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Positive solutions of a discrete nonlinear third-order three-point eigenvalue problem with sign-changing Green's function

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Abstract

In this paper, by using the Krasnosel'skii fixed point theorem in a cone, we discuss the existence of positive solutions to the discrete third-order three-point boundary value problem

$$\begin{cases} \Delta^3 u(t-1) = \lambda a(t) f(t, u(t)), & t \in [1, T-2]_{\mathbb{Z}}, \\ \Delta u(0) = u(T) = \Delta^2 u(\eta) = 0, \end{cases}$$

where $T > 4$ is an integer, $[1, T-2]_{\mathbb{Z}} = \{1, 2, \ldots, T-2\}$, $\lambda > 0$ is a parameter, $\eta \in \{\frac{T-1}{2}, \ldots, T-2\}$ for odd T, and $\eta \in \{\frac{T-2}{2}, \ldots, T-2\}$ for even T. Despite the sign-changing Green's function, we also give the explicit interval for λ to guarantee the existence of positive solutions of the problem when f satisfies different growth assumptions.

MSC: 39A10; 39A12

Keywords: third-order difference equation; three-point eigenvalue problem; sign-changing Green's function; positive solutions; Krasnosel'skii's fixed point theorem

1 Introduction

Let $[a, b]_{\mathbb{Z}}$ denote the integer set $\{a, a+1, \ldots, b\}$ with $b > a$. In this paper, we consider the existence of positive solutions for the discrete nonlinear third-order three-point BVP

$$\begin{cases} \Delta^3 u(t-1) = \lambda a(t) f(t, u(t)), & t \in [1, T-2]_{\mathbb{Z}}, \\ \Delta u(0) = u(T) = \Delta^2 u(\eta) = 0, \end{cases} \tag{1.1}$$

where $T > 4$ is an integer, $\lambda > 0$ is a parameter, $f : [1, T-2]_{\mathbb{Z}} \times [0, +\infty) \to [0, +\infty)$ is continuous, $a : [1, T-2]_{\mathbb{Z}} \to (0, \infty)$, and η satisfies the condition:

(H_0) $\eta \in [\frac{T-1}{2}, T-2]_{\mathbb{Z}}$, if T is an odd number, or $\eta \in [\frac{T-2}{2}, T-2]_{\mathbb{Z}}$, if T is an even number.

Boundary value problems for third-order differential or difference equations arise in many problems of physics, control system and applied mathematics, such as the deflection of a curved beam having a constant or varying cross section, three-layer beam and the electromagnetic wave incident on a system of charges sets them into motion, *etc.* [1]. In recent years, the existence of positive solutions of third-order boundary value problems has been discussed by several authors. For example, in [2–10], by using different methods, such as the Krasnosel'skii's fixed point theorem in a cone, the iterative technique, and the fixed point theory, the authors obtained the existence of positive solutions of the boundary value problems for third-order differential equations. For the discrete case, there are also several excellent results on the existence of positive solutions of the discrete third-order boundary value problems; see, for instance, [11–16] and the references therein. Specially, in [12], Agarwal and Henderson considered the following discrete third-order nonlinear eigenvalue problems:

$$\begin{cases} \Delta^3 u(t) = \lambda a(t) f(t, u(t)), & t \in [2, T]_{\mathbb{Z}}, \\ u(0) = u(1) = u(T+3) = 0. \end{cases} \tag{1.2}$$

By using the Krasnosel'skii fixed point theorem in a cone, they obtained the existence of positive solutions of (1.2) under both the case that $\lambda = 1$ and that $\lambda \neq 1$. Later, by using the Krasnosel'skii fixed point theorem, Anderson [2] obtained the existence of positive solutions for a kind of discrete nonlinear third-order eigenvalue problems, Ji and Yang [13] discussed the existence of positive solutions of a discrete third-order three-point right-focal boundary value problems, Karaca [14] discussed the existence of positive solutions under more general boundary conditions, and Kong *et al.* [15] obtained the existence of positive solutions for a discrete boundary value problem of a third-order functional difference equation. It can be seen that, in these papers, the key condition for obtaining the existence of positive solution is that the Green's functions is positive. This condition guarantees the positivity of corresponding summation operator. Now, the question is: when the Green's function changes its sign, how can we guarantee the positivity of the corresponding summation operator? In 2015, Wang and Gao [16] discussed the existence of positive solutions of the following third-order difference equation boundary value problems:

$$\begin{cases} \Delta^3 u(t-1) = a(t) f(t, u(t)), & t \in [1, T-1]_{\mathbb{Z}}, \\ u(0) = \Delta u(T) = \Delta^2 u(\eta) = 0. \end{cases} \tag{1.3}$$

It is worth to notice that the Green's function changes its sign in this paper. Inspired by the work of the above papers, we try to establish some criteria for the existence of positive solutions of (1.1) in this paper. The Green's function we construct in Section 2 changes its sign and is more complicated than in the continuous case. This will takes lots of difficulties for us to obtain the existence of positive solutions. To overcome it, a new cone is introduced to overcome these obstacles. Meanwhile, we will point out that the condition (H_0) is optimal for η to obtain the existence of a positive solution of (1.1); see Remark 2.4 and Remark 3.6.

The main tool we will use is the following fixed point theorem in a cone. See, for example [17] and [18].

