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Multi-dimensional discrete Halanay inequalities and the global stability of the disease free equilibrium of a discrete delayed malaria model

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Abstract

In this paper, we establish two *multi-dimensional discrete* Halanay-type inequalities. As an application, we consider a malaria transmission model with two delays and employ the two discrete Halanay-type inequalities to establish the global (exponential) asymptotical stability of the disease free equilibrium of the model. It is further shown that the disease free equilibrium of the delayed model is globally asymptotically stable when the basic reproduction number of the model is less than 1.

Keywords: discrete Halanay-type inequalities; multi-dimension; global asymptotical stability; nonsingular M-matrix; malaria transmission; time delay

1 Introduction

The *Halanay inequality* was first introduced in [1] and many generalizations of this inequality have been obtained due to their significance in the analysis of delayed dynamical systems [2–9], and especially in proving the global exponential stability of the equilibria of mathematical models proposed in neural networks and biology with time delays [8–11]. The original form of the Halanay inequality and some of its generalizations are for *continuous* dynamical systems with only one unknown scalar function involved in the system [1, 6, 9], *i.e.*, the *one-dimensional* case. In [9], Halanay inequality was generalized to a *multi-dimensional* form to deal with the stability of the equilibria of continuous dynamical systems in neural networks with impulses.

Along with the development of Halanay-type inequalities for continuous-time dynamical systems, *discrete-time* Halanay inequalities have also been established to handle the stability of discrete dynamical systems with delays [12–20]. For example, the following result was obtained in [15].

Theorem 1.1 (Theorem 1 in [15]) *Let $r > 0$ be a natural number, and let $\{x(n)\}_{n\geq -r}$ be a sequence of real numbers satisfying the inequality*

$$\Delta x(n) \leq -ax_n + b\max\{x(n), x(n-1), \dots, x(n-r)\}, \quad n \geq 0, \tag{1.1}$$

where Δ *is the forward difference operator and* $\Delta x(n) = x(n+1) - x(n)$. *If* $0 < b < a \leq 1$, *then there exists a constant* $\lambda_0 \in (0,1)$ *such that*

$$x(n) \leq \max\{0, x(0), x(-1), \ldots, x(-r)\}\lambda_0^n, \quad n \geq 0, \tag{1.2}$$

where λ_0 *is the smallest root in the interval* $(0,1)$ *of the equation*

$$\lambda^{r+1} + (a-1)\lambda^r - b = 0. \tag{1.3}$$

Based on Theorem 1.1, different generalizations of discrete Halanay-type inequalities have been developed [12, 13, 16, 19]. However, the generalizations are mostly dealing with *one-dimensional* functions, such as the sequence $\{x(n)\}$ in Theorem 1.1; while generalizations of discrete Halanay-type inequalities for *multi-dimensional* sequences $\{X(n)\} = \{(x_1(n), x_2(n), \ldots, x_m(n))^T\}$ are hardly developed. Motivated by this, the first aim of our present work is to develop certain *multi-dimensional discrete* Halanay-type inequalities.

On the other hand, the global stability of the equilibria of epidemic models is a key research topic in the quantitative analysis of the transmission of infectious diseases. Much research on this topic has been done on compartmental models of infectious diseases such as influenza, malaria, dengue, cholera, etc. [21–25]. Usually, a compartmental infectious disease model has a disease free equilibrium, which is globally asymptotically stable when the basic reproduction number R_0 of the model is less than 1, and it has a positive equilibrium which is globally asymptotically stable when $R_0 > 1$ [26]. Time delays are frequently involved in these infectious disease models [21–23, 27] and their presence poses great difficulty in stability analysis, especially in analyzing the global asymptotical stability of the equilibria of these models. For infectious disease models with two or more delays involved, the *local* asymptotical stability of the equilibrium can be obtained through the analysis of the eigenvalues of the Jacobian matrix of the linearized model at the corresponding equilibrium, but the *global* asymptotical stability of the equilibria of some of these infectious disease models with several delays remains unsolved at present [22, 23]. Hence, another aim of this work is to establish the *global asymptotical stability* of the disease free equilibrium of a *malaria transmission model with two delays*. We shall tackle this by applying the multi-dimensional discrete Halanay inequalities developed in this paper. Moreover, we shall show that the disease free equilibrium of the model is globally asymptotically stable when the basic reproduction number R_0 is less than 1, which is well consistent with the threshold property of the basic reproduction number.

The paper is organized as follows. In Section 2, we present two generalizations of Theorem 1.1 to multi-dimensional case. Applications of these two generalizations are given in Section 3 to obtain the global asymptotical stability of the disease free equilibrium of a malaria transmission model with two delays. Finally, some concluding remarks are given in Section 4.

