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Existence and uniqueness results for nonlocal integral boundary value problems for fractional differential equations

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Abstract
In this work, we investigate the existence and uniqueness of the solutions for a class of nonlinear fractional differential equations with nonlocal integral boundary conditions. Our analysis relies on the fixed point index theory and a u_0-positive operator. Examples are discussed for the illustration of the main work.

Keywords: fractional differential equations; integral boundary conditions; u_0-positive operator; fixed point theorem

1 Introduction

We consider a class of nonlinear fractional differential equations with nonlocal integral boundary value conditions of this form:

$$\begin{cases} D^{\alpha}_{0^+}u(t) + p(t)f(t,u(t)) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \\ u(1) = \lambda I^{\beta}_{0^+}u(\eta) = \lambda \int_0^{\eta} \frac{(\eta-s)^{\beta-1}u(s)}{\Gamma(\beta)}\,ds, \end{cases} \tag{1.1}$$

where $\alpha \in (n-1, n]$ is a real number, $n \geq 3$ is an integer, $0 < \eta \leq 1$, $\lambda, \beta > 0$, $0 \leq \frac{\lambda\Gamma(\alpha)\eta^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} < 1$, and $D^{\alpha}_{0^+}$ is the standard Riemann-Liouville differential operator.

Here, we emphasize that the integral boundary condition of (1.1) can be understood in the sense that the value of the unknown function at the position $t = 1$ is proportional to the Riemann-Liouville fractional integral of the unknown function $\lambda \int_0^{\eta} \frac{(\eta-s)^{\beta-1}u(s)}{\Gamma(\beta)}\,ds$, where $0 < \eta \leq 1$. Furthermore, for $\beta = 1$, the integral boundary condition reduces to the usual form of nonlocal integral condition $u(1) = \lambda \int_0^{\eta} u(s)\,ds$.

Fractional calculus has been investigated in diverse by several researchers. The recent development covers the theoretical as well as potential applications of the subject in physical and technical science. Fractional differential equations have been of great interest recently (see, *e.g.*, [1–5]). There are many results dealing with the existence and multiplicity of solutions of nonlinear fractional differential equations by the means of techniques of nonlinear analysis (see, *e.g.*, [6–12] and the references therein).

Wang and Zhang [13] studied the existence and multiplicity of positive solutions for the following nonlinear fractional differential equations:

$$\begin{cases} D_{0^+}^{\alpha} u(t) + h(t)f(t,u(t)) = 0, \quad 0 < t < 1, n-1 < \alpha \leq n, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \qquad u(1) = \sum_{i=1}^{m-2} \eta_i u(\xi_i), \end{cases} \tag{1.2}$$

where $\alpha \geq 2$, $\eta_i \geq 0$ $(i = 1, 2, \ldots, m-2)$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, $\sum_{i=1}^{m-2} \eta_i \xi_i^{\alpha-1} < 1$, $D_{0^+}^{\alpha}$ is the standard Riemann-Liouville derivative.

Zhang [14] studied the existence of positive solutions of the following nonlinear fractional differential equation with infinite-point boundary value conditions:

$$\begin{cases} D_{0^+}^{\alpha} u(t) + q(t)f(t,u(t)) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \\ u^{(i)}(1) = \sum_{j=1}^{\infty} \alpha_j u(\xi_j), \end{cases} \tag{1.3}$$

where $\alpha > 2$, $n-1 < \alpha \leq n$, $i \in [1, n-2]$ is a fixed integer, $\alpha_j \geq 0$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{j-1} < \xi j < \cdots < 1$ $(j = 1, 2, \ldots)$, $\Delta - \sum_{j=1}^{\infty} \alpha_j \xi_j^{\alpha-1} > 0$, $\Delta = (\alpha-1)(\alpha-2)\cdots(\alpha-i)$, $D_{0^+}^{\alpha}$ is the standard Riemann-Liouville derivative. By introducing height functions of the nonlinear term on some bounded sets and considering integrations of these height functions, several local existence and multiplicity of positive solutions theorems were obtained.

In [15], Ahmad and Agarwal considered the existence and the uniqueness of solutions to a class of Caputo type fractional differential equations of order $q \in (n-1, n]$ with slit-strips type boundary conditions:

$$\begin{cases} {}^cD^q u(t) = f(t,u(t)), \quad 0 < t < 1, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \\ u(\eta) = a \int_0^{\xi} u(s)\,\mathrm{d}s + b \int_{\zeta}^1 u(s)\,\mathrm{d}s, \end{cases} \tag{1.4}$$

where $0 < \xi < \eta < \zeta < 1$, a and b are positive constants.

Liu *et al.* [16] studied the existence of positive solutions and constructed two successively iterative sequences to approximate the solutions for the following fractional boundary value problem:

$$\begin{cases} D_{0^+}^{\alpha} u(t) + f(t,u(t)) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = u''(0) = 0, \\ u(1) = \lambda I_{0^+}^{\beta} u(\eta), \end{cases} \tag{1.5}$$

where $3 < \alpha \leq 4$, $0 < \eta \leq 1$, $\lambda, \beta > 0$, $0 \leq \frac{\lambda\Gamma(\alpha)\eta^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} < 1$, and $D_{0^+}^{\alpha}$ is the standard Riemann-Liouville differential operator.

For $n = 4$, $p(t) \equiv 1$, the fractional boundary value problem (1.1) reduces to the problem (1.5). For $n = 4$, $\beta = 1$, the boundary conditions of (1.1) reduce to $u(0) = u'(0) = u''(0) = 0$, $u(1) = \lambda \int_0^{\eta} u(s)\,\mathrm{d}s$, which had been considered in [17].