Theorem 1.1 *Let E be a Banach space and K a cone in E. Assume that Ω_1 and Ω_2 are bounded open subsets of E such that $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and $A : K \cap (\overline{\Omega}_2 \backslash \Omega_1) \to K$ is a completely continuous operator such that either*

(i) *$\|Au\| \leq \|u\|$ for $u \in K \cap \partial\Omega_1$ and $\|Au\| \geq \|u\|$ for $u \in K \cap \partial\Omega_2$, or*

(ii) $\|Au\| \geq \|u\|$ *for* $u \in K \cap \partial\Omega_1$ *and* $\|Au\| \leq \|u\|$ *for* $u \in K \cap \partial\Omega_2$.
Then A *has a fixed point in* $K \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

2 Preliminaries

First, let us consider the following linear problem:

$$\begin{cases} \Delta^3 u(t-1) = y(t), & t \in [1, T-2]_{\mathbb{Z}}, \\ \Delta u(0) = u(T) = \Delta^2 u(\eta) = 0. \end{cases} \tag{2.1}$$

Define the Green's function $G(t,s)$ as follows.

If $(t,s) \in [2,T]_{\mathbb{Z}} \times [\eta+1, T-2]_{\mathbb{Z}}$, then

$$G(t,s) = \begin{cases} \frac{-(T-s)(T-s-1)}{2}, & 0 \leq t-2 < s \leq T-2, \\ \frac{(t-T)(t+T-1-2s)}{2}, & \eta < s \leq t-2 \leq T-2. \end{cases} \tag{2.2}$$

If $(t,s) \in [2,T]_{\mathbb{Z}} \times [1,\eta]_{\mathbb{Z}}$, then

$$G(t,s) = \begin{cases} \frac{t-t^2-s^2-s+2sT}{2}, & 0 \leq t-2 < s \leq \eta, \\ s(T-t), & 1 \leq s \leq t-2 \leq T-2. \end{cases} \tag{2.3}$$

Meanwhile,

$$G(0,s) = G(1,s) = \begin{cases} \frac{-(T-s)(T-s-1)}{2}, & \eta < s \leq T-2, \\ \frac{2sT-s-s^2}{2}, & 1 \leq s \leq \eta. \end{cases} \tag{2.4}$$

If $\eta = T-2$, then the Green's function $G(t,s)$ is defined only by (2.3) and (2.4).

In the rest of this paper, we always suppose that for two integers a, b with $a < b$ and a function f defined on $[a,b]_{\mathbb{Z}}$, $\sum_{t=b}^{a} f(t) = 0$.

Lemma 2.1 *The problem* (2.1) *has a unique solution*

$$u(t) = \sum_{s=1}^{T-2} G(t,s)y(s), \tag{2.5}$$

where $G(t,s)$ *is defined in* (2.2), (2.3), *and* (2.4).

Proof Summing from $s = 1$ to $s = t-1$ at both sides of the equation in (2.1), then we get

$$\Delta^2 u(t-1) = \Delta^2 u(0) + \sum_{s=1}^{t-1} y(s).$$

Repeating the above process, we obtain

$$\Delta u(t-1) = (t-1)\Delta^2 u(0) + \sum_{s=1}^{t-2} (t-s-1)y(s).$$

Summing from $s = 1$ to $s = t$ at both sides of the above equation, we have

$$u(t) = u(0) + \frac{t(t-1)}{2}\Delta^2 u(0) + \sum_{s=1}^{t-2} \frac{(t-s)(t-s-1)}{2} y(s).$$

By using the boundary condition $\Delta u(0) = u(T) = \Delta^2 u(\eta) = 0$, we get

$$\begin{cases} \Delta^2 u(0) + \sum_{s=1}^{\eta} y(s) = 0, \\ u(0) = \sum_{s=1}^{\eta} \frac{T(T-1)}{2} y(s) - \sum_{s=1}^{T-2} \frac{(T-s)(T-s-1)}{2} y(s). \end{cases}$$

Therefore,

$$\begin{aligned} u(t) = {} & \sum_{s=1}^{\eta} \frac{T(T-1) - t(t-1)}{2} y(s) - \sum_{s=1}^{T-2} \frac{(T-s)(T-s-1)}{2} y(s) \\ & + \sum_{s=1}^{t-2} \frac{(t-s)(t-s-1)}{2} y(s). \end{aligned} \tag{2.6}$$

This implies that (2.5) holds. □

Lemma 2.2 *Suppose that* (H_0) *holds. Then the Green's function* $G(t,s)$ *has the following properties*:

(i) *If* $s \in [1,\eta]_{\mathbb{Z}}$, *then* $G(t,s)$ *is nonincreasing with respect to* $t \in [0,T]_{\mathbb{Z}}$.
If $s \in [\eta+1, T-2]_{\mathbb{Z}}$, *then* $G(t,s)$ *is nondecreasing with respect to* $t \in [0,T]_{\mathbb{Z}}$.

(ii) $G(t,s)$ *changes its sign on* $[0,T]_{\mathbb{Z}} \times [1,T-2]_{\mathbb{Z}}$. *In details, if* $(t,s) \in [0,T]_{\mathbb{Z}} \times [1,\eta]_{\mathbb{Z}}$, *then* $G(t,s) \geq 0$. *If* $(t,s) \in [0,T]_{\mathbb{Z}} \times [\eta+1,T-2]_{\mathbb{Z}}$, *then* $G(t,s) \leq 0$.

