [\varphi^-, \varphi^+] \in L([a,b],\mathcal{K})$, $C = [c^-, c^+] \in \mathcal{K}$.*

*Proof* By Proposition 4 in [17] and Lemma 1.1 in [4], we can obtain

$$\begin{aligned} F \in AC\bigl([a,b],\mathcal{K}\bigr) \quad &\Leftrightarrow \quad f^-, f^+ \in AC[a,b] \\ &\Leftrightarrow \quad f^{\mp}(t) = \int_a^t \varphi^{\mp}(s)\,ds + c^{\mp} \\ &\Leftrightarrow \quad F(t) = \bigl[f^-(t), f^+(t)\bigr] = \left[\int_a^t \varphi^-(s)\,ds + c^-, \int_a^t \varphi^+(s)\,ds + c^+\right] \\ &\Leftrightarrow \quad F(t) = \left[\int_a^t \varphi^-(s)\,ds, \int_a^t \varphi^+(s)\,ds\right] + \bigl[c^-, c^+\bigr] \\ &\Leftrightarrow \quad F(t) = \int_a^t \Phi(s)\,ds + C. \end{aligned}$$

□

**Lemma 2.4** *Let $f \in L[a,b]$ with $f(t) \ge 0$ or $f(t) \le 0$ and let $C = [c^-, c^+] \in \mathcal{K}$. If $\alpha > 0$ and $0 < \beta \le 1$, then*

(i) $\mathcal{J}^{\alpha}_{a+}\bigl(f(t)\cdot C\bigr) = I^{\alpha}_{a+} f(t) \cdot C$,

(ii) $\mathcal{D}^{\beta}_{a+}\bigl(f(t)\cdot C\bigr) = D^{\beta}_{a+} f(t) \cdot C$,

*where $D^{\beta}_{a+}$ denotes the classical Riemann-Liouville fractional differential operator of order $\beta$.*

*Proof* (i) If $f(t) \ge 0$, by Definition 2.3, we have

$$\begin{aligned} \mathcal{J}^{\alpha}_{a+}\bigl(f(t)\cdot C\bigr) &= \bigl[I^{\alpha}_{a+}\bigl(c^- f(t)\bigr), I^{\alpha}_{a+}\bigl(c^+ f(t)\bigr)\bigr] \\ &= \bigl[c^- I^{\alpha}_{a+} f(t), c^+ I^{\alpha}_{a+} f(t)\bigr] \\ &= I^{\alpha}_{a+} f(t) \cdot \bigl[c^-, c^+\bigr] \\ &= I^{\alpha}_{a+} f(t) \cdot C. \end{aligned}$$

Similarly, if $f(t) \leq 0$, then we can obtain

$$\begin{aligned}\mathcal{J}_{a+}^{\alpha}\big(f(t)\cdot C\big) &= \big[I_{a+}^{\alpha}\big(c^{+}f(t)\big), I_{a+}^{\alpha}\big(c^{-}f(t)\big)\big] \\ &= \big[c^{+}I_{a+}^{\alpha}f(t), c^{-}I_{a+}^{\alpha}f(t)\big] \\ &= I_{a+}^{\alpha}f(t)\cdot\big[c^{-},c^{+}\big] \\ &= I_{a+}^{\alpha}f(t)\cdot C.\end{aligned}$$

(ii) According to Definition 2.4 and Lemma 2.1, we get

$$\begin{aligned}\mathcal{D}_{a+}^{\beta}\big(f(t)\cdot C\big) &= \big(\mathcal{J}_{a+}^{1-\beta}\big(f(t)\cdot C\big)\big)' \\ &= \big(I_{a+}^{1-\beta}f(t)\cdot C\big)' \\ &= \big(I_{a+}^{1-\beta}f(t)\big)'\cdot C \\ &= D_{a+}^{\beta}f(t)\cdot C.\end{aligned}$$

□

**Lemma 2.5** *Let $C = [c^{-},c^{+}] \in \mathcal{K}$ and let $\alpha \in (0,1]$. Define the interval-valued function $G(t) := \frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)}C$ on $(a,b]$. Then $w(G_{1-\alpha}(t)) = c^{+} - c^{-}$ is a constant function on $[a,b]$.*

*Proof* By Definition 2.4, we get

$$G_{1-\alpha}(t) = \frac{1}{\Gamma(1-\alpha)}\int_a^t (t-s)^{-\alpha}G(s)\,ds.$$

