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Nonlocal boundary value problems for impulsive fractional q_k-difference equations

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Abstract

In this paper, we investigate the existence and uniqueness of solutions for a nonlocal boundary value problem of impulsive fractional q_k-difference equations involving a new q_k-shifting operator ${}_a\Phi_{q_k}(m) = q_k m + (1 - q_k)a$. Our main results rely on Banach's contraction mapping principle, Leray-Schauder nonlinear alternative, and Rothe fixed point theorem. Examples illustrating the obtained results are also presented.

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Keywords: quantum calculus; impulsive fractional q_k-difference equations; existence; uniqueness; fixed point theorem

1 Introduction

The main purpose of this manuscript is to study the existence and uniqueness of solutions for impulsive boundary value problems of fractional q_k-difference equations of the form

$$\begin{cases} {}_{t_k}D_{q_k}^{\alpha_k} x(t) = f(t, x(t)), \quad t \in J_k \subseteq [0, T], t \neq t_k, \\ {}_{t_k}I_{q_k}^{1-\alpha_k} x(t_k^+) - x(t_k) = \varphi_k(x(t_k)), \quad k = 1, 2, \ldots, m, \\ a_{t_0}I_{q_0}^{1-\alpha_0} x(0) = bx(T) + \sum_{l=0}^{m} c_{l\,t_l}I_{q_l}^{\gamma_l} x(t_{l+1}), \end{cases} \tag{1.1}$$

where $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, ${}_{t_k}D_{q_k}^{\alpha_k}$ denotes the Riemann-Liouville q_k-fractional derivative of order α_k on J_k, $0 < \alpha_k \leq 1$, $0 < q_k < 1$, $J_k = (t_k, t_{k+1}]$, $J_0 = [0, t_1]$, $k = 0, 1, \ldots, m$, $J = [0, T]$, $f \in C(J \times \mathbb{R}, \mathbb{R})$, $\varphi_k \in C(\mathbb{R}, \mathbb{R})$, $k = 1, 2, \ldots, m$, ${}_{t_k}I_{q_k}^{\alpha_k}$ denotes the Riemann-Liouville q_k-fractional integral of order $\alpha_k > 0$ on J_k, $a, b, c_l \in \mathbb{R}$, $\gamma_l > 0$, $l = 0, 1, 2, \ldots, m$.

The quantum calculus is known as the calculus without limits and provides a descent approach to deal with sets of nondifferentiable functions by considering difference operators. Quantum difference operators play an important role in several mathematical areas such as orthogonal polynomials, basic hyper-geometric functions, combinatorics, the calculus of variations, mechanics, and the theory of relativity. The book by Kac and Cheung [1] covers many fundamental aspects of the quantum calculus.

In recent years, the topic of q-calculus has attracted the attention of several researchers, and a variety of new results can be found in the papers [2–11] and the references therein.

In [12], the notions of q_k-derivative and q_k-integral for a function $f : J_k := [t_k, t_{k+1}] \to \mathbb{R}$, were introduced, and several their properties were obtained. Also, the existence and uniqueness results for initial value problems of first- and second-order impulsive

q_k-difference equations were studied. q_k-calculus analogues of some classical integral inequalities such as Hölder, Hermite-Hadamard, trapezoid, Ostrowski, Cauchy-Bunyakovsky-Schwarz, Grüss and Grüss-Čebyšev were proved in [13].

In [14], new concepts of fractional quantum calculus were defined by introducing a new q-shifting operator ${}_a\Phi_q(m) = qm + (1-q)a$. After giving the basic properties of the new q-shifting operator, the q-derivative and q-integral were defined. New definitions of the Riemann-Liouville fractional q-integral and q-difference on an interval $[a, b]$ were given, and their basic properties were discussed. As applications of the new concepts, existence and uniqueness results for first- and second-order initial value problems for impulsive fractional q-difference equations were presented. Recently, the existence of solutions for impulsive fractional q-difference equations with antiperiodic boundary conditions was discussed in [15], whereas the existence results for a nonlinear impulsive q_k-integral boundary value problem were obtained in [16].