(iii) *If* $s > \eta$, *then* $\max_{t\in[0,T]_{\mathbb{Z}}} G(t,s) = G(T,s) = 0$ *and*

$$\min_{t\in[0,T]_{\mathbb{Z}}} G(t,s) = G(0,s) = \frac{-(T-s)(T-s-1)}{2} \geq \frac{-(T-\eta)(T-\eta-1)}{2}. \tag{2.7}$$

If $s \leq \eta$, *then* $\min_{t\in[0,T]_{\mathbb{Z}}} G(t,s) = G(T,s) = 0$ *and*

$$\max_{t\in[0,T]_{\mathbb{Z}}} G(t,s) = G(0,s) = \frac{-s^2 - s + 2sT}{2} \leq \frac{-\eta^2 - \eta + 2\eta T}{2}. \tag{2.8}$$

Proof (i) As we know, if $\Delta_t G(t,s) \geq 0$ (≤ 0), then $G(t,s)$ is nondecreasing (nonincreasing) with respect to t. Now, we discuss the sign of $\Delta_t G(t,s)$. From (2.4), we know that $\Delta_t G(0,s) = 0$ whenever $s \leq \eta$ or $s > \eta$. Moreover, by (2.2) and (2.3),

$$G(2,s) = \begin{cases} \frac{2sT-s^2-s-2}{2}, & s \leq \eta, \\ \frac{-(T-s)(T-s-1)}{2}, & s > \eta. \end{cases}$$

Then $\Delta_t G(1,s) = -1$ if $s \leq \eta$ and $\Delta_t G(1,s) = 0$ if $s > \eta$. If $t \geq 2$, then the proof will be divided into two cases.

Case 1. $(t,s) \in [2,T-1]_{\mathbb{Z}} \times [\eta+1,T-2]_{\mathbb{Z}}$. If $s > t-2$, then by (2.2), $\Delta_t G(t,s) = 0$. If $s \leq t-2$, then $\Delta_t G(t,s) = t-s > 0$. Consequently, $G(t,s)$ is nondecreasing with respect to $t \in [0,T]_{\mathbb{Z}}$.

Case 2. $(t,s) \in [2,T-1]_{\mathbb{Z}} \times [1,\eta]_{\mathbb{Z}}$. If $s > t-2$, then $\Delta_t G(t,s) = -t < 0$. If $s \leq t-2$, then $\Delta_t G(t,s) = -s < 0$. Therefore, $G(t,s)$ is nonincreasing with respect to $t \in [0,T]_{\mathbb{Z}}$.

(ii) These results hold from (i) and the boundary condition $u(T) = 0$.

(iii) If $s > \eta$, then, from (i), we know $G(t,s)$ is nondecreasing with respect to t, then we have $\max_{t\in[0,T]_{\mathbb{Z}}} G(t,s) = G(T,s) = 0$ and

$$\min_{t\in[0,T]_{\mathbb{Z}}} G(t,s) = G(0,s) = \frac{-(T-s)(T-s-1)}{2}.$$

Furthermore, the function $z(s) = -(T-s)(T-s-1)$ is increasing for $s < (2T-1)/2$. Combining this with $\eta \leq T-2$ and $(2T-1)/2 > s > T-2$, we get the inequality in (2.7).

Meanwhile, if $s \leq \eta$, $G(t,s)$ is nonincreasing with respect to t. Then $\min_{t\in[0,T]} G(t,s) = G(T,s) = 0$ and

$$\max_{t\in[0,T]_{\mathbb{Z}}} G(t,s) = G(0,s) = \frac{-s^2 - s + 2sT}{2}.$$

Moreover, since η satisfies (H_0) and $(2T-1)/2 < s < \eta$, we obtain that $G(0,s) \leq \frac{-\eta^2-\eta+2\eta T}{2}$. □

Remark 2.3 If $\eta = T-2$, then we will find that $G(t,s) \geq 0$ and $G(t,s) \not\equiv 0$. This case has been discussed by several authors; see, for instance, [11–15]. So, in the rest of this paper, we could suppose that $\eta < T-2$.

Remark 2.4 Before we consider the existence of positive solutions of (2.1), we may discuss the existence of positive solution of a more special problem

$$\begin{cases} \Delta^3 u(t-1) = 1, & t \in [1, T-2]_{\mathbb{Z}}, \\ \Delta u(0) = u(T) = \Delta^2 u(\eta) = 0. \end{cases} \tag{2.9}$$

We will see that (H_0) is a necessary and sufficient condition for the existence of positive solutions to (2.9). To some extent, this explains why we choose η which satisfies (H_0).

From Lemma 2.1, we know that (2.9) has a solution $u(t)$ as follows:

$$u(t) = \frac{t^3 - 3(1+\eta)t^2 + (3\eta+2)t - T^3 + 3(1+\eta)T^2 - (3\eta+2)T}{6}.$$

For the sake of convenience, let

$$\phi(t) = t^3 - 3(1+\eta)t^2 + (3\eta+2)t - T^3 + 3(1+\eta)T^2 - (3\eta+2)T.$$

Obviously, $u(t) \geq 0 \Leftrightarrow \phi(t) \geq 0$. By direct computation, we get $\Delta\phi(t) = t(3t-3-6\eta)$, $\Delta\phi(t) \geq 0$ for $t > 1+2\eta$ and $\Delta\phi(t) \leq 0$ for $0 < t \leq 1+2\eta$. Furthermore, if $1+2\eta$ is an integer, then $\Delta\phi(1+2\eta) = 0$. Now, we prove that (H_0) is a necessary and sufficient condition of $\phi(t) \geq 0$, $t \in [0,T]_{\mathbb{Z}}$. In fact, if $\phi(t) \geq 0$, then $\phi(T) = 0$ implies that $\phi(T-1) \geq 0$. If $\phi(T-1) > 0$, then $\Delta\phi(T-1) < 0$. This implies that $\eta > \frac{T-2}{2}$. If $\phi(T-1) = 0$, then $\phi(T-2) > 0$. Otherwise, $\phi(t) \equiv 0$, $t \in [0,T]_{\mathbb{Z}}$. This contradicts $\phi(t) \not\equiv 0$. Therefore, $\Delta\phi(T-2) < 0$ and $\Delta\phi(T-1) = 0$. These two equations imply that T is an even number and $\eta = \frac{T-2}{2}$. Conversely, if $\eta \geq \frac{T-2}{2}$, then $T-1 \leq 1+2\eta$ and $\Delta\phi(T-1) \leq 0$. This combined with the condition $\phi(T) = 0$ implies that $\phi(t) \geq 0$, $t \in [0,T]_{\mathbb{Z}}$.