2 Generalized discrete Halanay-type inequalities

In this section, we shall establish two generalizations of Theorem 1.1 to multi-dimensional case. First, we introduce some notations as follows. Throughout, we denote

$\mathbb{I} = \{1, 2, \ldots, m\}$,

$\mathbb{S} = \{-r, -r+1, \ldots, -1, 0, 1, \ldots\}$ where r is a positive integer,

$$\mathbb{S}_1 = \{-r, -r+1, \dots, -1, 0\},$$

$$\mathbb{S}_2 = \{n_0 - r, n_0 - r + 1, \dots, n_0\}.$$

Let $X(n) = (x_1(n), x_2(n), \dots, x_m(n))^T$ where $\{x_i(n)\}$, $i \in \mathbb{I}$ is a real sequence. Define

$$\big[x_i(n)\big]_r = \max\big\{x_i(n), x_i(n-1), \dots, x_i(n-r)\big\}, \quad i \in \mathbb{I},$$

and

$$\big[X(n)\big]_r = \big(\big[x_1(n)\big]_r, \big[x_2(n)\big]_r, \dots, \big[x_m(n)\big]_r\big)^T.$$

Moreover, as usual the forward difference operator Δ is defined by $\Delta X(n) = X(n+1) - X(n)$. Let $Y(n) = (y_1(n), y_2(n), \dots, y_m(n))^T$. The notion $X \le (\ge) Y$ means $x_i(n) \le (\ge) y_i(n)$ for $i \in \mathbb{I}$ and $n \in \mathbb{S}$. Let $E = \operatorname{diag}(1, 1, \dots, 1)$ be the unitary matrix of suitable order and δ_{ij} be the Kronecker symbol, *i.e.*, $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if otherwise.

Our first generalization of Theorem 1.1 is the following result.

Theorem 2.1 *Let $\{X(n)\}_{n \ge -r}$ be a nonnegative sequence satisfying*

$$\Delta X(n) \le AX(n) + B\big[X(n)\big]_r, \quad n \ge 0 \tag{2.1}$$

and the initial conditions (*also known as initial strings in* [15])

$$x_i(s) = x_i^{(s)}, \quad i \in \mathbb{I}, s \in \mathbb{S}_1, \tag{2.2}$$

where $A = (a_{ij})_{m \times m}$ and $B = (b_{ij})_{m \times m}$ are two matrices satisfying

$$a_{ii} \ge -1, \qquad a_{ij} \ge 0, \quad i \ne j; \qquad b_{ij} \ge 0, \qquad \sum_{j=1}^{m} b_{ij} > 0, \quad i, j \in \mathbb{I}, \tag{2.3}$$

and

$$\sum_{j=1}^{m} (a_{ij} + b_{ij}) < 0, \quad i \in \mathbb{I}. \tag{2.4}$$

Then there exists $\lambda_i \in (0,1)$, $i \in \mathbb{I}$, such that

$$x_i(n) \le \max_{s \in \mathbb{S}_1}\big\{0, x_i^{(s)}\big\}\lambda_i^n, \quad n \ge 0, i \in \mathbb{I},$$

where $\lambda_i \in (0,1)$ is the root of the equation

$$\lambda^{r+1} - \sum_{j=1}^{m} (a_{ij} + \delta_{ij})\lambda^r - \sum_{j=1}^{m} b_{ij} = 0, \tag{2.5}$$

for each $i \in \mathbb{I}$.

Proof Consider the following system:

$$\Delta Y(n) = AY(n) + B\big[Y(n)\big]_r \tag{2.6}$$

with initial values

$$y_i(s) = y_i^{(s)}, \quad i \in \mathbb{I}, s \in \mathbb{S}_1, \tag{2.7}$$

such that

$$x_i^{(s)} \le y_i^{(s)}, \quad i \in \mathbb{I}, s \in \mathbb{S}_1. \tag{2.8}$$

For each $i \in \mathbb{I}$, the ith component of (2.1) gives

$$x_i(n+1) \le \sum_{j=1}^{m}(a_{ij} + \delta_{ij})x_j(n) + \sum_{j=1}^{m} b_{ij}\big[x_j(n)\big]_r, \tag{2.9}$$

while the ith component of (2.6) is

$$y_i(n+1) = \sum_{j=1}^{m}(a_{ij} + \delta_{ij})y_j(n) + \sum_{j=1}^{m} b_{ij}\big[y_j(n)\big]_r. \tag{2.10}$$

Noting that $a_{ij} + \delta_{ij} \ge 0$, $b_{ij} \ge 0$, $i, j \in \mathbb{I}$ (from (2.3)) and the initial values satisfy (2.8), by comparison arguments [28] we get

$$x_i(n) \le y_i(n), \quad n \ge 0. \tag{2.11}$$

Now, define for each $i \in \mathbb{I}$,

$$F_i(\lambda) = \lambda^{r+1} - \sum_{j=1}^{m}(a_{ij} + \delta_{ij})\lambda^r - \sum_{j=1}^{m} b_{ij}.$$

We observe that $F_i(\lambda)$ is continuous in the interval $[0,1]$ (with respect to λ). Further, in view of (2.3) and (2.4),

$$F_i(0) = -\sum_{j=1}^{m} b_{ij} < 0$$

and

$$F_i(1) = 1 - \sum_{j=1}^{m}(a_{ij} + \delta_{ij}) - \sum_{j=1}^{m} b_{ij} = -\sum_{j=1}^{m}(a_{ij} + b_{ij}) > 0.$$

Consequently, there exists $\lambda_i \in (0,1)$ such that $F_i(\lambda_i) = 0$, *i.e.*, λ_i is a root of equation (2.5).