In this paper, we consider a boundary value problem of impulsive fractional q_k-difference equations (1.1) by introducing a new q_k-shifting operator ${}_a\Phi_{q_k}(m) = q_k m + (1-q_k)a$ and establish some existence results for the new problem. The rest of this paper is organized as follows: In Section 2, we recall some known facts about fractional q_k-calculus, present an auxiliary lemma, which is used to convert problem (1.1) into a fixed point problem, and a lemma dealing with useful bounds. Section 3 contains the main results, whereas some illustrative examples are presented in Section 4.

2 Preliminaries

For any positive integer k, the q_k-shifting operator: ${}_a\Phi_{q_k}(m) = q_k m + (1-q_k)a$ [14] satisfies the relation

$$ {}_a\Phi_{q_k}^k(m) = {}_a\Phi_{q_k}^{k-1}\big({}_a\Phi_{q_k}(m)\big) \quad \text{with } {}_a\Phi_{q_k}^0(m) = m. $$

We define the power of q_k-shifting operator as

$$ {}_a(n-m)_{q_k}^{(0)} = 1, \qquad {}_a(n-m)_{q_k}^{(k)} = \prod_{i=0}^{k-1}\big(n - {}_a\Phi_{q_k}^i(m)\big), \quad k \in \mathbb{N} \cup \{\infty\}. $$

If $\gamma \in \mathbb{R}$, then

$$ {}_a(n-m)_{q_k}^{(\gamma)} = n^{(\gamma)} \prod_{i=0}^{\infty} \frac{1 - \frac{a}{n}\Phi_{q_k}^i(m/n)}{1 - \frac{a}{n}\Phi_{q_k}^{\gamma+i}(m/n)}, \quad n \neq 0. $$

The q_k-derivative of a function f on interval $[a, b]$ is defined by

$$ ({}_aD_{q_k}f)(t) = \frac{f(t) - f({}_a\Phi_{q_k}(t))}{(1-q_k)(t-a)}, \quad t \neq a \quad \text{and} \quad ({}_aD_{q_k}f)(a) = \lim_{t\to a}({}_aD_{q_k}f)(t), $$

and the q_k-derivative of higher order is given by

$$ \big({}_aD_{q_k}^k f\big)(t) = {}_aD_{q_k}^{k-1}({}_aD_{q_k}f)(t), \qquad \big({}_aD_{q_k}^0 f\big)(t) = f(t), \quad k \in \mathbb{N}. $$

The q_k-integral of a function f defined on the interval $[a,b]$ is given by

$$({}_aI_{q_k}f)(t) = \int_a^t f(s)_a\,ds = (1-q_k)(t-a)\sum_{i=0}^{\infty} q_k^i f\big({}_a\Phi_{q_k^i}(t)\big), \quad t \in [a,b]$$

and

$$\big({}_aI_{q_k}^k f\big)(t) = {}_aI_{q_k}^{k-1}({}_aI_{q_k}f)(t), \qquad \big({}_aI_{q_k}^0 f\big)(t) = f(t), \quad k \in \mathbb{N}.$$

The fundamental theorem of q_k-calculus applies to the operator ${}_aD_{q_k}$ and ${}_aI_{q_k}$ as follows:

$$({}_aD_{q_k}\,{}_aI_{q_k}f)(t) = f(t).$$

If f is continuous at $t = a$, then

$$({}_aI_{q_k}\,{}_aD_{q_k}f)(t) = f(t) - f(a).$$

The formula of q_k-integration by parts on the interval $[a,b]$ is

$$\int_a^b f(s)_a D_{q_k} g(s)_a\,d_{q_k}s = (fg)(t)\Big|_a^b - \int_a^b g\big({}_a\Phi_{q_k}(s)\big)_a D_{q_k} f(s)_a\,d_{q_k}s.$$

Now we recall the definitions of the Riemann-Liouville fractional q_k-integral and q_k-difference on interval $[a,b]$.