Let $E = \{u : [0,T]_{\mathbb{Z}} \to R \mid \Delta u(0) = u(T) = \Delta^2 u(\eta) = 0\}$. Then E is a Banach space under the norm $\|u\| = \max_{t\in[0,T]_{\mathbb{Z}}} |u(t)|$. Define a subset $\widehat{P} \subset E$ as follows:

$$\widehat{P} = \bigl\{y \in E : y(t) \ge 0, t \in [0,T]_{\mathbb{Z}}, \Delta y(t) \le 0, t \in [0,T-1]_{\mathbb{Z}}\bigr\}.$$

Then $\widehat{P}$ is a cone in E.

Lemma 2.5 *Assume that* (H_0) *holds. If* $y \in \widehat{P}$, *then the unique solution* $u(t)$ *of* (2.1) *belongs to* $\widehat{P}$, *where* $u(t)$ *is defined as* (2.5). *Moreover,* $u(t)$ *is concave on* $[0,\eta+2]_{\mathbb{Z}}$.

Proof First, if $0 \le t-2 \le \eta$, then

$$u(t) = \sum_{s=1}^{t-2} s(T-t)y(s) + \sum_{s=t-1}^{\eta} \frac{t-t^2-s^2-s+2sT}{2} y(s) - \sum_{s=\eta+1}^{T-2} \frac{(T-s)(T-s-1)}{2} y(s),$$

$$\Delta u(t) = -\sum_{s=1}^{t-2} sy(s) + (1-t)y(t-1) - \sum_{s=t}^{\eta} ty(s) \le 0, \tag{2.10}$$

and

$$\Delta^2 u(t-1) = -\sum_{s=t}^{\eta} y(s) \le 0. \tag{2.11}$$

Second, if $\eta < t-2 \le T-2$, then

$$u(t) = \sum_{s=1}^{\eta} s(T-t)y(s) + \sum_{s=\eta+1}^{t-2} \frac{(t-T)(t+T-1-2s)}{2} y(s) - \sum_{s=t-1}^{T-2} \frac{(T-s)(T-s-1)}{2} y(s).$$

Since η satisfies (H_0), we obtain

$$\Delta u(t) = -\sum_{s=1}^{\eta} sy(s) + \sum_{s=\eta+1}^{t-2} (t-s)y(s) + y(t-1) \le y(\eta)\frac{t(t-1-2\eta)}{2} \le 0 \tag{2.12}$$

and

$$\Delta^2 u(t-1) = \sum_{s=\eta+1}^{t-3} y(s) + y(t-1) + y(t-2) \ge 0.$$

By (2.10) and (2.12), we get $\Delta u(t) \le 0$ for all $t \in [0,T-1]_{\mathbb{Z}}$. Combining this with the boundary condition $u(T) = 0$, we get $u(t) \ge 0$ for $t \in [0,T]_{\mathbb{Z}}$, which implies $u \in \widehat{P}$. Moreover, by (2.11) and the condition $\Delta^2 u(\eta) = 0$, we get $\Delta^2 u(t-1) \le 0$ for $t \in [1,\eta+1]_{\mathbb{Z}}$. Therefore, $u(t)$ is concave on $[0,\eta+2]_{\mathbb{Z}}$. □

Lemma 2.6 *Suppose that* (H_0) *holds. If* $y \in \widehat{P}$, *then the unique solution* $u(t)$, *defined in* (2.5), *satisfies the following inequality*:

$$\min_{t\in[T-\theta,\theta]_{\mathbb{Z}}} u(t) \ge \theta^* \|u\|, \tag{2.13}$$

where $\theta \in [\frac{T+1}{2}, \eta+1]_{\mathbb{Z}}$ *for odd* T *and* $\theta \in [\frac{T}{2}, \eta+1]_{\mathbb{Z}}$ *for even* T. *Moreover,* $\theta^* = \frac{\eta+2-\theta}{\eta+2}$.

Proof By Lemma 2.5, $u(t)$ is concave for $t \in [0, \eta + 2]_{\mathbb{Z}}$. Then

$$\frac{u(t) - u(0)}{t} \geq \frac{u(\eta + 2) - u(0)}{\eta + 2}, \quad t \in [0, \eta + 1]_{\mathbb{Z}}. \tag{2.14}$$

From Lemma 2.5, we see that $u(t)$ is nonincreasing for $t \in [0, T]_{\mathbb{Z}}$, which implies that $u(0) = \|u\|$. Combining this with (2.14), we obtain

$$u(t) \geq \frac{\eta + 2 - t}{\eta + 2} u(0) = \frac{\eta + 2 - t}{\eta + 2} \|u\|, \quad t \in [0, \eta + 1]_{\mathbb{Z}}.$$

This implies

$$\min_{t \in [T-\theta, \theta]_{\mathbb{Z}}} u(t) = u(\theta) \geq \frac{\eta + 2 - \theta}{\eta + 2} \|u\| = \theta^* \|u\|. \qquad \square$$

3 Existence results

In this section, we are concerned with the existence of at least one positive solution of the problem (1.1). Assume that

(H_1) $f : [1, T - 2]_{\mathbb{Z}} \times [0, \infty) \to [0, \infty)$ is continuous, the mapping $t \to f(t, u)$ is decreasing for each $u \in [0, \infty)$ and the mapping $u \to f(t, u)$ is increasing for each $t \in [1, T - 2]_{\mathbb{Z}}$;
(H_2) $a : [1, T - 2]_{\mathbb{Z}} \to [0, \infty)$ is decreasing.