It is also noted that $F_i(\lambda) = 0$ is the characteristic equation of (2.10) when $0 < \lambda < 1$. Hence, $y_i(n) = K\lambda_i^n$, $n \ge 0$ is a solution of (2.10), where K is a constant. In particular, we

set $K = \max_{s\in\mathbb{S}_1}\{0, x_i^{(s)}\}$ such that (2.8) is fulfilled for $s \in \mathbb{S}_1$. Thus, it follows from (2.11) that

$$x_i(n) \le y_i(n) = \max_{s\in\mathbb{S}_1}\left\{0, x_i^{(s)}\right\}\lambda_i^n, \quad n \ge 0, i \in \mathbb{I}.$$

The proof is complete. □

Corollary 2.2 *Let $\{X(n)\}_{n\ge -r}$ be a nonnegative sequence satisfying* (2.1) *and* (2.2). *Suppose that A and B also satisfy* (2.3) *and* (2.4). *Then there exists $\mathcal{K} = (k_1, k_2, \dots, k_m)^T$, where $k_i, i \in \mathbb{I}$ is a positive constant, and $\lambda \in (0,1)$ such that*

$$X(n) \le \mathcal{K}\lambda^n, \quad n \ge 0.$$

Proof Set $k_i = \max_{s\in\mathbb{S}_1}\{0, x_i^{(s)}\}$, $i \in \mathbb{I}$ and $\lambda = \max\{\lambda_1, \lambda_2, \dots, \lambda_m\}$, where λ_i has the same meaning as in Theorem 2.1. It is immediate from Theorem 2.1 that $X(n) \le \mathcal{K}\lambda^n$, $n \ge 0$. □

Corollary 2.3 *Let $\{X(n)\}_{n\ge n_0-r}$ be a nonnegative sequence satisfying*

$$\Delta X(n) \le AX(n) + B\left[X(n)\right]_r, \quad n \ge n_0 \tag{2.12}$$

and the initial conditions

$$x_i(s) = x_i^{(s)}, \quad i \in \mathbb{I}, s \in \mathbb{S}_2, \tag{2.13}$$

where $A = (a_{ij})_{m\times m}$ and $B = (b_{ij})_{m\times m}$ are two matrices satisfying (2.3) *and* (2.4). *Then there exists $\mathcal{K} = (k_1, k_2, \dots, k_m)^T$, where $k_i = \max_{s\in\mathbb{S}_2}\{0, x_i^{(s)}\}$, $i \in \mathbb{I}$, and $\lambda \in (0,1)$ such that*

$$X(n) \le \mathcal{K}\lambda^{n-n_0}, \quad n \ge n_0.$$

Proof Set $n - n_0 = \bar{n}$ and it follows from Theorem 2.1 and Corollary 2.2 that

$$X(\bar{n}) \le \mathcal{K}\lambda^{\bar{n}}, \quad \bar{n} \ge 0,$$

i.e., $X(n) \le \mathcal{K}\lambda^{n-n_0}$, $n \ge n_0$. □

In the sequel, we shall generalize Theorem 1.1 to another multi-dimensional case by applying the theory of nonsingular M-matrix. The following lemma on nonsingular M-matrix [29] will be needed later.

Lemma 2.4 [29] *Suppose that the matrix $C = (c_{ij})_{m\times m}$ satisfies $c_{ij} \le 0$, $i \ne j$, $i,j \in \mathbb{I}$. The following statements are equivalent to C is a nonsingular M-matrix.*

(1) *All the successive principal minors of C are positive.*
(2) *$c_{ii} > 0$ and there exists a positive vector $z > 0$ such that $Cz > 0$.*
(3) *$C = D - M$ and $\rho(D^{-1}M) < 1$, where $M \ge 0$, $D = \operatorname{diag}(d_1, d_2, \dots, d_m)$ and $\rho(D^{-1}M)$ is the spectral radius of the matrix $D^{-1}M$.*

Remark 2.5 For a nonsingular M-matrix C, we denote

$$\Gamma(C) = \{z \in \mathbb{R}^m | Cz > 0, z > 0\}.$$

The set $\Gamma(C)$ is not empty in view of (2) of Lemma 2.4. Also, if $z \in \Gamma(C)$, then $kz \in \Gamma(C)$ for any constant $k > 0$.

The second generalization of Theorem 1.1 is stated as follows.