Definition 2.1 Let $\nu \geq 0$, and let f be a function defined on $[a,b]$. The fractional q_k-integral of Riemann-Liouville type is given by $({}_aI_{q_k}^0 f)(t) = h(t)$ and

$$\big({}_aI_{q_k}^{\nu} f\big)(t) = \frac{1}{\Gamma_{q_k}(\nu)}\int_a^t {}_a\big(t - {}_a\Phi_{q_k}(s)\big)_{q_k}^{(\nu-1)} f(s)_a\,d_{q_k}s, \quad \nu > 0, t \in [a,b].$$

Definition 2.2 The fractional q_k-derivative of Riemann-Liouville type of order $\nu \geq 0$ on the interval $[a,b]$ is defined by $({}_aD_{q_k}^0 f)(t) = f(t)$ and

$$\big({}_aD_{q_k}^{\nu} f\big)(t) = \big({}_aD_{q_k}^{l}\,{}_aI_{q_k}^{l-\nu} f\big)(t), \quad \nu > 0,$$

where l is the smallest integer greater than or equal to ν.

Lemma 2.1 *Let $\alpha, \beta \in \mathbb{R}^+$, and let f be a continuous function on $[a,b]$, $a \geq 0$. The Riemann-Liouville fractional q_k-integral has the following semigroup property*:

$${}_aI_{q_k}^{\beta}\,{}_aI_{q_k}^{\alpha} f(t) = {}_aI_{q_k}^{\alpha}\,{}_aI_{q_k}^{\beta} f(t) = {}_aI_{q_k}^{\alpha+\beta} f(t).$$

Lemma 2.2 *Let f be a q_k-integrable function on $[a,b]$. Then*

$${}_aD_{q_k}^{\alpha}\,{}_aI_{q_k}^{\alpha} f(t) = f(t) \quad \textit{for } \alpha > 0, t \in [a,b].$$

Lemma 2.3 *Let $\alpha > 0$, and let p be a positive integer. Then, for $t \in [a,b]$,*

$$ {}_aI^{\alpha}_{q_k}\,{}_aD^{p}_{q_k} f(t) = {}_aD^{p}_{q_k}\,{}_aI^{\alpha}_{q_k} f(t) - \sum_{k=0}^{p-1} \frac{(t-a)^{\alpha-p+k}}{\Gamma_{q_k}(\alpha+k-p+1)}\,{}_aD^{k}_{q_k} f(a). $$

From [14] we have the following formulas

$$ {}_aD^{\alpha}_{q_k}(t-a)^{\beta} = \frac{\Gamma_{q_k}(\beta+1)}{\Gamma_{q_k}(\beta-\alpha+1)}(t-a)^{\beta-\alpha}, \tag{2.1} $$

$$ {}_aI^{\alpha}_{q_k}(t-a)^{\beta} = \frac{\Gamma_{q_k}(\beta+1)}{\Gamma_{q_k}(\beta+\alpha+1)}(t-a)^{\beta+\alpha}. \tag{2.2} $$

In the sequel, we define $PC(J,\mathbb{R}) = \{x : J \to \mathbb{R}$, $x(t)$ is continuous everywhere except for some t_k at which $x(t_k^+)$ and $x(t_k^-)$ exist and $x(t_k^-) = x(t_k)$, $k = 1,2,3,\dots,m\}$. For $\beta \in \mathbb{R}_+$, we introduce the space $C_{\beta,k}(J_k,\mathbb{R}) = \{x : J_k \to \mathbb{R} : (t-t_k)^{\beta}x(t) \in C(J_k,\mathbb{R})\}$ with the norm $\|x\|_{C_{\beta,k}} = \sup_{t\in J_k} |(t-t_k)^{\beta}x(t)|$ and $PC_{\beta} = \{x : J \to \mathbb{R} :$ for each $t \in J_k$, $(t-t_k)^{\beta}x(t) \in C(J_k,\mathbb{R})$, $k = 0,1,2,\dots,m\}$ with the norm

$$ \|x\|_{PC_{\beta}} = \max\Big\{\sup_{t\in J_k}\big|(t-t_k)^{\beta}x(t)\big| : k = 0,1,2,\dots,m\Big\}. $$

Clearly, PC_{β} is a Banach space.