Substituting $G(t)$ into the above equality gives

$$G_{1-\alpha}(t) = \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_a^t (t-s)^{-\alpha}(s-a)^{\alpha-1}\,ds\cdot C.$$

Setting $M(t) = \int_a^t (t-s)^{-\alpha}(s-a)^{\alpha-1}\,ds$. Make the substitution $s = a + \theta(t-a)$, we obtain

$$M(t) = \int_0^1 \theta^{\alpha-1}(1-\theta)^{-\alpha}\,d\theta = \mathbf{B}(\alpha, 1-\alpha),$$

where $\mathbf{B}(\cdot,\cdot)$ denotes the Beta function. This implies that $M(t)$ is constant function on $(a,b]$. Using the relation $\mathbf{B}(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$, we get

$$G_{1-\alpha}(t) = \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\mathbf{B}(\alpha,1-\alpha)\cdot C = C.$$

Since $C$ is a constant interval, it is easy to know that $w(G_{1-\alpha}(t))$ is a constant function on $(a,b]$. □

## 3 The Cauchy problem for interval-valued fractional differential equations

This section is devoted to deriving the relationship between the solutions to the Cauchy type problem for interval-valued differential equations of fractional order and the solutions to the corresponding interval-valued integral equation.

Let $F \in L([a,b] \times \mathcal{K}, \mathcal{K})$. Consider the following interval-valued fractional differential equation of order $\alpha \in (0,1]$:

$$\mathcal{D}^{\alpha}_{a+} Y(t) = F\bigl(t, Y(t)\bigr), \quad t \in T, \tag{1}$$

with the initial condition

$$\begin{aligned} &\mathcal{D}^{\alpha-1}_{a+} Y(a+) = \lim_{t\to a+} \mathcal{J}^{1-\alpha}_{a+} Y(t) = B \in \mathcal{K} \quad (0 < \alpha < 1); \\ &\mathcal{D}^{0}_{a+} Y(a+) = Y(a) = B \in \mathcal{K} \quad (\alpha = 1). \end{aligned} \tag{2}$$

In particular, if $\alpha = 1$, then, according to Definition 2.4 and (2), the problem in (1)-(2) is reduced to the Cauchy problem for the interval-valued differential equation:

$$Y'(t) = F\bigl(t, Y(t)\bigr), \qquad Y(a) = B \in \mathcal{K}. \tag{3}$$

Therefore, the problem (1)-(2) is referred to as a *Cauchy type problem* for the interval-valued fractional differential equation.

First we introduce the following interval-valued integral equation in order to discuss the solution of the Cauchy type problem (1)-(2):

$$Y(t) \ominus_g \frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)} B = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} F\bigl(s, Y(s)\bigr)\, ds. \tag{4}$$

**Theorem 3.1** *Let $G$ be an open set in $\mathcal{K}$ and let $F : [a,b] \times G \to \mathcal{K}$ be an interval-valued function such that $F(t, Y(t)) \in L([a,b], \mathcal{K})$ for any $Y \in G$. If $Y(t) \in L([a,b], \mathcal{K})$ satisfies a.e. the relations* (1) *and* (2) (*i.e.*, *$Y(t)$ is a solution of the problem* (1)-(2)), *and it satisfies either $\frac{d}{dt} w(Y_{1-\alpha}(t)) \geq 0$ for a.e. $t \in [a,b]$ or $\frac{d}{dt} w(Y_{1-\alpha}(t)) \leq 0$ for a.e. $t \in [a,b]$, then $Y(t)$ is also a solution of the integral equation* (4).

*Proof* Suppose that $Y(t)$ satisfies a.e. equations (1) and (2). Since $F(t, Y) \in L([a,b], \mathcal{K})$, it follows that the interval-valued fractional gH-derivative $\mathcal{D}^{\alpha}_{a+} Y(t) \in L([a,b], \mathcal{K})$ exists a.e. on $[a,b]$. By Definition 2.4, we have

$$\mathcal{D}^{\alpha}_{a+} Y(t) = \bigl(\mathcal{J}^{1-\alpha}_{a+} Y\bigr)'(t), \qquad \mathcal{J}^{0}_{a+} Y(t) = Y(t).$$