Define the cone K by

$$K = \Big\{ u \in \widehat{P} \Big| \min_{t \in [T-\theta, \theta]_{\mathbb{Z}}} u(t) \geq \theta^* \|u\| \Big\}.$$

Define the operator $T_\lambda : K \to E$ by

$$T_\lambda u(t) = \lambda \sum_{s=1}^{T-2} G(t, s) a(s) f\big(s, u(s)\big).$$

From Lemma 2.5 to Lemma 2.6, we know that $T_\lambda : K \to K$. Meanwhile, since E is finite dimensional, $T_\lambda : K \to K$ is completely continuous. Therefore, if u is a fixed point of T_λ in K, then u is positive solution of (1.1).

Since $-\eta^2 - \eta + 2\eta T > 0$, $G(\theta, s) \geq 0$, and $G(\theta, s) \not\equiv 0$ for $s \in [T - \theta, \theta]_{\mathbb{Z}}$, we could define two positive constants as follows:

$$A = \sum_{s=1}^{T-2} \frac{-\eta^2 - \eta + 2\eta T}{2} a(s), \qquad B = \sum_{s=T-\theta}^{\theta} G(\theta, s) a(s).$$

Lemma 3.1 *Suppose that* (H_0), (H_1), *and* (H_2) *hold. If there exist two positive constants r and R with* $r \neq R$ *such that*

(A_1) $f(t, u) \leq \frac{r}{\lambda A}$, $(t, u) \in [1, T - 2]_{\mathbb{Z}} \times [0, r]$;
(A_2) $f(t, u) \geq \frac{R}{\lambda B}$, $(t, u) \in [1, T - 2]_{\mathbb{Z}} \times [\theta^* R, R]$.

Then (1.1) *has at least one positive solution* $u \in K$ *with* $\min\{r, R\} \leq \|u\| \leq \max\{r, R\}$.

Proof We only deal with the case $r < R$, the case that $r > R$ could be treated similarly.

Let $\Omega_1 = \{u \in E : \|u\| < r\}$. From Lemma 2.2(ii), we know that $G(t,s) \leq 0$ for $s \in [\eta+1, T-2]_{\mathbb{Z}}$ and $G(t,s) \geq 0$ for $s \in [1,\eta]_{\mathbb{Z}}$. Then by ($A_1$), for $u \in K \cap \partial\Omega_1$,

$$\begin{aligned}
\|T_\lambda u\| &= \lambda \max_{t\in[0,T]_{\mathbb{Z}}} \left| \sum_{s=1}^{T-2} G(t,s)a(s)f\big(s,u(s)\big) \right| \\
&\leq \lambda \max_{t\in[0,T]_{\mathbb{Z}}} \left| \sum_{s=1}^{\eta} G(t,s)a(s)f\big(s,u(s)\big) \right| + \lambda \max_{t\in[0,T]_{\mathbb{Z}}} \left| \sum_{s=\eta+1}^{T-2} G(t,s)a(s)f\big(s,u(s)\big) \right| \\
&\leq \lambda \frac{-\eta^2-\eta+2\eta T}{2} \sum_{s=1}^{\eta} a(s)f\big(s,u(s)\big) + \lambda \frac{(T-\eta)(T-\eta-1)}{2} \sum_{s=\eta+1}^{T-2} a(s)f\big(s,u(s)\big).
\end{aligned}$$

Since η satisfies (H_0), we get $-\eta^2-\eta+2\eta T \geq (T-\eta)(T-\eta-1)$. Then

$$\|T_\lambda u\| \leq \lambda \frac{-\eta^2-\eta+2\eta T}{2} \sum_{s=1}^{T-2} a(s)f\big(s,u(s)\big) \leq \lambda \frac{-\eta^2-\eta+2\eta T}{2} \sum_{s=1}^{T-2} a(s)\frac{r}{\lambda A} = r.$$

Therefore,

$$\|T_\lambda u\| \leq \|u\|, \quad \text{for } u \in K \cap \partial\Omega_1. \tag{3.1}$$

Let $\Omega_2 = \{u \in E : \|u\| < R\}$. By (2.3) and Lemma 2.2, we know that $G(t,s) \geq 0$ for $t-2 < s \leq \eta$. Then for $u \in K$ and $t \in [T-\theta,\theta]_{\mathbb{Z}}$, we have

$$\begin{aligned}
&\sum_{s=1}^{T-\theta-1} G(t,s)a(s)f\big(s,u(s)\big) + \sum_{s=\theta+1}^{\eta} G(t,s)a(s)f\big(s,u(s)\big) + \sum_{s=\eta+1}^{T-2} G(t,s)a(s)f\big(s,u(s)\big) \\
&\quad\geq \sum_{s=1}^{T-\theta-1} s(T-t)a(s)f\big(s,u(s)\big) - \sum_{s=\eta+1}^{T-2} \frac{(T-s)(T-s-1)}{2} a(s)f\big(s,u(s)\big) \\
&\quad\geq a(\eta)f\big(\eta,u(\eta)\big)\left[\sum_{s=1}^{T-\theta-1} s(T-t) - \sum_{s=\theta+1}^{T-2} \frac{(T-s)(T-s-1)}{2}\right] \\
&\quad\geq a(\eta)f\big(\eta,u(\eta)\big)(T-\theta)(T-\theta-1) \geq 0.
\end{aligned} \tag{3.2}$$