Lemma 2.4 *Let $y \in AC(J,\mathbb{R})$. Then $x \in PC(J,\mathbb{R})$ is a solution of*

$$ \begin{cases} {}_{t_k}D^{\alpha_k}_{q_k} x(t) = y(t), \quad t \in J, t \neq t_k, \\ {}_{t_k}I^{1-\alpha_k}_{q_k} x(t_k^+) - x(t_k) = \varphi_k(x(t_k)), \quad k = 1,2,\dots,m, \\ a_{t_0}I^{1-\alpha_0}_{q_0} x(0) = bx(T) + \sum_{l=0}^{m} c_{l\,t_l}I^{\gamma_l}_{q_l} x(t_{l+1}), \end{cases} \tag{2.3} $$

if and only if

$$ \begin{aligned} x(t) = {}& \frac{(t-t_k)^{\alpha_k-1}}{\Gamma_{q_k}(\alpha_k)}\left(\prod_{j=0}^{k-1}\frac{(t_{j+1}-t_j)^{\alpha_j-1}}{\Gamma_{q_j}(\alpha_j)}\right)\left\{\frac{b}{\Omega}\left[\sum_{j=0}^{m-1}\left(\prod_{j<i\le m}\frac{(t_{i+1}-t_i)^{\alpha_i-1}}{\Gamma_{q_i}(\alpha_i)}\right)\right.\right. \\ & \left.\times\left\{{}_{t_j}I^{\alpha_j}_{q_j} y(t_{j+1}) + \varphi_{j+1}\big(x(t_{j+1})\big)\right\}\right] + \frac{b}{\Omega}\,{}_{t_m}I^{\alpha_m}_{q_m} y(T) \\ & + \sum_{l=0}^{m}\frac{c_l(t_{l+1}-t_l)^{\alpha_l+\gamma_l-1}}{\Omega\Gamma_{q_l}(\alpha_l+\gamma_l)}\left[\sum_{j=0}^{l-1}\left(\prod_{j<i\le l-1}\frac{(t_{i+1}-t_i)^{\alpha_i-1}}{\Gamma_{q_i}(\alpha_i)}\right)\right. \\ & \left.\left.\times\left\{{}_{t_j}I^{\alpha_j}_{q_j} y(t_{j+1}) + \varphi_{j+1}\big(x(t_{j+1})\big)\right\}\right] + \sum_{l=0}^{m}\frac{c_l}{\Omega}\,{}_{t_l}I^{\alpha_l+\gamma_l}_{q_l} y(t_{l+1})\right\} \\ & + \frac{(t-t_k)^{\alpha_k-1}}{\Gamma_{q_k}(\alpha_k)}\left[\sum_{j=0}^{k-1}\left(\prod_{j<i\le k-1}\frac{(t_{i+1}-t_i)^{\alpha_i-1}}{\Gamma_{q_i}(\alpha_i)}\right)\left\{{}_{t_j}I^{\alpha_j}_{q_j} y(t_{j+1}) + \varphi_{j+1}\big(x(t_{j+1})\big)\right\}\right] \\ & + {}_{t_k}I^{\alpha_k}_{q_k} y(t), \end{aligned} \tag{2.4} $$

where $\sum_a^b(\cdot) = 0$, $\prod_a^b(\cdot) = 1$ *for* $a > b$, $\prod_{a<a}(\cdot) = 1$, *and the nonzero constant* Ω *is defined by*

$$\Omega = a - b\left(\prod_{j=0}^{m} \frac{(t_{j+1}-t_j)^{\alpha_j-1}}{\Gamma_{q_j}(\alpha_j)}\right) - \sum_{l=0}^{m} \frac{c_l(t_{l+1}-t_l)^{\alpha_l+\gamma_l-1}}{\Gamma_{q_l}(\alpha_l+\gamma_l)}\left(\prod_{j=0}^{l-1} \frac{(t_{j+1}-t_j)^{\alpha_j-1}}{\Gamma_{q_j}(\alpha_j)}\right). \tag{2.5}$$