Theorem 2.6 *Let $\{X(n)\}_{n\geq -r}$ be a nonnegative sequence satisfying* (2.1), *where $A = (a_{ij})_{m\times m}$ and $B = (b_{ij})_{m\times m}$ are two matrices satisfying*

$$a_{ii} \geq -1, \qquad a_{ij} \geq 0, \quad i \neq j; \qquad b_{ij} \geq 0, \quad i,j \in \mathbb{I}, \tag{2.14}$$

and $C = -(A + B)$ is a nonsingular M-matrix. Moreover, suppose the nonnegative sequence $\{X(n)\}_{n\geq -r}$ satisfies

$$X(s) \leq z\lambda^s, \quad s \in \mathbb{S}_1, \tag{2.15}$$

where $z = (z_1, z_2, \ldots, z_m)^T > 0$ and $0 < \lambda < 1$ is the solution of

$$\left(A\lambda^r + \lambda^r E + B - \lambda^{r+1}E\right)z \leq 0. \tag{2.16}$$

Then

$$X(n) \leq z\lambda^n, \quad n \geq 0. \tag{2.17}$$

Proof Since $C = -(A + B)$ is a nonsingular M-matrix, from (2) of Lemma 2.4 there exists $z > 0$ such that $-(A + B)z > 0$, or equivalently,

$$(A + B)z < 0. \tag{2.18}$$

Noting that $A\lambda^r + \lambda^r E + B - \lambda^{r+1}E = A + B$ when $\lambda = 1$, it follows from (2.18) that (2.16) has a solution $z > 0$ when $\lambda = 1$. Consequently, (2.16) has a solution $z > 0$ when $0 < \lambda < 1$ due to the continuity.

We now claim that

$$X(n) \leq (1 + \epsilon)z\lambda^n \triangleq U(n) = \left(u_1(n), u_2(n), \ldots, u_m(n)\right)^T, \quad n \geq 0, \tag{2.19}$$

for any $\epsilon > 0$ arbitrarily given, where $z > 0$ and $\lambda \in (0,1)$ satisfy (2.16).

Suppose that (2.19) is not true, then there exist a positive integer h, $1 \leq h \leq m$, and a positive integer N such that

$$x_h(N) \leq u_h(N), \qquad x_h(N + 1) > u_h(N + 1) \tag{2.20}$$

and

$$x_i(n) \leq u_i(n) = (1 + \epsilon)z_i\lambda^n, \quad -r \leq n \leq N, i \in \mathbb{I}. \tag{2.21}$$

Using (2.9) (when $i = h$ and $n = N$) and (2.21), we find

$$\begin{aligned} x_h(N+1) &\le \sum_{j=1}^{m}(a_{hj}+\delta_{hj})x_j(N) + \sum_{j=1}^{m} b_{hj}\big[x_j(N)\big]_r \\ &\le \sum_{j=1}^{m}(a_{hj}+\delta_{hj})u_j(N) + \sum_{j=1}^{m} b_{hj}\big[u_j(N)\big]_r \\ &= \sum_{j=1}^{m}(a_{hj}+\delta_{hj})(1+\epsilon)z_j\lambda^N + \sum_{j=1}^{m} b_{hj}(1+\epsilon)z_j\lambda^{N-r} \\ &= \sum_{j=1}^{m}\big[(a_{hj}+\delta_{hj})\lambda^r + b_{hj}\big](1+\epsilon)z_j\lambda^{N-r}. \end{aligned}$$

On the other hand, the hth component of (2.16) gives

$$\sum_{j=1}^{m}\big[(a_{hj}+\delta_{hj})\lambda^r + b_{hj} - \lambda^{r+1}\delta_{hj}\big]z_j \le 0,$$

or equivalently

$$\sum_{j=1}^{m}\big[(a_{hj}+\delta_{hj})\lambda^r + b_{hj}\big]z_j \le \lambda^{r+1}z_h.$$

Using the above inequality in the earlier inequality, we get

$$\begin{aligned} x_h(N+1) &\le \sum_{j=1}^{m}\big[(a_{hj}+\delta_{hj})\lambda^r + b_{hj}\big](1+\epsilon)z_j\lambda^{N-r} \\ &\le \lambda^{r+1}z_h(1+\epsilon)\lambda^{N-r} = (1+\epsilon)z_h\lambda^{N+1} = u_h(N+1), \end{aligned}$$

which contradicts the second inequality of (2.20). Therefore, (2.19) holds for $n \ge 0$. Now letting $\epsilon \to 0$ in (2.19), we obtain

$$X(n) \le z\lambda^n, \quad n \ge 0.$$

The proof is complete. □

Corollary 2.7 *Let* $\{X(n)\}_{n\ge n_0-r}$ *be a nonnegative sequence satisfying* (2.12), *where* $A = (a_{ij})_{m\times m}$ *and* $B = (b_{ij})_{m\times m}$ *are two matrices satisfying* (2.14) *and* $C = -(A+B)$ *is a nonsingular M-matrix. Moreover, suppose the nonnegative sequence* $\{X(n)\}_{n\ge n_0-r}$ *satisfies*

$$X(s) \le z\lambda^s, \quad s \in \mathbb{S}_2,$$

where $z = (z_1, z_2, \ldots, z_m)^T > 0$ *and* $0 < \lambda < 1$ *satisfy* (2.16). *Then*

$$X(n) \le z\lambda^n, \quad n \ge n_0.$$

Proof By setting $\bar{n} = n - n_0$, the result is an immediate consequence of Theorem 2.6. □