Motivated by the work mentioned above, in this article we study the differential equations (1.1) by using u_0-positive operator and fixed point index theory under some conditions concerning the first eigenvalue with respect to the relevant linear operator. The methods are different from those in previous work; we not only obtain the existence and multiplicity of positive solutions for (1.1) under sublinear and superlinear cases, but we also get the uniqueness existence of solutions for (1.1). Moreover, we construct successively iterative sequences to approximate the unique solutions.

This paper is arranged as follows. Some lemmas needed below are listed in Section 2. The existence and multiplicity of the positive solutions to the problem (1.1) are proved in the first part of Section 3. In the second part, one shows the existence and uniqueness of positive solutions and constructs successively iterative sequences to approximate the solutions. Finally, in Section 4, examples are given for the illustration of the main work.

2 Some lemmas

Let Banach space $E = C([0,1])$ be endowed with the norm $\|u\|_\infty = \max_{0\le t\le 1}|u(t)|$, and θ is the zero function in E. Define a closed cone $P_c \subset E$ by $P_c = \{u \in E \mid u(t) \ge 0, t \in [0,1]\}$. For any $0 < r < R < +\infty$, let $B_r = \{u \in P_c : \|u\| < r\}$, $\partial B_r = \{u \in P_c : \|u\| = r\}$, $B_R = \{u \in P_c : \|u\| < R\}$, $\overline{B}_R \setminus B_r = \{u \in P_c : r \le \|u\| \le R\}$.

We list the following assumptions adopted in this paper:

(A_1) $p : [0,1] \to [0,+\infty)$ is continuous and $0 < \int_0^1 p(s)\,\mathrm{d}s < +\infty$;
(A_2) $f : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous and $f(t,0) \not\equiv 0$.

For the convenience of the reader, we present here some necessary definitions of the fractional calculus. These definitions can be found in the recent literature [1–3].

Definition 2.1 The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f : (0,\infty) \to R$ is given by

$$I_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\,\mathrm{d}s,$$

provided that the right-hand side is pointwise defined on $(0,\infty)$.

Definition 2.2 The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $f : (0,\infty) \to R$ is given by

$$D_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} f(s)\,\mathrm{d}s,$$

where $n-1 \le \alpha < n$, provided that the right-hand side is pointwise defined on $(0,\infty)$.

Lemma 2.1 *Assume that $y(t) \in C([0,1])$. The problem*

$$\begin{cases} D_{0^+}^{\alpha} u(t) + y(t) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \\ u(1) = \lambda I_{0^+}^{\beta} u(\eta) = \lambda \int_0^{\eta} \frac{(\eta-s)^{\beta-1} u(s)}{\Gamma(\beta)}\,\mathrm{d}s, \end{cases} \tag{2.1}$$

where $\alpha \in (n-1,n]$, $n \ge 3$, $n \in N$, $0 < \eta \le 1$, $\lambda, \beta > 0$, $0 \le \frac{\lambda\Gamma(\alpha)\eta^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} < 1$, is equivalent to

$$u(t) = \int_0^1 G(t,s) y(s)\,\mathrm{d}s,$$

where

$$G(t,s) = \begin{cases} \frac{-P\Gamma(\alpha+\beta)(t-s)^{\alpha-1}+\Gamma(\alpha+\beta)(1-s)^{\alpha-1}t^{\alpha-1}-\Gamma(\alpha)\lambda(\eta-s)^{\alpha+\beta-1}t^{\alpha-1}}{P\Gamma(\alpha)\Gamma(\alpha+\beta)}, & 0 \le s \le t \le 1, s \le \eta, \\ \frac{\Gamma(\alpha+\beta)(1-s)^{\alpha-1}t^{\alpha-1}-\Gamma(\alpha)\lambda(\eta-s)^{\alpha+\beta-1}t^{\alpha-1}}{P\Gamma(\alpha)\Gamma(\alpha+\beta)}, & 0 \le t \le s \le \eta \le 1, \\ \frac{-P\Gamma(\alpha+\beta)(t-s)^{\alpha-1}+\Gamma(\alpha+\beta)(1-s)^{\alpha-1}t^{\alpha-1}}{P\Gamma(\alpha)\Gamma(\alpha+\beta)}, & 0 \le \eta \le s \le t \le 1, \\ \frac{\Gamma(\alpha+\beta)(1-s)^{\alpha-1}t^{\alpha-1}}{P\Gamma(\alpha)\Gamma(\alpha+\beta)}, & 0 \le t \le s \le 1, s \ge \eta, \end{cases}$$

with $P = 1 - \frac{\lambda\Gamma(\alpha)}{\Gamma(\alpha+\beta)}\eta^{\alpha+\beta-1}$, $0 < P \le 1$. *$G(t,s)$ is called the Green's function of boundary value problem* (2.1). *Obviously, $G(t,s)$ is a continuous function on* $[0,1] \times [0,1]$.