Proof Applying the Riemann-Liouville fractional q_0-integral of order α_0 to both sides of the first equation of (2.3) for $t \in J_0$, we obtain

$${}_{t_0}I_{q_0}^{\alpha_0}\,{}_{t_0}D_{q_0}^{\alpha_0}x(t) = {}_{t_0}I_{q_0}^{\alpha_0}\,{}_{t_0}D_{q_0}\,{}_{t_0}I_{q_0}^{1-\alpha_0}x(t) = {}_{t_0}I_{q_0}^{\alpha_0}y(t). \tag{2.6}$$

From Lemmas 2.1, 2.2, and 2.3 for $t \in J_0$, we have

$$x(t) = \frac{t^{\alpha_0-1}}{\Gamma_{q_0}(\alpha_0)}\,{}_{t_0}I_{q_0}^{1-\alpha_0}x(0) + {}_{t_0}I_{q_0}^{\alpha_0}y(t).$$

For $t \in J_1$, applying the Riemann-Liouville fractional q_1-integral of order α_1 again to the first equation in (2.3) and using the previous process, we get

$$x(t) = \frac{(t-t_1)^{\alpha_1-1}}{\Gamma_{q_1}(\alpha_1)}\,{}_{t_1}I_{q_1}^{1-\alpha_1}x\bigl(t_1^+\bigr) + {}_{t_1}I_{q_1}^{\alpha_1}y(t). \tag{2.7}$$

The impulsive condition implies that

$$x(t) = \frac{(t-t_1)^{\alpha_1-1}}{\Gamma_{q_1}(\alpha_1)}\left[\frac{t_1^{\alpha_0-1}}{\Gamma_{q_0}(\alpha_0)}\,{}_{t_0}I_{q_0}^{1-\alpha_0}x(0) + {}_{t_0}I_{q_0}^{\alpha_0}y(t_1) + \varphi_1\bigl(x(t_1)\bigr)\right] + {}_{t_1}I_{q_1}^{\alpha_1}y(t).$$

Similarly, for $t \in J_2$, we have

$$x(t) = \frac{(t-t_2)^{\alpha_2-1}}{\Gamma_{q_2}(\alpha_2)}\left[\frac{(t_2-t_1)^{\alpha_1-1}}{\Gamma_{q_1}(\alpha_1)}\left(\frac{t_1^{\alpha_0-1}}{\Gamma_{q_0}(\alpha_0)}\,{}_{t_0}I_{q_0}^{1-\alpha_0}x(0) + {}_{t_0}I_{q_0}^{\alpha_0}y(t_1) + \varphi_1\bigl(x(t_1)\bigr)\right) + {}_{t_1}I_{q_1}^{\alpha_1}y(t_2) + \varphi_2\bigl(x(t_2)\bigr)\right] + {}_{t_2}I_{q_2}^{\alpha_2}y(t).$$

Repeating this process for $t \in J_k \subseteq J$, $k = 0, 1, 2, \ldots, m$, we obtain

$$x(t) = \frac{(t-t_k)^{\alpha_k-1}}{\Gamma_{q_k}(\alpha_k)}\left(\prod_{j=0}^{k-1} \frac{(t_{j+1}-t_j)^{\alpha_j-1}}{\Gamma_{q_j}(\alpha_j)}\right)\bigl({}_{t_0}I_{q_0}^{1-\alpha_0}x(0)\bigr) + \frac{(t-t_k)^{\alpha_k-1}}{\Gamma_{q_k}(\alpha_k)}\left[\sum_{j=0}^{k-1}\left(\prod_{j<i\le k-1} \frac{(t_{i+1}-t_i)^{\alpha_i-1}}{\Gamma_{q_i}(\alpha_i)}\right)\bigl\{{}_{t_j}I_{q_j}^{\alpha_j}y(t_{j+1}) + \varphi_{j+1}\bigl(x(t_{j+1})\bigr)\bigr\}\right] + {}_{t_k}I_{q_k}^{\alpha_k}y(t). \tag{2.8}$$