Remark 2.8 A similar version of Corollary 2.7 can be found in [20]. Under the assumptions

$$P, Q \in \mathbb{R}_+^{m\times m}, \qquad \rho(P+Q) < 1, \tag{2.22}$$

it is established in Theorem 1 of [20] that if the nonnegative sequence $\{U(n)\}$ satisfies

$$U(n) \le PU(n) + QU\big(n - \tau(n)\big), \quad n \ge n_0, \tag{2.23}$$

and

$$U(s) \le z\exp\big(-\mu(s - n_0)\big), \quad s = n_0 - \tau, n_0 - \tau + 1, \dots, n_0, \tag{2.24}$$

where $\tau(n) = (\tau_{ij})_{m\times m}$, $0 \le \tau_{ij} \le \tau$, $i,j \in \mathbb{I}$, τ is a nonnegative integer and the constant $\mu > 0$, the vector $z > 0$ are determined by

$$\big[\exp(\mu)\big(P + Q\exp(\mu\tau)\big) - E\big]z \le 0,$$

then $U(n) \le z\exp(-\mu(n - n_0))$, $n \ge n_0$.

Comparing (2.23) with (2.1), we see that $P = A + E$ and $Q = B$. If we set $C = -(A + B)$, $D = E$, and $M = E - (-(A + B))$, then $D^{-1}M = M = E + A + B = P + Q$. Hence, the condition $\rho(P + Q) < 1$ is equivalent to the condition that $-(A + B)$ is a nonsingular M-matrix (according to (3) of Lemma 2.4).

The next theorem shows that the estimation of $\{X(n)\}$ similar to (2.17) still holds without the assumption (2.15), which imposes a condition on the initial values of (2.1).

Theorem 2.9 *Let* $\{X(n)\}_{n\ge -r}$ *be a nonnegative sequence satisfying* (2.1) *and* (2.2). *Suppose that A and B also satisfy* (2.14) *and* $-(A + B)$ *is a nonsingular M-matrix. Further,* $z > 0$ *and* $0 < \lambda < 1$ *satisfy* (2.16). *Then there exists a constant* $\omega > 0$ *such that*

$$X(n) \le \omega z\lambda^n, \quad n \ge 0. \tag{2.25}$$

Proof As in the proof of Theorem 2.6, we can choose $z = (z_1, z_2, \dots, z_m)^T > 0$ and $\lambda \in (0,1)$ satisfying (2.16). Noting the conditions of Theorem 2.6, we only need to show

$$X(s) \le \omega z\lambda^s, \quad s \in \mathbb{S}_1. \tag{2.26}$$

Setting

$$\omega_1 = \max_{i\in\mathbb{I}, s\in\mathbb{S}_1}\big\{\big|x_i(s)\big|\big\}, \qquad \omega_2 = \frac{1}{\min_{i\in\mathbb{I}}\{z_i\}},$$

it is easy to verify that (2.26) holds for $\omega = \omega_1\omega_2$. Noting Remark 2.5, $\omega z > 0$ and $\lambda \in (0,1)$ also fulfill (2.16). Thus, (2.25) is obtained as in the proof of Theorem 2.6 by setting $u_i(n)$ in (2.21) as $u_i(n) = (1 + \epsilon)\omega z_i\lambda^n$. □

Remark 2.10 The results of Theorems 2.1, 2.6, and 2.9 can be applied to obtain the global exponential stability of the zero solution of (2.1). As such, we can make use of these theorems to analyze the global exponential stability [27, 30] of the equilibrium of a discrete dynamical system with time delays. It is well known that the equilibrium is *globally asymptotically stable* if it is *globally exponentially stable* [30]. Hence, the global asymptotical stability of the equilibria of some discrete dynamical systems with time delays may be established via the results obtained in this section. We shall give such applications in the next section.

3 Global asymptotical stability of equilibrium

In this section, we shall obtain the global stability of the disease free equilibrium of an infectious disease model with two delays that describes malaria transmission. Our proofs employ the generalizations of Halanay-type inequalities obtained in Section 2.

In [22], the following delayed Ross-Macdonald model was established to describe the malaria transmission between human and mosquito:

$$\begin{cases} \frac{dx}{dt} = -\gamma x + abm \exp(-\gamma \tau_1)(1 - x(t - \tau_1))y(t - \tau_1), \\ \frac{dy}{dt} = -\mu y + ac \exp(-\mu \tau_2)(1 - y(t - \tau_2))x(t - \tau_2). \end{cases} \tag{3.1}$$

In model (3.1), $x(y)$ is the ratio of the number of infected human (mosquito) to the total number of human (mosquito) population at time t (days). The total number of human (mosquito) population is H (M) and H (M) is a constant. Define $m = M/H$. Moreover, γ is the average recovery rate of human from malaria infection, μ is the natural death rate of mosquito, a is the average number of bites of a mosquito per human per day, b is the rate of a susceptible human becoming infectious after being bitten by an infected mosquito, c is the rate of a susceptible mosquito becoming infectious after it bites an infected human, and τ_1 and τ_2 are the average incubation times of the parasites in the body of human and mosquito, respectively. For more details of (3.1), one can refer to [22].