Proof A function $u \in C^{n-1}[0,1] \cap C^n(0,1)$ is called a solution of FBVP (2.1) if it satisfies (2.1). It is shown in [1–3] that problem (2.1) is equivalent to the following integral equation:

$$u(t) = -I_{0^+}^{\alpha} y(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_n t^{\alpha-n}.$$

By $u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0$, we have

$$C_2 = C_3 = \cdots = C_n = 0.$$

Then we get

$$u(t) = -I_{0^+}^{\alpha} y(t) + C_1 t^{\alpha-1}.$$

By $u(1) = \lambda I_{0^+}^{\beta} u(\eta)$, we have

$$-I_{0^+}^{\alpha} y(1) + C_1 = -\lambda I_{0^+}^{\alpha+\beta} y(\eta) + \lambda C_1 \frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta)} \eta^{\alpha+\beta-1}.$$

When $1 - \frac{\lambda\Gamma(\alpha)}{\Gamma(\alpha+\beta)}\eta^{\alpha+\beta-1} \neq 0$, we obtain

$$C_1 = \frac{1}{1 - \frac{\lambda\Gamma(\alpha)}{\Gamma(\alpha+\beta)}\eta^{\alpha+\beta-1}} \bigl(I_{0^+}^{\alpha} y(1) - \lambda I_{0^+}^{\alpha+\beta} y(\eta)\bigr) =: \frac{1}{P} \bigl(I_{0^+}^{\alpha} y(1) - \lambda I_{0^+}^{\alpha+\beta} y(\eta)\bigr).$$

Therefore, the solution to problem (2.1) is

$$\begin{aligned} u(t) = {} & -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\,\mathrm{d}s + \frac{t^{\alpha-1}}{P\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} y(s)\,\mathrm{d}s \\ & - \frac{\lambda t^{\alpha-1}}{P\Gamma(\alpha+\beta)} \int_0^{\eta} (\eta-s)^{\alpha+\beta-1} y(s)\,\mathrm{d}s. \end{aligned}$$

For $t \le \eta$, one has

$$\begin{aligned} u(t) = {} & -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\,\mathrm{d}s + \frac{t^{\alpha-1}}{P\Gamma(\alpha)} \left\{ \int_0^t + \int_t^{\eta} + \int_{\eta}^1 \right\} (1-s)^{\alpha-1} y(s)\,\mathrm{d}s \\ & - \frac{\lambda t^{\alpha-1}}{P\Gamma(\alpha+\beta)} \left\{ \int_0^t + \int_t^{\eta} \right\} (\eta-s)^{\alpha+\beta-1} y(s)\,\mathrm{d}s \\ = {} & \int_0^t \frac{-P\Gamma(\alpha+\beta)(t-s)^{\alpha-1} + \Gamma(\alpha+\beta)(1-s)^{\alpha-1}t^{\alpha-1} - \Gamma(\alpha)\lambda(\eta-s)^{\alpha+\beta-1}t^{\alpha-1}}{P\Gamma(\alpha)\Gamma(\alpha+\beta)} y(s)\,\mathrm{d}s \end{aligned}$$

$$\begin{aligned}
&+\int_t^\eta \frac{\Gamma(\alpha+\beta)(1-s)^{\alpha-1}t^{\alpha-1}-\Gamma(\alpha)\lambda(\eta-s)^{\alpha+\beta-1}t^{\alpha-1}}{P\Gamma(\alpha)\Gamma(\alpha+\beta)}y(s)\,\mathrm{d}s \\
&+\int_\eta^1 \frac{\Gamma(\alpha+\beta)(1-s)^{\alpha-1}t^{\alpha-1}}{P\Gamma(\alpha)\Gamma(\alpha+\beta)}y(s)\,\mathrm{d}s \\
={}&\int_0^1 G(t,s)y(s)\,\mathrm{d}s.
\end{aligned}$$

For $t \geq \eta$, one has

$$\begin{aligned}
u(t) ={}& -\frac{1}{\Gamma(\alpha)}\left\{\int_0^\eta + \int_\eta^t\right\}(t-s)^{\alpha-1}y(s)\,\mathrm{d}s + \frac{t^{\alpha-1}}{P\Gamma(\alpha)}\left\{\int_0^\eta + \int_\eta^t + \int_t^1\right\}(1-s)^{\alpha-1}y(s)\,\mathrm{d}s \\
&- \frac{\lambda t^{\alpha-1}}{P\Gamma(\alpha+\beta)}\int_0^\eta (\eta-s)^{\alpha+\beta-1}y(s)\,\mathrm{d}s \\
={}& \int_0^\eta \frac{-P\Gamma(\alpha+\beta)(t-s)^{\alpha-1}+\Gamma(\alpha+\beta)(1-s)^{\alpha-1}t^{\alpha-1}-\Gamma(\alpha)\lambda(\eta-s)^{\alpha+\beta-1}t^{\alpha-1}}{P\Gamma(\alpha)\Gamma(\alpha+\beta)}y(s)\,\mathrm{d}s \\
&+ \int_\eta^t \frac{-P\Gamma(\alpha+\beta)(t-s)^{\alpha-1}+\Gamma(\alpha+\beta)(1-s)^{\alpha-1}t^{\alpha-1}}{P\Gamma(\alpha)\Gamma(\alpha+\beta)}y(s)\,\mathrm{d}s \\
&+ \int_t^1 \frac{\Gamma(\alpha+\beta)(1-s)^{\alpha-1}t^{\alpha-1}}{P\Gamma(\alpha)\Gamma(\alpha+\beta)}y(s)\,\mathrm{d}s \\
={}& \int_0^1 G(t,s)y(s)\,\mathrm{d}s.
\end{aligned}$$

The proof is finished. □

Lemma 2.2 ([16]) *The Green's function $G(t,s)$ has the following properties*:

(1) $G(t,s) > 0$, $\forall t,s \in (0,1)$;
(2) $G(t,s) \leq \frac{(1-s)^{\alpha-1}t^{\alpha-1}}{P\Gamma(\alpha)}$, $\forall t,s \in (0,1)$;
(3) $G(t,s) \geq \frac{\lambda t^{\alpha-1}\eta^{\alpha+\beta-1}}{P\Gamma(\alpha+\beta)}\{(1-s)^{\alpha-1}-(1-s)^{\alpha+\beta-1}\}$, $\forall t,s \in (0,1)$;
(4) $t^{\alpha-1}w_1(s) \leq G(t,s) \leq t^{\alpha-1}w_2(s)$, $\forall t,s \in (0,1)$,

with $w_1(s) = \frac{\lambda\eta^{\alpha+\beta-1}}{P\Gamma(\alpha+\beta)}\{(1-s)^{\alpha-1}-(1-s)^{\alpha+\beta-1}\}$, $w_2(s) = \frac{(1-s)^{\alpha-1}}{P\Gamma(\alpha)}$.