Meanwhile, by (2.3) and Lemma 2.2, we get $G(\theta,s) \geq 0$ for $s \leq \theta \leq \eta$. Combining this with (3.2), we get

$$T_\lambda u(\theta) = \lambda \sum_{s=1}^{T-2} G(\theta,s)a(s)f\big(s,u(s)\big) \geq \lambda \sum_{s=T-\theta}^{\theta} G(\theta,s)a(s)f\big(s,u(s)\big). \tag{3.3}$$

Moreover, for $u \in K \cap \partial\Omega_2$, $\theta^* R \leq u(t) \leq R$ for $t \in [T-\theta,\theta]_{\mathbb{Z}}$ by Lemma 2.6. Combining this with (A_2), we have

$$T_\lambda u(\theta) \geq \lambda \sum_{s=T-\theta}^{\theta} G(\theta,s)a(s)\frac{R}{\lambda B} = R.$$

This combines with the fact that $T_\lambda u(t) \geq 0$ for $t \in [0, T]_{\mathbb{Z}}$, we have

$$\|T_\lambda u\| \geq \|u\|, \quad \text{for } u \in K \cap \partial\Omega_2. \tag{3.4}$$

Applying Theorem 1.1(i) to (3.1) and (3.4), we get T_λ has a fixed point $u \in K \cap (\overline{\Omega}_2 \backslash \Omega_1)$, and then u is a positive solution of (1.1) with $r \leq \|u\| \leq R$. □

Theorem 3.2 *Suppose that* (H_0), (H_1), *and* (H_2) *hold. Assume that*

$$(A_3) \quad f^0 := \lim_{u\to 0^+} \max_{t\in[1,T-2]_{\mathbb{Z}}} \frac{f(t,u)}{u} = 0,$$

$$f_\infty := \lim_{u\to \infty} \min_{t\in[1,T-2]_{\mathbb{Z}}} \frac{f(t,u)}{u} = \infty \quad (\textit{super linear case})$$

or

$$(A_4) \quad f_0 := \lim_{u\to 0^+} \min_{t\in[1,T-2]_{\mathbb{Z}}} \frac{f(t,u)}{u} = \infty,$$

$$f^\infty := \lim_{u\to \infty} \max_{t\in[1,T-2]_{\mathbb{Z}}} \frac{f(t,u)}{u} = 0 \quad (\textit{sublinear case}).$$

Then for any $\lambda \in (0, \infty)$, (1.1) *has at least one positive solution.*

Proof *Super linear case.* From (A_3), $f^0 = 0$, then there exists a constant $R_1 > 0$ such that

$$f(t,u) \leq \frac{R_1}{\lambda A}, \quad (t,u) \in [1, T-2]_{\mathbb{Z}} \times [0, R_1].$$

Furthermore, since $f_\infty = \infty$, there exist $R_2 > R_1$ such that

$$f(t,u) \geq \frac{u}{\theta^*\lambda B} \geq \frac{\theta^* R_2}{\theta^*\lambda B} = \frac{R_2}{\lambda B}, \quad (t,u) \in [1, T-2]_{\mathbb{Z}} \times \big[\theta^* R_2, R_2\big].$$

Now, by Lemma 3.1, (1.1) has a positive solution $u \in K$.

Sublinear case. Since $f_0 = \infty$, there exists $R_1' > 0$ such that

$$f(t,u) \geq \frac{u}{\theta^*\lambda B}, \quad (t,u) \in [1, T-2]_{\mathbb{Z}} \times \big[0, R_1'\big]. \tag{3.5}$$

Set $\Omega_1 = \{u \in E : \|u\| < R_1'\}$. Then, if $u \in K \cap \partial\Omega_1$,

$$\min_{s\in[T-\theta,\theta]_{\mathbb{Z}}} u(s) \geq \theta^* \|u\| = \theta^* R_1'. \tag{3.6}$$

Then, by (3.3)-(3.6) and the fact that $G(\theta, s) \geq 0$ for $s \in [T-\theta, \theta]_{\mathbb{Z}}$, we get

$$T_\lambda u(\theta) = \lambda \sum_{s=1}^{T-2} G(\theta,s)a(s)f\big(s,u(s)\big) \geq \lambda \sum_{s=T-\theta}^{\theta} G(\theta,s)a(s)\frac{u(s)}{\theta^*\lambda B} = R_1'.$$

This implies that

$$\|T_\lambda u\| \geq \|u\|, \quad \text{for } u \in K \cap \partial\Omega_1. \tag{3.7}$$

On the other hand, since $f^\infty = 0$, there exists $R_0 > 0$ such that

$$f(t,u) \leq \frac{u}{\lambda A}, \quad (t,u) \in [1, T-2]_{\mathbb{Z}} \times [R_0, \infty). \tag{3.8}$$

We consider two cases: f is bounded and f is unbounded. If f is bounded, *i.e.*, there exists a constant $M > 0$, such that $f \leq M$, then we take $R_2' = \max\{2R_1', \lambda MA\}$ and $\Omega_2 = \{u \in K : \|u\| < R_2'\}$. If f is unbounded, then we take $R_2' > \max\{2R_1', R_0\}$ such that $f(t,u) \leq f(t,R_2)$, for $(t,u) \in [1, T-2]_{\mathbb{Z}} \times [0, R_2]$ and $\Omega_2 = \{u \in K : \|u\| < R_2'\}$. Now, similar to the proof of (3.1), we will get

$$\|T_\lambda u\| \leq \|u\|, \quad \text{for } u \in K \cap \partial\Omega_2. \tag{3.9}$$