We now consider a discrete version of (3.1) as follows:

$$\begin{cases} x(n+1) = x(n) - \gamma x(n) + abm \exp(-\gamma r_1)(1 - x(n - r_1))y(n - r_1), \\ y(n+1) = y(n) - \mu y(n) + ac \exp(-\mu r_2)(1 - y(n - r_2))x(n - r_2). \end{cases} \tag{3.2}$$

In model (3.2), $n \geq 0$ and $x(n)$ ($y(n)$) is the ratio of the number of infected humans (mosquitos) to the total number of humans (mosquitos) of the population on the nth day. Moreover, r_1 and r_2 are now positive integers to reflect the incubation time of the parasites in the body of human and mosquito, respectively. The implications of the other parameters are the same as in (3.1) and a, b, c, m, γ, and μ are all positive constants.

Denote $r = \max\{r_1, r_2\}$. The initial values of (3.2) are

$$x(s) = x^{(s)}, \qquad y(s) = y^{(s)}, \quad s = -r, -r+1, \ldots, 0.$$

In view of the practical background of model (3.2), it is natural to assume that

$$0 < x^{(s)} < 1, \qquad 0 < y^{(s)} < 1, \quad s = -r, -r+1, \ldots, 0. \tag{3.3}$$

Remark 3.1 The *basic reproduction number* R_0 of model (3.2) (refer to [22, 23]) is defined as

$$R_0 = \frac{a^2 bcm\alpha\beta}{\gamma\mu}, \tag{3.4}$$

where $\alpha = \exp(-\gamma r_1)$ and $\beta = \exp(-\mu r_2)$. If we denote $R_{01} = abm\alpha/\gamma$ and $R_{02} = ac\beta/\mu$, then $R_0 = R_{01}R_{02}$.

Remark 3.2 Since γ is the recovery rate of human from malaria infection, it follows that $1/\gamma$ is the average recovery time of human from the infection. As it takes 7-30 days for a human to recover from the infection [22, 23], it is reasonable to assume that

$$0 < \gamma < 1. \tag{3.5}$$

Moreover, since the natural lifespan of mosquito is 14-60 days [22, 23], it also makes sense to assume that

$$0 < \mu < 1. \tag{3.6}$$

Lemma 3.3 *Suppose that* (3.5), (3.6) *are satisfied and*

$$R_{01} < 1, \qquad R_{02} < 1. \tag{3.7}$$

Then the solutions $\{x(n), y(n)\}$ *of* (3.2) *with* (3.3) *satisfy*

$$0 < x(n) < 1, \qquad 0 < y(n) < 1, \quad n \geq 0.$$

Proof We shall first prove that $0 < x(n) < 1$ for $0 < n \leq r$. Suppose that there exists $0 \leq N_1 < r$ such that $x(N_1) > 0$ and $x(N_1 + 1) \leq 0$. From (3.2), noting (3.3) and (3.5) we find

$$\begin{aligned} x(N_1 + 1) &= x(N_1) - \gamma x(N_1) + abm\alpha\big(1 - x(N_1 - r_1)\big)y(N_1 - r_1) \\ &> (1 - \gamma)x(N_1) > 0, \end{aligned}$$

which is a contradiction. Hence, $x(n) > 0$ for $0 < n \leq r$.

Next, suppose that there exists $0 \leq N_2 < r$ such that $x(N_2) < 1$ and $x(N_2 + 1) \geq 1$. In view of (3.3), (3.5), and (3.7), it follows from (3.2) that

$$\begin{aligned} x(N_2 + 1) &= x(N_2) - \gamma x(N_2) + abm\alpha\big(1 - x(N_2 - r_1)\big)y(N_2 - r_1) \\ &< 1 - \gamma + abm\alpha = 1 - \gamma(1 - abm\alpha/\gamma) = 1 - \gamma(1 - R_{01}) < 1. \end{aligned}$$

This is a contradiction and hence $x(n) < 1$ for $0 < n \leq r$.

Similarly, we can prove that $0 < y(n) < 1$ for $0 < n \leq r$. By assuming that $0 < x(n) < 1$ and $0 < y(n) < 1$ for $(k - 1)r < n \leq kr$, using a similar technique we can show that $0 < x(n) < 1$ and $0 < y(n) < 1$ for $kr < n \leq (k + 1)r$. Hence, it is shown by induction that $0 < x(n) < 1$ and $0 < y(n) < 1$ for $n \geq 0$. □

Remark 3.4 Lemma 3.3 gives sufficient conditions to guarantee that the solutions of (3.2) with (3.3) are positive and bounded. As pointed out in [31], sometimes these sufficient conditions may be quite rigorous. Indeed, noting Remark 3.1 we see that condition (3.7) implies $R_0 < 1$, but in fact numerical simulations show that solutions of (3.2) with (3.3) can be positive and bounded even when $R_0 > 1$. In view of this, we shall introduce a set Ω of parameters that ensures the positivity and boundedness of the solutions of (3.2) with (3.3), let

$$\begin{aligned}\Omega = \big\{&(a,b,c,m,\gamma,\mu,r_1,r_2) | a,b,c,m,\gamma,\mu,r_1,r_2 \text{ are such that}\\ &\text{solutions of (3.2) with (3.3) are positive and bounded}\big\}.\end{aligned} \tag{3.8}$$

The following lemma is about the existence of the equilibrium of model (3.2), which is obtained by direct computation.