Definition 2.3 ([18]) We say that a bounded linear operator $T : E \to E$ is u_0-positive on the cone P_c if there exists $u_0 \in P_c \setminus \{\theta\}$ such that for every $x \in P_c \setminus \{\theta\}$ there exist a natural number n and positive functions $\alpha(x) > 0$, $\beta(x) > 0$ such that

$$\alpha(x)u_0 \leq T^n x \leq \beta(x)u_0,$$

where θ is the zero function in E. Furthermore, if $u_0 = \varphi_1$, the positive eigenfunction of T corresponding to its first eigenvalue λ_1, then T is a φ_1-positive operator.

Lemma 2.3 (Krein-Rutmann theorem [19]) *Suppose that $T : E \to E$ is a completely continuous linear operator and $T(P_c) \subseteq P_c$. If there exist $\psi \in C[0,1]\backslash(-P_c)$ and a constant $c > 0$ such that $cT\psi \geq \psi$, then the spectral radius $r(T) \neq 0$ and A has a positive eigenfunction φ_1 corresponding to its first eigenvalue $\lambda_1 = (r(T))^{-1}$.*

Define the operator $A : C[0,1] \to C[0,1]$ by

$$(Au)(t) = \int_0^1 G(t,s)p(s)f\bigl(s,u(s)\bigr)\,\mathrm{d}s, \quad t \in [0,1]. \tag{2.2}$$

It is not hard to see that the fixed points of operator A coincide with the solutions to the problem (1.1). It is obvious that $A(P_c) \subseteq P_c$ when the assumptions (A_1) and (A_2) hold. Applying the Arzel-Ascoli theorem, we conclude that A is a completely continuous operator.

Define the operator $T : C[0,1] \to C[0,1]$ by

$$(Tu)(t) = \int_0^1 G(t,s)p(s)u(s)\,\mathrm{d}s, \quad t \in [0,1]. \tag{2.3}$$

It is not difficult to verify that $T : P_c \to P_c$ is a completely continuous linear operator. By virtue of the Krein-Rutman theorem, we have the following lemma.

Lemma 2.4 *Suppose T is defined by* (2.3), *then the spectral radius $r(T) \neq 0$ and A has a positive eigenfunction φ_1 corresponding to its first eigenvalue $\lambda_1 = (r(T))^{-1}$.*

Proof By Lemma 2.2, $G(t,s) > 0$ for all $t,s \in (0,1)$. Take $[t_1,t_2] \subset (0,1)$, $p(t) > 0$, $\forall t \in [t_1,t_2]$, choose $\psi \in E$ such that $\psi(t) \geq 0$ for all $t \in [0,1]$, $\psi(t) > 0$ for all $t \in [t_1,t_2]$, $\psi(t) = 0$ for all $t \in [0,t_1) \cup (t_2,1]$. Thus, we have

$$(T\psi)(t) = \int_0^1 G(t,s)p(s)\psi(s)\,\mathrm{d}s = \int_{t_1}^{t_2} G(t,s)p(s)\psi(s)\,\mathrm{d}s > 0.$$

So, there exists a constant $c > 0$ such that $c(T\psi)(t) \geq \psi(t)$, $\forall t \in [0,1]$. By Lemma 2.3, we complete the proof. □

Lemma 2.5 *T is a u_0-positive operator with $u_0(t) = t^{\alpha-1}$. In addition, T is a φ_1-positive operator, where φ_1 is the positive eigenfunction corresponding to its first eigenvalue.*

Proof For any $x \in P_c\backslash\{\theta\}$, by Lemma 2.2, we have

$$\begin{aligned}(Tx)(t) &= \int_0^1 G(t,s)p(s)x(s)\,\mathrm{d}s \\ &\leq \int_0^1 w_2(s)p(s)x(s)\,\mathrm{d}s \cdot t^{\alpha-1}.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}(Tx)(t) &= \int_0^1 G(t,s)p(s)x(s)\,\mathrm{d}s \\ &\geq \int_0^1 w_1(s)p(s)x(s)\,\mathrm{d}s \cdot t^{\alpha-1}.\end{aligned}$$

Therefore, T is a u_0-positive operator with $u_0(t) = t^{\alpha-1}$, *i.e.*

$$\alpha(x)u_0 \leq Tx \leq \beta(x)u_0, \quad \forall x \in P_c\backslash\{\theta\}.$$

Let φ_1 is the positive eigenfunction of T corresponding to λ_1, *i.e.* $\varphi_1 = \lambda_1 T\varphi_1$. Then there exist $\tilde{\alpha}(\varphi_1), \tilde{\beta}(\varphi_1) > 0$ such that

$$\tilde{\alpha}(\varphi_1)u_0 \leq T\varphi_1 = \frac{1}{\lambda_1}\varphi_1 \leq \tilde{\beta}(\varphi_1)u_0.$$

Hence, we see that T is a φ_1-positive operator.