Therefore, by Theorem 1.1, we obtain a positive solution u of problem (1.1). □

Theorem 3.3 *Suppose that* (H_0), (H_1), *and* (H_2) *hold. If* $0 < Af^0 < \theta^* Bf_\infty < \infty$, *then for each* $\lambda \in (\frac{1}{\theta^* Bf_\infty}, \frac{1}{Af^0})$, (1.1) *has at least one positive solution.*

Proof For any $\lambda \in (\frac{1}{\theta^* Bf_\infty}, \frac{1}{Af^0})$, there exists $\varepsilon > 0$ such that

$$\frac{1}{\theta^* B(f_\infty - \varepsilon)} \leq \lambda \leq \frac{1}{A(f^0 + \varepsilon)}. \tag{3.10}$$

By the definition of f^0, there exists $R_3 > 0$ such that $f(t,u) \leq (f^0 + \varepsilon)u$, for $(t,u) \in [1, T-2]_{\mathbb{Z}} \times [0, R_3]$. Let $\Omega_1 = \{u \in K : \|u\| < R_1\}$, then similar to the proof of (3.1), if $u \in K \cap \partial\Omega_1$, then

$$\|T_\lambda u\| \leq \lambda \frac{-\eta^2 - \eta + 2\eta T}{2} \sum_{s=1}^{T-2} a(s)\bigl(f^0 + \varepsilon\bigr)\|u\|.$$

Furthermore, by (3.10), we get

$$\|T_\lambda u\| \leq \|u\|, \quad \text{for } u \in K \cap \partial\Omega_1. \tag{3.11}$$

On the other hand, by the definition of f_∞, there exists $\overline{R}_2$ such that $f(t,u) \geq (f_\infty - \varepsilon)u$, for $(t,u) \in [1, T-2]_{\mathbb{Z}} \times [\overline{R}_2, \infty)$. Let $R_2 = \max\{2R_1, \frac{\overline{R}_2}{\theta^*}\}$ and $\Omega_2 = \{u \in K : \|u\| < R_2\}$. If $u \in K$ with $\|u\| = R_2$, then $\min_{s \in [T-\theta,\theta]_{\mathbb{Z}}} u(s) \geq \theta^*\|u\| \geq \overline{R}_2$. Therefore, similar to the discussion from (3.2)-(3.4), we get

$$T_\lambda u(\theta) \geq \lambda \sum_{s=T-\theta}^{\theta} G(\theta,s)(f_\infty - \varepsilon)\theta^*\|u\|a(s) = \lambda\theta^* B(f_\infty - \varepsilon).$$

Combining this with (3.10), we get

$$\|T_\lambda u\| \geq \|u\|, \quad \text{for } u \in K \cap \partial\Omega_2. \tag{3.12}$$

By Theorem 1.1(i), (1.1) has at least one positive solution $u \in K$. □

Similar to the discussion of Lemma 3.1, Theorem 3.2, and Theorem 3.3, we could obtain the following two theorems. So, we just state them here without any proof.

Theorem 3.4 *Suppose that* (H_0), (H_1), *and* (H_2) *hold. If* $0 < Af^{\infty} < \theta^* Bf_0 < \infty$, *then for each* $\lambda \in (\frac{1}{\theta^* Bf_0}, \frac{1}{Af^{\infty}})$, (1.1) *has at least one positive solution.*

Theorem 3.5 *Suppose that* (H_0), (H_1), *and* (H_2) *hold. Then the following results hold.*

(1) *If* $f_\infty = \infty$, $0 < f^0 < \infty$ *then for each* $\lambda \in (0, \frac{1}{Af^0})$, (1.1) *has at least one positive solution.*
(2) *If* $f_0 = \infty$, $0 < f_\infty < \infty$ *then for each* $\lambda \in (0, \frac{1}{Af^{\infty}})$, (1.1) *has at least one positive solution.*
(3) *If* $f^0 = \infty$, $0 < f_\infty < \infty$ *then for each* $\lambda \in (\frac{1}{\theta^* Bf_\infty}, \infty)$, (1.1) *has at least one positive solution.*
(4) *If* $f^\infty = 0$, $0 < f_0 < \infty$ *then for each* $\lambda \in (\frac{1}{\theta^* Bf_0}, \infty)$, (1.1) *has at least one positive solution.*

Remark 3.6 The η we choose in this paper is optimal for obtaining the existence of positive solution. Otherwise, suppose that $\eta \in [1, \frac{T-2}{2} - 1]_{\mathbb{Z}}$ for even T and $\eta \in [1, [\frac{T-2}{2}]]_{\mathbb{Z}}$ for odd T. We will see that even the linear problem (2.9) will not have a positive solution. In fact, by the analysis in Remark 2.4, we know that if $\eta < \frac{T-2}{2}$, then $\Delta\phi(T-1) > 0$. Combining this with the boundary condition $\phi(T) = 0$, we get $\phi(T-1) < 0$. Therefore, $\phi(t)$ is not a positive solution of (2.9). Therefore, (H_0) is an optimal condition for obtaining the existence of a positive solution.