Then, by Lemma 2.3, we obtain $\mathcal{J}^{1-\alpha}_{a+} Y(t) = Y_{1-\alpha}(t) \in AC([a,b], \mathcal{K})$. Using Theorem 7 in [17], we can infer that

$$\mathcal{J}^{\alpha}_{a+} \mathcal{D}^{\alpha}_{a+} Y(t) = Y(t) \ominus_g \frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)} Y_{1-\alpha}(a) \tag{5}$$

for a.e. $t \in [a,b]$. According to Definition 2.4, we know that

$$Y_{1-\alpha}(t) = \mathcal{J}^{1-\alpha}_{a+} Y(t) = \mathcal{D}^{\alpha-1}_{a+} Y(t).$$

Using the above equality and (2), equation (5) can be rewritten as follows:

$$\mathcal{J}^{\alpha}_{a+} \mathcal{D}^{\alpha}_{a+} Y(t) = Y(t) \ominus_g \frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)} B. \tag{6}$$

Applying the operator $\mathcal{J}_{a+}^{\alpha}$ to both sides of (1) and using Definition 2.3 and (6), we can obtain the integral equation (4). □

**Theorem 3.2** *Let $G$ be an open set in $\mathcal{K}$ and let $F:[a,b]\times G\to\mathcal{K}$ be an interval-valued function such that $F(t,Y(t))\in L([a,b],\mathcal{K})$ for any $Y\in G$. Assume that $Y(t)\in L([a,b],\mathcal{K})$ is w-monotonic and satisfies a.e. the interval-valued integral equation* (4) *with $w(Y(t)) - \frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)}w(B)$ has a constant sign on $[a,b]$. If $Y_{1-\alpha}(t)$ is w-monotonic, then $Y(t)$ is also a solution of the problem* (1)-(2).

*Proof* Assume that $Y(t)\in L([a,b],\mathcal{K})$ satisfies the integral equation (4). Applying the operator $\mathcal{D}_{a+}^{\alpha}$ to both sides of (4) gives

$$\mathcal{D}_{a+}^{\alpha}\left(Y(t)\ominus_g\frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)}B\right)=\mathcal{D}_{a+}^{\alpha}\mathcal{J}_{a+}^{\alpha}F\bigl(t,Y(t)\bigr).$$

According to Theorems 4 and 5 in [17] and using Lemma 2.5, we can obtain the following two possible equalities:

$$\mathcal{D}_{a+}^{\alpha}Y(t)\ominus_g\mathcal{D}_{a+}^{\alpha}\left(\frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)}B\right)=F\bigl(t,Y(t)\bigr)$$

or

$$\mathcal{D}_{a+}^{\alpha}Y(t)+\left(-\mathcal{D}_{a+}^{\alpha}\left(\frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)}B\right)\right)=F\bigl(t,Y(t)\bigr).$$

Further, in accordance with (2.1.21) in [4] and Lemma 2.4, the second term on the left side of the previous two equalities are equal to $\{0\}$. Thus, we arrive at the equation (1).

Now we show that equation (2) also holds. Applying the operator $\mathcal{J}_{a+}^{1-\alpha}$ to both sides of (4) gives

$$\mathcal{J}_{a+}^{1-\alpha}\left(Y(t)\ominus_g\frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)}B\right)=\mathcal{J}_{a+}^{1-\alpha}\mathcal{J}_{a+}^{\alpha}F\bigl(t,Y(t)\bigr).$$

By Theorem 1 and Remark 5 in [17], we can infer that

$$\mathcal{J}_{a+}^{1-\alpha}Y(t)\ominus_g\mathcal{J}_{a+}^{1-\alpha}\left(\frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)}B\right)=\mathcal{J}_{a+}^{1}F\bigl(t,Y(t)\bigr).$$

Using Lemma 2.4 and Definition 2.3, it follows that

$$\mathcal{J}_{a+}^{1-\alpha}Y(t)\ominus_g\frac{1}{\Gamma(\alpha)}\bigl(I_{a+}^{1-\alpha}(t-a)^{\alpha-1}\bigr)\cdot B=\int_a^t F\bigl(s,Y(s)\bigr)\,ds.$$

In accordance with (2.1.16) in [4], we get

$$\mathcal{J}_{a+}^{1-\alpha}Y(t)\ominus_g B=\int_a^t F\bigl(s,Y(s)\bigr)\,ds.$$

Taking the limit as $t\to a+$, we have $\mathcal{D}_{a+}^{\alpha-1}Y(a+)\ominus_g B=\{0\}$, and hence equation (2) holds. □

**Remark 1** Theorems 3.1 and 3.2 show that, in general, the Cauchy type problem (1)-(2) for interval-valued fractional differential equations and the corresponding interval-valued integral equation (4) are not equivalent in the sense that, if $Y(t) \in L([a,b],\mathcal{K})$ satisfies one of these relations, then it also satisfies the other, unless the solutions satisfy some strict conditions.