The proof is completed. □

Lemma 2.6 ([18, 19]) *Let $\Omega \subset E$ be a bounded open set, and $A : \overline{\Omega} \cap P_c \to P_c$ is completely continuous.*

(i) *If there exists $u_0 \in P_c \setminus \{\theta\}$ such that $u - Au \neq \mu u_0$, $\forall \mu \geq 0$, $u \in \partial\Omega \cap P_c$, then $i(A, \Omega \cap P_c, P_c) = 0$.*

(ii) *Let $\theta \in \Omega \subset E$, if $u \neq \mu Au$, $\forall u \in \partial\Omega \cap P_c$, $0 \leq \mu \leq 1$, then $i(A, \Omega \cap P_c, P_c) = 1$.*

Lemma 2.7 ([18, 19]) *Let Ω_1 and Ω_2 be two bounded open sets in E such that $\theta \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Let operator $A : (\overline{\Omega}_2 \setminus \Omega_1) \cap P_c \to P_c$ is completely continuous. Suppose that one of the two conditions is satisfied*:

(i) *$Au \ngeq u$, $\forall u \in P_c \cap \partial\Omega_1$, $Au \nleq u$, $\forall u \in P_c \cap \partial\Omega_2$;*

(ii) *$Au \nleq u$, $\forall u \in P_c \cap \partial\Omega_1$, $Au \ngeq u$, $\forall u \in P_c \cap \partial\Omega_2$.*

Then A has at least one fixed point in $(\Omega_2 \setminus \overline{\Omega}_1) \cap P_c$.

3 Main results

3.1 Existence and multiplicity results

For convenience, we list the following assumptions:

(H_1) $\underline{\lim}_{u\to 0^+} \min_{t\in[0,1]} \frac{f(t,u)}{u} > \lambda_1$;
(H_2) $\overline{\lim}_{u\to \infty} \max_{t\in[0,1]} \frac{f(t,u)}{u} < \lambda_1$;
(H_3) $\overline{\lim}_{u\to 0^+} \max_{t\in[0,1]} \frac{f(t,u)}{u} < \lambda_1$;
(H_4) $\underline{\lim}_{u\to \infty} \min_{t\in[0,1]} \frac{f(t,u)}{u} > \lambda_1$;
(H_5) there exist $r_0 > 0$, such that

$$f\bigl(t,u(t)\bigr) < \left[\int_0^1 w_2(s)p(s)\,\mathrm{d}s\right]^{-1} r_0, \quad \forall 0 < u \leq r_0, t \in [0,1];$$

(H_6) there exist $\bar{r}_0 > 0$, such that

$$f\bigl(t,u(t)\bigr) > \left[\int_\tau^{1-\tau} \tau^{\alpha-1} w_1(s)p(s)\,\mathrm{d}s\right]^{-1} \bar{r}_0,$$

$\forall 0 < u \leq \bar{r}_0$, $t \in [\tau, 1-\tau]$, where $\tau \in (0,1)$, such that $p(t) \not\equiv 0$, $t \in [\tau, 1-\tau]$.

Theorem 3.1 *Suppose that conditions* (H_1)-(H_2) *hold, then the FBVP* (1.1) *has at least one positive solution.*

Proof By (H_1), there exist $r > 0$, $\varepsilon > 0$ such that

$$f(t,u) \geq (\lambda_1 + \varepsilon)u, \quad t \in [0,1], u \in [0,r].$$

Let φ_1 be the positive eigenfunction of T corresponding to λ_1, *i.e.* $\varphi_1 = \lambda_1 T\varphi_1$. For all $u \in \overline{B}_r \cap P_c$, we have

$$(Au)(t) \geq (\lambda_1 + \varepsilon)\int_0^1 G(t,s)p(s)u(s)\,\mathrm{d}s = (\lambda_1 + \varepsilon)(Tu)(t), \quad t \in [0,1].$$

Without loss of generality, we may assume that A has no fixed points on $\partial B_r \cap P_c$. Now we claim that

$$u - Au \neq \mu\varphi_1, \quad \forall u \in \partial B_r \cap P_c, \mu \geq 0. \tag{3.1}$$

Otherwise, there exist $u_1 \in \partial B_r \cap P_c$ and $\mu_1 \geq 0$ such that $u_1 - Au_1 = \mu_1\varphi_1$, then $u_1 \geq \mu_1\varphi_1$. Put $\tau^* = \sup\{\tau \mid u_1 \geq \tau\varphi_1\}$. T is a positively linear operator, then we have

$$(\lambda_1 + \varepsilon)T(u_1) \geq \lambda_1 T(u_1) \geq \tau^*\lambda_1 T(\varphi_1) = \tau^*\varphi_1.$$

Thus, $u_1 = Au_1 + \mu_1\varphi_1 \geq (\lambda_1 + \varepsilon)Tu_1 + \mu_1\varphi_1 \geq (\tau^* + \mu_1)\varphi_1$, which contradicts the definition of τ^*. Hence, (3.1) holds. By Lemma 2.6 yields

$$i(A, B_r \cap P_c, P_c) = 0. \tag{3.2}$$

In addition, by (H_2), there exist $\varepsilon \in (0, \lambda_1)$, $m > 0$ such that $f(t,u) \leq (\lambda_1 - \varepsilon)u + m$, $\forall u \geq R_1$, $t \in [0,1]$. Let