In particular, for $t = T$, we get

$$x(T) = \left(\prod_{j=0}^{m} \frac{(t_{j+1}-t_j)^{\alpha_j-1}}{\Gamma_{q_j}(\alpha_j)}\right)\left({}_{t_0}I_{q_0}^{1-\alpha_0}x(0)\right) + \left[\sum_{j=0}^{m-1}\left(\prod_{j<i\leq m} \frac{(t_{i+1}-t_i)^{\alpha_i-1}}{\Gamma_{q_i}(\alpha_i)}\right)\left\{{}_{t_j}I_{q_j}^{\alpha_j}y(t_{j+1}) + \varphi_{j+1}\left(x(t_{j+1})\right)\right\}\right] + {}_{t_m}I_{q_m}^{\alpha_m}y(T).$$

Taking the Riemann-Liouville fractional q_l-integral of order γ_l on (2.8) from t_l to t_{l+1} and using (2.2), we have

$$\begin{aligned} {}_{t_l}I_{q_l}^{\gamma_l}x(t_{l+1}) &= \frac{(t_{l+1}-t_l)^{\alpha_l+\gamma_l-1}}{\Gamma_{q_l}(\alpha_l+\gamma_l)}\left(\prod_{j=0}^{l-1} \frac{(t_{j+1}-t_j)^{\alpha_j-1}}{\Gamma_{q_j}(\alpha_j)}\right)\left({}_{t_0}I_{q_0}^{1-\alpha_0}x(0)\right) \\ &\quad + \frac{(t_{l+1}-t_l)^{\alpha_l+\gamma_l-1}}{\Gamma_{q_l}(\alpha_l+\gamma_l)}\left[\sum_{j=0}^{l-1}\left(\prod_{j<i\leq l-1} \frac{(t_{i+1}-t_i)^{\alpha_i-1}}{\Gamma_{q_i}(\alpha_i)}\right)\right. \\ &\quad \left.\times\left\{{}_{t_j}I_{q_j}^{\alpha_j}y(t_{j+1}) + \varphi_{j+1}\left(x(t_{j+1})\right)\right\}\right] + {}_{t_l}I_{q_l}^{\alpha_l+\gamma_l}y(t_{l+1}). \end{aligned}$$

By the boundary condition of (2.3) we find that

$$\begin{aligned} {}_{t_0}I_{q_0}^{1-\alpha_0}x(0) &= \frac{b}{\Omega}\left[\sum_{j=0}^{m-1}\left(\prod_{j<i\leq m} \frac{(t_{i+1}-t_i)^{\alpha_i-1}}{\Gamma_{q_i}(\alpha_i)}\right)\right. \\ &\quad \left.\times\left\{{}_{t_j}I_{q_j}^{\alpha_j}y(t_{j+1}) + \varphi_{j+1}\left(x(t_{j+1})\right)\right\}\right] + \frac{b}{\Omega}{}_{t_m}I_{q_m}^{\alpha_m}y(T) \\ &\quad + \sum_{l=0}^{m} \frac{c_l(t_{l+1}-t_l)^{\alpha_l+\gamma_l-1}}{\Omega\Gamma_{q_l}(\alpha_l+\gamma_l)}\left[\sum_{j=0}^{l-1}\left(\prod_{j<i\leq l-1} \frac{(t_{i+1}-t_i)^{\alpha_i-1}}{\Gamma_{q_i}(\alpha_i)}\right)\right. \\ &\quad \left.\times\left\{{}_{t_j}I_{q_j}^{\alpha_j}y(t_{j+1}) + \varphi_{j+1}\left(x(t_{j+1})\right)\right\}\right] + \sum_{l=0}^{m} \frac{c_l}{\Omega}{}_{t_l}I_{q_l}^{\alpha_l+\gamma_l}y(t_{l+1}). \end{aligned}$$

Substituting the value of ${}_{t_0}I_{q_0}^{1-\alpha_0}x(0)$ into (2.8) yields (2.4). The converse follows by direct computation. This completes the proof. □