Next we shall establish an important result related to the existence of a solution to the interval-valued integral equation (4), and then we can obtain the existence of a solution to the Cauchy type problem (1)-(2) under certain conditions.

**Theorem 3.3** *Let $G$ be an open set in $\mathcal{K}$ and let $F : [a,b] \times G \to \mathcal{K}$ be an interval-valued function such that $F(t,Y) \in L([a,b],\mathcal{K})$ for any $Y \in G$. Let $M > 0$ such that $w(F(t,Y(t))) \leq M$ for any $t \in [a,b]$. Assume that $F$ satisfies the Lipschitz condition*

$$H\bigl(F(t,Y_1),F(t,Y_2)\bigr) \leq LH(Y_1,Y_2) \quad (L > 0) \tag{7}$$

*for all $t \in [a,b]$ and all $Y_1, Y_2 \in L([a,b],\mathcal{K})$. Then there exist two unique solutions $\widetilde{Y}$, $\widehat{Y}$ to the interval-valued integral equation* (4) *in the space $\mathcal{L}^\alpha([a,t^*],\mathcal{K})$, where*

$$\mathcal{L}^\alpha\bigl(\bigl[a,t^*\bigr],\mathcal{K}\bigr) = \bigl\{Y \in L\bigl(\bigl[a,t^*\bigr],\mathcal{K}\bigr) | \mathcal{D}^\alpha_{a+}Y \in L\bigl(\bigl[a,t^*\bigr],\mathcal{K}\bigr)\bigr\}$$

$t^* = \min\{t_2, \frac{\alpha}{M}w(B), b\}$ *if* $w(B) > 0$ *and* $t^* = \min\{t_2, b\}$ *if* $w(B) = 0$, *while* $t_2 \in (a,b]$ *is chosen such that* $L\frac{(t_2-a)^\alpha}{\Gamma(\alpha+1)} < 1$.

*Proof* In essence, it follows from Definition 2.1 that the two cases of the existence of gH-difference imply that the interval-valued integral equation (4) is a unified formulation for one of the following integral equations:

$$Y(t) \ominus \frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)}B = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}F\bigl(s,Y(s)\bigr)\,ds \tag{8}$$

and

$$\frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)}B \ominus Y(t) = -\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}F\bigl(s,Y(s)\bigr)\,ds. \tag{9}$$

Setting $Y_0(t) = \frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)}B$. If $w(B) = 0$, then $Y_0(t)$ is an ordinary real valued function. This implies that the integral equation (9) is a classical single-valued integral equation. In this case, the integral equations (8) and (9) are identical. Without loss of generality, we assume that $w(B) > 0$. First, we choose $t_1 = \min\{a + \frac{\alpha}{M}w(B), b\}$, then we can obtain $0 \leq t-a \leq \frac{\alpha}{M}w(B)$ for any $t \in [a,t_1]$. Further, we can infer that

$$(t-a)^{\alpha-1}w(B) \geq \frac{M(t-a)^\alpha}{\alpha} \tag{10}$$

for any $t \in [a, t_1]$. Therefore, it follows from (10) that

$$\begin{aligned} & w\left(-\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}F\big(s, Y(s)\big)\,ds\right) \\ &\quad = \frac{1}{\Gamma(\alpha)} w\left(\int_a^t (t-s)^{\alpha-1}F\big(s, Y(s)\big)\,ds\right) \\ &\quad \le \frac{M}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\,ds \\ &\quad = \frac{1}{\Gamma(\alpha)}\frac{M(t-a)^{\alpha}}{\alpha} \\ &\quad \le \frac{1}{\Gamma(\alpha)}(t-a)^{\alpha-1}w(B) \\ &\quad = w\big(Y_0(t)\big). \end{aligned} \tag{11}$$

Next, we choose $t_2 \in (a, b]$ such that the inequality

$$L\frac{(t_2-a)^{\alpha}}{\Gamma(\alpha+1)} < 1 \tag{12}$$

holds. Take $t^* = \min\{t_1, t_2\}$. Then we shall prove the existence of two unique solutions $\widetilde{Y}, \widehat{Y} \in L([a, b], \mathcal{K})$ to the integral equations (8) and (9) on the interval $[a, t^*]$, respectively.