$$W := \{u \in P_c \mid u = \mu Au, \mu \in [0,1]\}; \tag{3.3}$$

$$\begin{aligned} u(t) = \mu(Au)(t) &\leq \int_0^1 G(t,s)p(s)f\bigl(s, u(s)\bigr)\,\mathrm{d}s \\ &\leq \int_0^1 G(t,s)p(s)\bigl((\lambda_1 - \varepsilon)u(s) + m\bigr)\,\mathrm{d}s \\ &= (\lambda_1 - \varepsilon)(Tu)(t) + u_0, \end{aligned} \tag{3.4}$$

where $u_0 := m\int_0^1 G(t,s)q(s)\,\mathrm{d}s$. Notice that $r((\lambda_1 - \varepsilon)T) < 1$, thus the operator $I - (\lambda_1 - \varepsilon)T$ is invertible. Combining this with (3.4) yields $u \leq ((\lambda_1 - \varepsilon)T)^{-1}u_0$, this implies that W is bounded. Choose $R > \max\{R_1, \sup W\}$, we have

$$u \neq \mu Au, \quad \forall u \in \partial B_R \cap P_c, \mu \in [0,1].$$

By Lemma 2.6,

$$i(A, B_R \cap P_c, P_c) = 1. \tag{3.5}$$

By (3.2) and (3.5), we have $i(A, (B_R \setminus \overline{B}_r) \cap P_c, P_c) = i(A, B_R \cap P_c, P_c) - i(A, B_r \cap P_c, P_c) = 1$. Hence the operator A has at least one fixed point on $(B_R \setminus \overline{B}_r) \cap P_c$.

The proof is finished. □

Theorem 3.2 *Suppose that conditions* (H_3), (H_4) *hold, then the FBVP* (1.1) *has at least one positive solution.*

Proof The proof is similar to Theorem 3.2 of [13, 17], so we omit the details.
By (H_3) and Lemma 2.7, we get

$$i(A, B_r \cap P_c, P_c) = 1. \tag{3.6}$$

By (H_4) and Lemma 2.7, we get

$$i(A, B_R \cap P_c, P_c) = 0. \tag{3.7}$$

Hence the operator A has at least one fixed point on $(B_R \setminus \overline{B}_r) \cap P_c$. □

Theorem 3.3 *Suppose that conditions* (H_1), (H_5) *hold, then the FBVP* (1.1) *has at least one positive solution.*

Proof Similar to the proof of Theorem 3.1, by (H_1), we have

$$i(A, B_r \cap P_c, P_c) = 0. \tag{3.8}$$

By (H_5), choose $r_0 > r$, we have

$$f\bigl(t, u(t)\bigr) < \left[\int_0^1 w_2(s)p(s)\,\mathrm{d}s\right]^{-1} r_0, \quad \forall 0 < u \le r_0, t \in [0,1].$$

Now we claim that

$$u \ne \mu A u, \quad \forall u \in \partial B_{r_0} \cap P_c, \mu \in [0,1]. \tag{3.9}$$

If otherwise, there exist $u_1 \in \partial B_{r_0} \cap P_c$, and $\mu_1 \in [0,1]$ such that $u_1 = \mu_1 A u_1$. Noticing

$$Au_1 = \int_0^1 G(t,s)p(s)f\bigl(s, u_1(s)\bigr)\,\mathrm{d}s < \int_0^1 w_2(s)p(s)f\bigl(s, u_1(s)\bigr)\,\mathrm{d}s < r_0 = \|u_1\|, \quad t \in [0,1].$$

Thus,

$$\|u_1\| > \|Au_1\| \ge \mu_1 \|Au_1\|,$$

which contradicts the fact of $u_1 = \mu_1 A u_1$. Then (3.9) holds. By Lemma 2.6,

$$i(A, B_{r_0} \cap P_c, P_c) = 1. \tag{3.10}$$

Therefore, $i(A, (B_{r_0} \setminus \overline{B}_r) \cap P_c, P_c) = i(A, B_{r_0} \cap P_c, P_c) - i(A, B_r \cap P_c, P_c) = 1$, then the FBVP (1.1) has at least one positive solution in $(B_{r_0} \setminus \overline{B}_r) \cap P_c$.
The proof is finished. □

Theorem 3.4 *Suppose that conditions* (H_2), (H_6) *hold, then the FBVP* (1.1) *has at least one positive solution.*

Proof By (H_2), we have

$$i(T, B_R \cap P_c, P_c) = 1. \tag{3.11}$$

By (H_6), choose $R > \bar{r}_0$, we have

$$f\bigl(t, u(t)\bigr) > \left[\int_\tau^{1-\tau} \tau^{\alpha-1} w_1(s) p(s)\,\mathrm{d}s\right]^{-1} \bar{r}_0,$$

$\forall 0 < u \leq \bar{r}_0$, $t \in [\tau, 1-\tau]$, where $\tau \in (0,1)$, such that $p(t) \not\equiv 0$, $t \in [\tau, 1-\tau]$. Now we claim that

$$Au \nleq u, \quad \forall u \in \partial B_{\bar{r}_0} \cap P_c. \tag{3.12}$$

If otherwise, there exist $u_1 \in \partial B_{\bar{r}_0} \cap P_c$ such that $Au_1 \leq u_1$. Noticing

$$\begin{aligned} Au_1 &= \int_0^1 G(t,s) p(s) f\bigl(s, u_1(s)\bigr)\,\mathrm{d}s \geq \int_0^1 s^{\alpha-1} w_1(s) p(s) f\bigl(s, u_1(s)\bigr)\,\mathrm{d}s \\ &> \int_\tau^{1-\tau} \tau^{\alpha-1} w_1(s) p(s) f\bigl(s, u_1(s)\bigr)\,\mathrm{d}s > \bar{r}_0 = \|u_1\|, \quad t \in [0,1]. \end{aligned}$$

Thus, $\|Au_1\| > \|u_1\|$, which contradicts the fact of $Au \nleq u$. Then (3.12) holds. By Lemma 2.7,

$$i(A, B_{\bar{r}_0} \cap P_c, P_c) = 0. \tag{3.13}$$

Thus the FBVP (1.1) has at least one positive solution in $(B_R \setminus \overline{B}_{\bar{r}_0}) \cap P_c$.