Lemma 2.5 *Assume that all conditions of Lemma* 2.4 *hold. In addition, assume that* $\sup_{t\in J}|y(t)| = N_1$ *and there exists a constant* N_2 *such that* $|\varphi_k(x)| \leq N_2$ *for* $k = 1, 2, \ldots, m$ *and* $x \in \mathbb{R}$. *Then the following inequality holds*:

$$\left|x(t)\right| \leq \Psi_1 N_1 + \Psi_2 N_2 \tag{2.9}$$

for all $t \in J$, *where*

$$\begin{aligned} \Psi_1 &= \left(\prod_{j=0}^{m} \frac{(t_{j+1}-t_j)^{\alpha_j-1}}{\Gamma_{q_j}(\alpha_j)}\right)\left\{\frac{|b|}{|\Omega|}\left[\sum_{j=0}^{m}\left(\prod_{j<i\leq m} \frac{(t_{i+1}-t_i)^{\alpha_i-1}}{\Gamma_{q_i}(\alpha_i)}\right)\frac{(t_{j+1}-t_j)^{\alpha_j}}{\Gamma_{q_j}(\alpha_j+1)}\right]\right. \\ &\quad + \sum_{l=0}^{m} \frac{|c_l|(t_{l+1}-t_l)^{\alpha_l+\gamma_l-1}}{|\Omega|\Gamma_{q_l}(\alpha_l+\gamma_l)}\left[\sum_{j=0}^{l-1}\left(\prod_{j<i\leq l-1} \frac{(t_{i+1}-t_i)^{\alpha_i-1}}{\Gamma_{q_i}(\alpha_i)}\right)\frac{(t_{j+1}-t_j)^{\alpha_j}}{\Gamma_{q_j}(\alpha_j+1)}\right] \end{aligned}$$

$$+\sum_{l=0}^{m}\frac{|c_l|}{|\Omega|}\frac{(t_{l+1}-t_l)^{\alpha_l+\gamma_l}}{\Gamma_{q_l}(\alpha_l+\gamma_l+1)}\Bigg\}$$

$$+\sum_{j=0}^{m}\left(\prod_{j<i\le m}\frac{(t_{i+1}-t_i)^{\alpha_i-1}}{\Gamma_{q_i}(\alpha_i)}\right)\frac{(t_{j+1}-t_j)^{\alpha_j}}{\Gamma_{q_j}(\alpha_j+1)}$$

and

$$\Psi_2=\left(\prod_{j=0}^{m}\frac{(t_{j+1}-t_j)^{\alpha_j-1}}{\Gamma_{q_j}(\alpha_j)}\right)\Bigg\{\frac{|b|}{|\Omega|}\sum_{j=0}^{m-1}\left(\prod_{j<i\le m}\frac{(t_{i+1}-t_i)^{\alpha_i-1}}{\Gamma_{q_i}(\alpha_i)}\right)$$
$$+\sum_{l=0}^{m}\frac{|c_l|(t_{l+1}-t_l)^{\alpha_l+\gamma_l-1}}{|\Omega|\Gamma_{q_l}(\alpha_l+\gamma_l)}\sum_{j=0}^{l-1}\left(\prod_{j<i\le l-1}\frac{(t_{i+1}-t_i)^{\alpha_i-1}}{\Gamma_{q_i}(\alpha_i)}\right)\Bigg\}$$
$$+\sum_{j=0}^{m-1}\left(\prod_{j<i\le m}\frac{(t_{i+1}-t_i)^{\alpha_i-1}}{\Gamma_{q_i}(\alpha_i)}\right).$$