Lemma 3.5

(1) *There exists only the disease free equilibrium* $E_0 = (0,0)$ *of* (3.2) *if* $R_0 \le 1$.

(2) *There exist two equilibria of* (3.2) *if* $R_0 > 1$, *namely the disease free equilibrium* E_0 *and the positive equilibrium* $E^* = (x^*, y^*)$, *where*

$$x^* = \frac{R_0 - 1}{R_0 + R_{01}}, \qquad y^* = \frac{R_0 - 1}{R_0 + R_{02}}. \tag{3.9}$$

We shall now employ Theorem 2.1 to obtain the global asymptotical stability of the disease free equilibrium E_0 of model (3.2).

Theorem 3.6 *Suppose that* (3.5), (3.6), *and* (3.7) *hold. Then the disease free equilibrium* E_0 *of* (3.2) *with* (3.3) *is globally asymptotically stable.*

Proof We rewrite (3.2) as

$$\begin{cases}\Delta x(n) = -\gamma x(n) + abm\alpha y(n - r_1) - abm\alpha x(n - r_1)y(n - r_1),\\ \Delta y(n) = -\mu y(n) + ac\beta x(n - r_2) - ac\beta x(n - r_2)y(n - r_2).\end{cases}$$

Since the conditions of Lemma 3.3 are satisfied, we have $x(n) > 0$ and $y(n) > 0$ for $n \ge -r$. Hence, we get

$$\begin{cases}\Delta x(n) \le -\gamma x(n) + abm\alpha y(n - r_1),\\ \Delta y(n) \le -\mu y(n) + ac\beta x(n - r_2),\end{cases}$$

which clearly leads to

$$\begin{cases}\Delta x(n) \le -\gamma x(n) + abm\alpha [y(n)]_r,\\ \Delta y(n) \le -\mu y(n) + ac\beta [x(n)]_r,\end{cases} \tag{3.10}$$

for $n \ge 0$.

Comparing (3.10) with (2.1), we see that

$$A = \begin{pmatrix}-\gamma & 0\\ 0 & -\mu\end{pmatrix}, \qquad B = \begin{pmatrix}0 & abm\alpha\\ ac\beta & 0\end{pmatrix}. \tag{3.11}$$

It is obvious that condition (2.3) is fulfilled. Next, noting from (3.7) that $R_{01} = abm\alpha/\gamma < 1$ and $R_{02} = ac\beta/\mu < 1$, we have

$$\sum_{j=1}^{2}(a_{1j} + b_{1j}) = -\gamma + abm\alpha < 0, \qquad \sum_{j=1}^{2}(a_{2j} + b_{2j}) = -\mu + ac\beta < 0.$$

Hence, condition (2.4) is also satisfied. Now, by Theorem 2.1, there exist $\lambda_1, \lambda_2 \in (0,1)$ such that

$$x(n) \le k_1\lambda_1^n, \qquad y(n) \le k_2\lambda_2^n, \tag{3.12}$$

for $n \ge 0$, where $k_1 = \max_{s\in\mathbb{S}_1}\{x^{(s)}\}$ and $k_2 = \max_{s\in\mathbb{S}_1}\{y^{(s)}\}$.

In view of (3.12), the zero solution of (3.2), which is the disease free equilibrium E_0, is globally exponentially stable. Noting Remark 2.10, it follows that E_0 is also globally asymptotically stable. □

Remark 3.7 When the conditions of Theorem 3.6 are satisfied, noting (3.12) the exponential convergence rate of the solutions of (3.2) with (3.3) to the disease free equilibrium is $\min\{-\ln\lambda_1, -\ln\lambda_2\}$.

Remark 3.8 Condition (3.7) implies that $R_0 = R_{01}R_{02} < 1$, but $R_0 < 1$ may not imply (3.7). In the literature of compartmental infectious disease models, usually E_0 is globally asymptotically stable when $R_0 < 1$. This expected result is not obtained when Theorem 2.1 is applied to model (3.2). As such, in the sequel we shall apply Theorem 2.9 to model (3.2) to see whether E_0 is globally asymptotically stable when $R_0 < 1$.

Theorem 3.9 *Suppose that* (3.5), (3.6) *are satisfied and*

$$R_0 < 1. \tag{3.13}$$

Then the disease free equilibrium E_0 *of* (3.2) *with* (3.3) *is globally asymptotically stable in* Ω.