Using the form of the integral equation (8) and (9), we define the operators $P, Q : L([a, t^*], \mathcal{K}) \to L([a, t^*], \mathcal{K})$, where

$$(PY)(t) = Y_0(t) + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}F\big(s, Y(s)\big)\,ds \tag{13}$$

and

$$(QY)(t) = Y_0(t) \ominus \left(-\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}F\big(s, Y(s)\big)\,ds\right). \tag{14}$$

Notice that the operator $P$ is always well defined, while the operator $Q$ is also well defined by the inequality (11).

To apply the Banach contraction principle, we have to prove the following:

(i) if $Y \in L([a, t^*], \mathcal{K})$, then $(PY)(t), (QY)(t) \in L([a, t^*], \mathcal{K})$;

(ii) for any $Y_1, Y_2 \in L([a, t^*], \mathcal{K})$, the following estimates hold:

$$\mathcal{H}_1(PY_1, PY_2) \le \omega\mathcal{H}_1(Y_1, Y_2), \tag{15}$$

$$\mathcal{H}_1(QY_1, QY_2) \le \omega\mathcal{H}_1(Y_1, Y_2), \tag{16}$$

where $\omega = L\frac{(t^*-a)^{\alpha}}{\Gamma(\alpha+1)}$.

Obviously, $Y_0(t) \in L([a, t^*], \mathcal{K})$. By Lemma 2.2, we know that the interval-valued integrals in the right-hand side of (13) and (14) belong to $L([a, t^*], \mathcal{K})$, since $F(t, Y(t)) \in L([a, t^*], \mathcal{K})$.

By the Lipschitz condition (7) and Lemma 3.1 in [4], we can from equations (13) and (14) infer that

$$\begin{aligned}
&\mathcal{H}_1(PY_1, PY_2) \\
&\quad= \int_a^{t^*} H\biggl(Y_0(t) + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} F\bigl(s, Y_1(s)\bigr)\,ds, \\
&\qquad Y_0(t) + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} F\bigl(s, Y_2(s)\bigr)\,ds\biggr)\,dt \\
&\quad= \int_a^{t^*} H\biggl(\frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} F\bigl(s, Y_1(s)\bigr)\,ds, \\
&\qquad \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} F\bigl(s, Y_2(s)\bigr)\,ds\biggr)\,dt \\
&\quad= \int_a^{t^*} H\bigl(\mathcal{J}_{a+}^{\alpha} F\bigl(t, Y_1(t)\bigr), \mathcal{J}_{a+}^{\alpha} F\bigl(t, Y_2(t)\bigr)\bigr)\,dt \\
&\quad\le \int_a^{t^*} I_{a+}^{\alpha} H\bigl(F\bigl(t, Y_1(t)\bigr), F\bigl(t, Y_2(t)\bigr)\bigr)\,dt \\
&\quad\le \int_a^{t^*} L I_{a+}^{\alpha} H\bigl(Y_1(t), Y_2(t)\bigr)\,dt \\
&\quad\le L\frac{(t^*-a)^{\alpha}}{\Gamma(\alpha+1)} \int_a^{t^*} H\bigl(Y_1(t), Y_2(t)\bigr)\,dt \\
&\quad= L\frac{(t^*-a)^{\alpha}}{\Gamma(\alpha+1)} \mathcal{H}_1(Y_1, Y_2).
\end{aligned}$$

Using the same argument, we can obtain

$$\mathcal{H}_1(QY_1, QY_2) \le L\frac{(t^*-a)^{\alpha}}{\Gamma(\alpha+1)} \mathcal{H}_1(Y_1, Y_2).$$

Therefore, the inequalities (15) and (16) hold. In accordance with (12), we know that $0 < \omega < 1$. So the Banach contraction principle implies that there exist a unique solution $\widetilde{Y} \in L([a,t^*], \mathcal{K})$ and a unique solution $\widehat{Y} \in L([a,t^*], \mathcal{K})$ to the integral equations (8) and (9), respectively. This completes the proof of the theorem. □