The proof is finished. □

Similarly, we can get the following result.

Corollary 3.1 *Suppose that conditions* (H_4), (H_5) *hold, then the FBVP* (1.1) *has at least one positive solution.*

Corollary 3.2 *Suppose that conditions* (H_3), (H_6) *hold, then the FBVP* (1.1) *has at least one positive solution.*

Corollary 3.3 *Suppose that conditions* (H_1), (H_4), (H_5) *hold, then the FBVP* (1.1) *has at least two positive solutions.*

Corollary 3.4 *Suppose that conditions* (H_2), (H_3), (H_6) *hold, then the FBVP* (1.1) *has at least two positive solutions.*

3.2 Uniqueness results

Theorem 3.5 *Suppose that there exists* $k \in [0,1)$ *such that*

$$\bigl|f(t,u) - f(t,v)\bigr| \leq k\lambda_1 |u - v|, \quad \forall t \in [0,1], u, v \in P_c,$$

where λ_1 is the first eigenvalue of T. Then the FBVP (1.1) *has a unique positive solution u^*, moreover, for any $u_0 \in P_c$, there exist iterative sequences $\{u_n\}_{n=0}^{\infty}$ with*

$$u_{n+1} = Au_n, \qquad \lim_{n\to\infty} u_n = u^*, n = 0, 1, 2, \ldots.$$

Proof First of all, it is not hard to see that the fixed points of operator A coincide with the solutions to the problem (1.1).

Second, we will show that A has fixed points in P_c. For any given $u_0 \in P_c$, let $u_{n+1} = Au_n$. By Lemma 2.5, there exists $\beta = \beta(|u_1 - u_0|) > 0$, such that

$$\bigl(T|u_1 - u_0|\bigr)(t) \leq \beta\varphi_1(t), \qquad t \in [0, 1].$$

Then, for any $m \in \mathbb{N}$, we have

$$\begin{aligned}
|u_{m+1} - u_m| &= \bigl|(Au_m)(t) - (Au_{m-1})(t)\bigr| \\
&= \left|\int_0^1 G(t,s)p(s)\bigl[f\bigl(s, u_m(s)\bigr) - f\bigl(s, u_{m-1}(s)\bigr)\bigr]\,\mathrm{d}s\right| \\
&\leq \int_0^1 G(t,s)p(s)\bigl|f\bigl(s, u_m(s)\bigr) - f\bigl(s, u_{m-1}(s)\bigr)\bigr|\,\mathrm{d}s \\
&\leq k\lambda_1 \int_0^1 G(t,s)p(s)|u_m - u_{m-1}|\,\mathrm{d}s \\
&= k\lambda_1 T\bigl(|u_m - u_{m-1}|\bigr)(t) \leq \cdots \\
&\leq k^m\lambda_1^m T^m\bigl(|u_1 - u_0|\bigr)(t) \\
&\leq k^m\lambda_1^m T^{m-1}\beta\varphi_1(t) = \beta k^m\lambda_1^{m-1}T^{m-2}\varphi_1(t) \\
&= \beta k^m\lambda_1\varphi_1(t).
\end{aligned}$$

Thus, for $n, m \in \mathbb{N}$, we have

$$\begin{aligned}
|u_{n+m+1} - u_n| &= |u_{n+m+1} - u_{n+m} + \cdots + u_{n+1} - u_n| \\
&\leq |u_{n+m+1} - u_{n+m}| + \cdots + |u_{n+1} - u_n| \\
&\leq \beta\bigl[k^{n+m} + \cdots + k^n\bigr]\lambda_1\varphi_1(t) \\
&= \beta\lambda_1\frac{k^n(1 - k^{m+1})}{1 - k}\varphi_1(t).
\end{aligned}$$

Therefore,

$$0 \leq \|u_{n+m+1} - u_n\| \leq \beta\lambda_1\frac{k^n(1 - k^{m+1})}{1 - k}\bigl\|\varphi_1(t)\bigr\| \to 0, \qquad \text{as } n, m \to 0.$$

By the completeness of E, there exist a $u^* \in P_c$ such that $\lim_{n\to\infty} u_n = u^*$.

Thus, $u^* = \lim_{n\to\infty} u_{n+1} = \lim_{n\to\infty} Au_n = Au^*$, A have fixed points in P_c.

Finally, we will show that A has at most one fixed point in P_c.

Suppose there exist two fixed points $u, v \in P_c$, $u = Au$, $v = Av$. By Lemma 2.5, there exists $\beta = \beta(|u - v|) > 0$, such that

$$\bigl(T|u - v|\bigr)(t) \le \beta\varphi_1(t), \quad t \in [0,1].$$

Then $\forall n \in \mathbb{N}$, the following hold:

$$|u - v| = \bigl|A^n u - A^n v\bigr| \le \beta k^m \lambda_1 \varphi_1.$$

This means that $u = v$, A has at most one fixed point in P_c.

The proof is completed. □

Remark 3.1 The iterative sequences in Theorem 3.5 starting with a simple function is helpful for calculating.

By the same method of [12], we have the following results. We omit the details.