Example 3.7 Consider the following discrete nonlinear third-order three-point eigenvalue problem:

$$\begin{cases} \Delta^3 u(t-1) = \lambda a(t) f(t, u(t)), & t \in [1,5]_{\mathbb{Z}}, \\ \Delta u(0) = u(7) = \Delta^2 u(4) = 0, \end{cases} \tag{3.13}$$

where $a(t) = t^2 - 11t + 32$, f satisfies the assumption (H_1), according to the hypotheses in Theorem 3.2-Theorem 3.5, we will give $f(t, u(t))$ the specific forms. Then finding the positive solutions of the problem (3.13) is equivalent to finding the fixed point the operator equation

$$Tu(t) := \lambda \sum_{s=1}^{5} G(t,s) a(s) f\bigl(s, u(s)\bigr).$$

Since $s \in [1,5]_{\mathbb{Z}}$, if $s > 4$ then $s = 5$. Furthermore, we get the expression of $G(t,s)$ as follows.

(i) $(t,s) \in [2,7]_{\mathbb{Z}} \times \{5\}$. Since $s = 5$, we know that if $s \le t-2$, then $t = 7$ and $G(t,s) = G(7,5) = 0$ in the case that $s \le t-2$. Therefore, the expression of $G(t,s)$ is

$$G(t,s) = \begin{cases} 0, & s \le t-2, \\ -1, & s > t-2. \end{cases}$$

(ii) $(t,s) \in [2,7]_{\mathbb{Z}} \times [1,4]_{\mathbb{Z}}$. Then

$$G(t,s) = \begin{cases} s(7-t), & s \le t-2, \\ \frac{t-t^2-s^2+13s}{2}, & s > t-2. \end{cases}$$

(iii)

$$G(0,s)=G(1,s)=\begin{cases}\frac{13s-s^2}{2}, & s\leq 4,\\ -1, & s>4.\end{cases}$$

Obviously, $G(t,s)$ changes its sign on $[0,T]_{\mathbb{Z}}\times[1,T-2]_{\mathbb{Z}}$. On the other hand, by direct computation, if $s>4$, *i.e.*, $s=5$, then $G(t,s)$ has only two values -1 and 0, which implies that $G(t,s)\leq 0$. Meanwhile, $\Delta_t G(t,s)=G(t+1,s)-G(t,s)\geq 0$. We see that $G(t,s)$ is increasing with respect to t in this case. If $(t,s)\in[0,7]_{\mathbb{Z}}\times[1,4]_{\mathbb{Z}}$, $G(t,s)>0$, and then $\Delta_t G(t,s)=G(t+1,s)-G(t,s)\leq 0$. This implies $G(t,s)$ is decreasing with respect to t in this case. Therefore, if $s>4$, then $\max_{t\in[0,7]_{\mathbb{Z}}}G(t,s)=0$. If $s\leq 4$, $\max_{t\in[0,7]_{\mathbb{Z}}}G(t,s)=\frac{13s-s^2}{2}\leq G(0,4)=18$.

Now, according to Lemma 2.6, take $\theta=4$, then $\theta^*=1/5$ and

$$A=\sum_{s=1}^{4}\frac{-4^2-4+13\times 4}{2}(6-s)=224,\qquad B=9\times\sum_{s=3}^{4}(6-s)=45.$$

If $f(t,s)=(7-t)h_1(s)$, $h_1(s)=\frac{u^3+2u^2+1}{5u+4}$, then f satisfies (H_1), $f^0=0$, and $f_\infty=\infty$. By Theorem 3.2, for $\lambda\in(0,\infty)$, (3.13) has at least one positive solution. If $f(t,s)=(7-t)h_2(s)$, $h_2(s)=\sqrt{1+s}$, then f satisfies (H_1), $f_0=\infty$, and $f^\infty=0$. By Theorem 3.2, for $\lambda\in(0,\infty)$, (3.13) has at least one positive solution.

If $f(t,s)=(7-t)h_3(s)$, where

$$h_3(s)=\begin{cases}\frac{\tan s}{7{,}320}, & (t,s)\in[1,5]_{\mathbb{Z}}\times[0,\pi/4],\\ \frac{7{,}319}{7{,}320-1{,}830\pi}(s-\frac{\pi}{4})+\frac{1}{7{,}320}, & (t,s)\in[1,5]_{\mathbb{Z}}\times[\pi/4,1],\\ s\arctan s, & (t,s)\in[1,5]_{\mathbb{Z}}\times[1,\infty),\end{cases}$$

then f satisfies (H_1), $f^0=1/1{,}220$, and $f_\infty=\pi$. Furthermore, $0<Af^0=1/5<\theta^*Bf_\infty=9\pi<\infty$. From Theorem 3.3, if $\lambda\in(1/(9\pi),5)$, then (3.13) has at least one positive solution.

If $f(t,s)=(7-t)h_4(s)$, where

$$h_4(s)=\begin{cases}\tan s, & (t,s)\in[1,5]_{\mathbb{Z}}\times[0,\pi/4],\\ \frac{4}{3\pi}(s-1)+2, & (t,s)\in[1,5]_{\mathbb{Z}}\times[\pi/4,1],\\ \frac{2s^2+980}{488s+3}, & (t,s)\in[1,5]_{\mathbb{Z}}\times[1,\infty),\end{cases}$$

then f satisfies (H_1), $f_0=2$ and $f^\infty=1/224$. This implies that $0<Af^\infty=1<\theta^*Bf_0=18<\infty$. By Theorem 3.4, if $\lambda\in(1/18,1)$, then (3.13) has at least one positive solution.

Finally, we could construct the suitable $f(t,s)$ according to $h_1(s)$ to $h_4(s)$ such that $f(t,s)$ satisfies the condition in Theorem 3.5. So, we omit this here.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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