**Remark 2** Actually, we can apply the method of successive approximations to obtain a unique solution $\widetilde{Y}(t)$ and $\widehat{Y}(t)$ to the integral equations (8) and (9) on the interval $[a,t^*]$, respectively. According to the Banach contraction principle, the solutions $\widetilde{Y}$ and $\widehat{Y}$ can be obtained as a limit of the convergent sequences $P^m Y_0^*$ and $Q^m Y_0^*$, respectively. Specifically, we have

$$\lim_{m\to\infty} \mathcal{H}_1\bigl(P^m Y_0^*, \widetilde{Y}\bigr) = 0, \tag{17}$$

$$\lim_{m\to\infty} \mathcal{H}_1\bigl(Q^m Y_0^*, \widehat{Y}\bigr) = 0, \tag{18}$$

where $Y_0^*$ is any interval-valued function in $L([a,t^*],\mathcal{K})$. By equations (13) and (14), the iterative sequences $\{(P^m Y_0^*)(t)\}$ and $\{(Q^m Y_0^*)(t)\}$ are defined by

$$\big(P^m Y_0^*\big)(t) = Y_0(t) + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} F\big(s, \big(P^{m-1} Y_0^*\big)(s)\big)\, ds, \quad m \in \mathbb{N},$$

$$\big(Q^m Y_0^*\big)(t) = Y_0(t) \ominus \left(-\frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} F\big(s, \big(Q^{m-1} Y_0^*\big)(s)\big)\, ds\right), \quad m \in \mathbb{N}.$$

Combining Theorem 3.2 with Theorem 3.3, we can formulate the following result associated with the existence of the solutions to the Cauchy type problem (1)-(2).

**Theorem 3.4** *Let $G$ be an open set in $\mathcal{K}$ and let $F : [a,b] \times G \to \mathcal{K}$ be an interval-valued function such that $F(t, Y(t)) \in L([a,b],\mathcal{K})$ for any $Y \in L([a,b],G)$. Let $M > 0$ such that $w(F(t,Y(t))) \le M$ for any $t \in [a,b]$. Assume that $F$ satisfies the Lipschitz condition* (7) *for all $t \in [a,b]$ and all $Y_1, Y_2 \in L([a,b],\mathcal{K})$. Then there exist two unique solutions $\widetilde{Y}$, $\widehat{Y}$ to the interval-valued integral equation* (4) *in the space $\mathcal{L}^\alpha([a,t^*],\mathcal{K})$.*

*Furthermore, if $Y(t) \in L([a,b],\mathcal{K})$ is $w$-monotonic and $w(\widetilde{Y}(t)) - w(B)$ and $w(\widehat{Y}(t)) - w(B)$ has a constant sign on $[a,t^*]$, $\widetilde{Y}_{1-\alpha}(t)$ and $\widehat{Y}_{1-\alpha}(t)$ are $w$-monotonic, then $\widehat{Y}(t)$ and $\widehat{Y}(t)$ are also two unique solutions to the Cauchy type problem* (1)-(2), *where $t^*$ is given as in Theorem* 3.3.

**Remark 3** According to Lemma 4.1 in [7], when $0 < \alpha \le 1$, the result of Theorem 3.4 remains true for the following weighted Cauchy type problem:

$$\mathcal{D}_{a+}^\alpha Y(t) = F\big(t, Y(t)\big), \qquad \lim_{t \to a+} (t-a)^{1-\alpha} Y(t) = B \in \mathcal{K}.$$

## 4 Conclusions

Usually, the existence of the solutions to the Cauchy problem (or initial value problem) for a differential equation is characterized by the existence of the solutions to the equivalent integral equation. Accordingly, it also becomes a fundamental way to construct the successive approximation sequence by means of the integral equation. In this paper, we note that, in general, the Cauchy type problem for interval-valued fractional differential equations and the corresponding integral equation are not equivalent. However, under certain conditions, we have derived the relationship between the solutions to the Cauchy type problem and the ones to the interval-valued integral equation. Therefore, these results provide the possibility for us to solve the Cauchy type problem for interval-valued fractional equations by the corresponding integral equation.

**Competing interests**
The author declares that he has no competing interests.

**Author's contributions**
YS drafted the manuscript and completed all proofs of the results in this paper.

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