Corollary 3.5 *Suppose that there exists $u_0 \in P_c$ such that*

$$\begin{aligned} &D_{0^+}^{\alpha} u_0(t) + p(t)f\bigl(t, u_0(t)\bigr) \ge 0, \quad 0 < t < 1, \\ &u_0(0) = u_0'(0) = \cdots = u_0^{(n-2)}(0) = 0, \qquad u_0(1) \le \lambda I_{0^+}^{\beta} u_0(\eta), \end{aligned}$$

and $\forall u, v \in \Omega$, $u(t) \ge v(t)$, there exist $k \in [0,1)$ such that

$$0 \le f\bigl(t, u(t)\bigr) - f\bigl(t, v(t)\bigr) \le k\lambda_1\bigl(u(t) - v(t)\bigr),$$

where $\Omega = \{u \in P_c \mid u \ge u_0\}$. Then FBVP (1.1) *has a unique solution u^* with $\lim_{n\to\infty} u_n = u^*$, $u_{n+1} = Au_n$, $n = 0, 1, 2, \ldots$.*

Corollary 3.6 *Suppose that there exists $u_0 \in P_c$ such that*

$$\begin{aligned} &D_{0^+}^{\alpha} u_0(t) + p(t)f\bigl(t, u_0(t)\bigr) \le 0, \quad 0 < t < 1, \\ &u_0(0) = u_0'(0) = \cdots = u_0^{(n-2)}(0) = 0, \qquad u_0(1) \ge \lambda I_{0^+}^{\beta} u_0(\eta), \end{aligned}$$

and $\forall u, v \in \Omega$, $u(t) \ge v(t)$, there exist $k \in [0,1)$ such that

$$0 \le f\bigl(t, u(t)\bigr) - f\bigl(t, v(t)\bigr) \le k\lambda_1\bigl(u(t) - v(t)\bigr),$$

where $\Omega = \{u \in P_c \mid u \le u_0\}$. Then FBVP (1.1) *has a unique solution u^* with $\lim_{n\to\infty} u_n = u^*$, $u_{n+1} = Au_n$, $n = 0, 1, 2, \ldots$.*

4 Examples

Example 4.1 Consider the following boundary value problem:

$$\begin{cases} D_{0+}^{7/2} u(t) + \frac{\lambda_0(1-t)^{-1/2}}{2(1+\lambda_0)}[3u + 1 + \sin u + u^2] = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = u''(0) = 0, \qquad u(1) = 5I_{0+}^{3/2} u(\frac{1}{2}), \end{cases} \tag{4.1}$$

where $\alpha = \frac{7}{2}$, $\beta = \frac{3}{2}$, $\eta = \frac{1}{2}$, $\lambda = 5$, $0 \leq \frac{\lambda\Gamma(\alpha)\eta^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \approx 0.0433 < 1$, and $p(t) = \frac{(1-t)^{-1/2}}{1+\lambda_0}$, $f(t,u(t)) = \frac{1}{2}\lambda_0[3u + 1 + \sin u + u^2]$, $\lambda_0 > \lambda_1$, λ_1 is the first eigenvalue of the operator T.

By simple computation, it is clear that (A_1), (A_2), (H_1), (H_4) hold.

Choose $r_0 = 1$, then $\forall 0 < u \leq 1$, $t \in [0,1]$, we have

$$f\bigl(t,u(t)\bigr) = \frac{1}{2}\lambda_0\bigl[3u + 1 + \sin u + u^2\bigr] \leq 3\lambda_0, \qquad \biggl[\int_0^1 w_2(s)p(s)\,ds\biggr]^{-1} r_0 \approx 9.538(1 + \lambda_0).$$

Thus, (H_5) holds.

It follows from Corollary 3.3 that FBVP (4.1) has at least two positive solutions.

Example 4.2 Consider the following boundary value problem:

$$\begin{cases} D_{0+}^{7/2}u(t) + \lambda_0(1-t)^{-1/2}[\frac{1}{3}u + 2 + t^3 + \sin t + \frac{1}{2}u^{1/2}] = 0, & 0 < t < 1, \\ u(0) = u'(0) = u''(0) = 0, & u(1) = 5I_{0+}^{3/2}u(\frac{1}{2}), \end{cases} \tag{4.2}$$

where $\alpha = \frac{7}{2}$, $\beta = \frac{3}{2}$, $\eta = \frac{1}{2}$, $\lambda = 5$, $0 \leq \frac{\lambda\Gamma(\alpha)\eta^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \approx 0.0433 < 1$, and $p(t) = (1-t)^{-1/2}$, $f(t,u(t)) = \lambda_0[\frac{1}{3}u + 2 + t^3 + \sin t + \frac{1}{2}u^{1/2}]$, $0 < \lambda_0 < \lambda_1$, λ_1 is the first eigenvalue of the operator T.

It is clear that (A_1), (A_2) hold.

For $\forall u, v \in P_c$, we have

$$\bigl|f\bigl(t,u(t)\bigr) - f\bigl(t,v(t)\bigr)\bigr| \leq \lambda_0 \left|\left[\frac{1}{3}(u - v) + \frac{1}{2}\bigl(u^{1/2} - v^{1/2}\bigr)\right]\right| \leq \lambda_1 \frac{5}{6}|u - v|.$$

It follows from Theorem 3.5 that FBVP (4.2) has a unique positive solution, moreover, for any $u_0 \in P_c$, there exist iterative sequences $\{u_n\}_{n=0}^{\infty}$ with

$$u_{n+1} = Au_n, \qquad \lim_{n\to\infty} u_n = u^*, n = 0, 1, 2, \ldots.$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript. All authors read and approved the final manuscript.

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