Proof Noting the definition of Ω (refer to (3.8)), (3.10) also holds true. With A and B defined as in (3.11), we see that condition (2.14) is satisfied. Moreover, since

$$-(A+B) = \begin{pmatrix} \gamma & -abm\alpha \\ -ac\beta & \mu \end{pmatrix},$$

we have $\gamma > 0$ and $\det(-(A+B)) = \gamma\mu - a^2bcm\alpha\beta = \gamma\mu(1 - R_0) > 0$ due to (3.13), *i.e.*, the successive principal minors of $-(A+B)$ are both positive. Therefore, according to (1) of Lemma 2.4, $-(A+B)$ is a nonsingular M-matrix. The conditions of Theorem 2.9 are all satisfied, hence the disease free equilibrium E_0 of (3.2) with (3.3) is globally exponentially stable and consequently it is globally asymptotically stable in Ω. □

We shall now give an example to illustrate Theorem 3.9 and Remark 3.4.

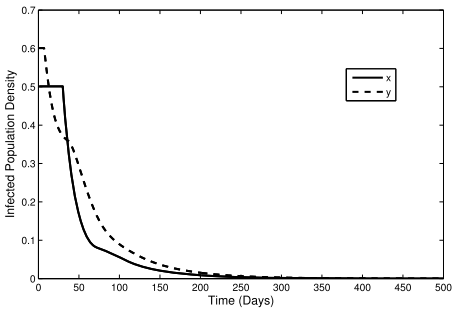

Figure 1 The dynamics of Example 3.10. The solid line is the variation of the infected human population density and the dashed line is the variation of the infected mosquito population.

Example 3.10 Consider a special case of (3.2) (with day as the time unit)

$$\begin{cases} x(n+1) = x(n) - 0.07x(n) + 0.15 \times 0.5 \\ \qquad\qquad \times 2\exp(-0.07 \times 30)(1 - x(n-30))y(n-30), \\ y(n+1) = y(n) - 0.05y(n) + 0.15 \times 0.5\exp(-0.05 \times 7)(1 - y(n-7))x(n-7). \end{cases} \tag{3.14}$$

Here, the parameters are

$$\gamma = 0.07, \qquad a = 0.15, \qquad b = c = 0.5, \qquad m = 2,$$
$$\mu = 0.05, \qquad r_1 = 30, \qquad r_2 = 7.$$

By direct computation, we have $R_{01} = 0.2624$, $R_{02} = 1.0570$, and $R_0 = 0.2774$. According to Theorem 3.9, the disease free equilibrium E_0 of (3.14) is globally asymptotically stable since (3.5), (3.6), and (3.13) are satisfied. Indeed, the dynamics of Example 3.10 is depicted in Figure 1 and we observe that both $x(n)$ and $y(n)$ tend to 0 as $n \to +\infty$, *i.e.*, the disease free equilibrium E_0 is globally asymptotically stable.

In this example, we note that (3.7) is not satisfied since $R_{02} > 1$. However, it is observed from Figure 1 that the solution of (3.14) is positive and bounded. This illustrates Remark 3.4 and shows that even if (3.7) does not hold, the set Ω (see (3.8)) may not be empty.

4 Concluding remarks

Halanay-type inequalities, whether continuous or discrete, have been widely applied to obtain the global exponential (asymptotical) stability of the equilibria of dynamical systems with several delays, especially dynamical systems of neural networks. In this paper, we have derived two generalizations of multi-dimensional discrete Halanay-type inequalities. Further, the generalizations are applied to a discrete malaria transmission model with two delays. We have shown that the disease free equilibrium is globally asymptotically stable when the basic reproduction number R_0 is less than 1, which is well consistent with the threshold property of the basic reproduction number.

The global asymptotical stability of the equilibria of infectious disease models with time delays is usually obtained via the construction of suitable Lyapunov functionals together with Razumikhin-type theorem and/or LaSalle invariant sets. However, a suitable Lyapunov functional is somewhat difficult to construct for a delayed dynamical system. Hence, it is reasonable to try other methods to obtain the global asymptotical stability of equilibria of dynamical systems with time delays. From our present work, we have ob-

served that it is direct and simple to obtain the global asymptotical stability of equilibria of dynamical systems with time delays via Halanay-type inequalities.

When using Halanay-type inequalities established in this paper to obtain the global asymptotical stability of the equilibrium of a discrete dynamical system with time delays, the positivity of the solutions should be initially guaranteed. This can be proved for many kinds of compartmental infectious disease models. Hence, it is direct to obtain the global asymptotical stability of the disease free equilibria of dynamical systems with time delays by applying Halanay-type inequalities. It is well known that, in order to obtain the global asymptotical stability of the positive equilibrium (x^*, y^*) of a dynamical system with time delays, the change of variables $\bar{x} = x - x^*$ and $\bar{y} = y - y^*$ is usually applied to transfer the global asymptotical stability of the positive equilibrium to the global asymptotical stability of the zero solution of the system with respect to $\bar{x}$ and $\bar{y}$. However, after the change of variables, the positivity of $\bar{x}$ and $\bar{y}$ cannot be guaranteed. Hence, if one intends to employ Halanay-type inequalities to obtain the global asymptotical stability of the positive equilibrium, new techniques are needed to deal with $\bar{x}$ and $\bar{y}$. This remains as future work.

Competing interests
None of the authors have any competing interests to the paper.

Authors' contributions
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