Proof For any $t\in J_k$, we have

$$\begin{aligned}
|x(t)| &\le \frac{(t-t_k)^{\alpha_k-1}}{\Gamma_{q_k}(\alpha_k)}\left(\prod_{j=0}^{k-1}\frac{(t_{j+1}-t_j)^{\alpha_j-1}}{\Gamma_{q_j}(\alpha_j)}\right)\Bigg\{\frac{|b|}{|\Omega|}\Bigg[\sum_{j=0}^{m-1}\left(\prod_{j<i\le m}\frac{(t_{i+1}-t_i)^{\alpha_i-1}}{\Gamma_{q_i}(\alpha_i)}\right)\\
&\quad\times\left\{{}_{t_j}I_{q_j}^{\alpha_j}|y(t_{j+1})|+|\varphi_{j+1}(x(t_{j+1}))|\right\}\Bigg]+\frac{|b|}{|\Omega|}{}_{t_m}I_{q_m}^{\alpha_m}|y(T)|\\
&\quad+\sum_{l=0}^{m}\frac{|c_l|(t_{l+1}-t_l)^{\alpha_l+\gamma_l-1}}{|\Omega|\Gamma_{q_l}(\alpha_l+\gamma_l)}\Bigg[\sum_{j=0}^{l-1}\left(\prod_{j<i\le l-1}\frac{(t_{i+1}-t_i)^{\alpha_i-1}}{\Gamma_{q_i}(\alpha_i)}\right)\\
&\quad\times\left\{{}_{t_j}I_{q_j}^{\alpha_j}|y(t_{j+1})|+|\varphi_{j+1}(x(t_{j+1}))|\right\}\Bigg]+\sum_{l=0}^{m}\frac{|c_l|}{|\Omega|}{}_{t_l}I_{q_l}^{\alpha_l+\gamma_l}|y(t_{l+1})|\Bigg\}\\
&\quad+\frac{(t-t_k)^{\alpha_k-1}}{\Gamma_{q_k}(\alpha_k)}\Bigg[\sum_{j=0}^{k-1}\left(\prod_{j<i\le k-1}\frac{(t_{i+1}-t_i)^{\alpha_i-1}}{\Gamma_{q_i}(\alpha_i)}\right)\left\{{}_{t_j}I_{q_j}^{\alpha_j}|y(t_{j+1})|+|\varphi_{j+1}(x(t_{j+1}))|\right\}\Bigg]\\
&\quad+{}_{t_k}I_{q_k}^{\alpha_k}|y(t)|\\
&\le \frac{(T-t_m)^{\alpha_m-1}}{\Gamma_{q_m}(\alpha_m)}\left(\prod_{j=0}^{m-1}\frac{(t_{j+1}-t_j)^{\alpha_j-1}}{\Gamma_{q_j}(\alpha_j)}\right)\Bigg\{\frac{|b|}{|\Omega|}\Bigg[\sum_{j=0}^{m-1}\left(\prod_{j<i\le m}\frac{(t_{i+1}-t_i)^{\alpha_i-1}}{\Gamma_{q_i}(\alpha_i)}\right)\\
&\quad\times\left\{N_1{}_{t_j}I_{q_j}^{\alpha_j}1+N_2\right\}\Bigg]+\frac{|b|}{|\Omega|}N_1{}_{t_m}I_{q_m}^{\alpha_m}1+\sum_{l=0}^{m}\frac{|c_l|(t_{l+1}-t_l)^{\alpha_l+\gamma_l-1}}{|\Omega|\Gamma_{q_l}(\alpha_l+\gamma_l)}\\
&\quad\times\Bigg[\sum_{j=0}^{l-1}\left(\prod_{j<i\le l-1}\frac{(t_{i+1}-t_i)^{\alpha_i-1}}{\Gamma_{q_i}(\alpha_i)}\right)\left\{N_1{}_{t_j}I_{q_j}^{\alpha_j}1+N_2\right\}\Bigg]+\sum_{l=0}^{m}\frac{|c_l|}{|\Omega|}N_1{}_{t_l}I_{q_l}^{\alpha_l+\gamma_l}1\Bigg\}\\
&\quad+\frac{(T-t_m)^{\alpha_m-1}}{\Gamma_{q_m}(\alpha_m)}\Bigg[\sum_{j=0}^{m-1}\left(\prod_{j<i\le m-1}\frac{(t_{i+1}-t_i)^{\alpha_i-1}}{\Gamma_{q_i}(\alpha_i)}\right)\left\{N_1{}_{t_j}I_{q_j}^{\alpha_j}1+N_2\right\}\Bigg]+N_1{}_{t_m}I_{q_m}^{\alpha_m}1\\
&=\left(\prod_{j=0}^{m}\frac{(t_{j+1}-t_j)^{\alpha_j-1}}{\Gamma_{q_j}(\alpha_j)}\right)\Bigg\{\frac{|b|}{|\Omega|}\Bigg[\sum_{j=0}^{m-1}\left(\prod_{j<i\le m}\frac{(t_{i+1}-t_i)^{\alpha_i-1}}{\Gamma_{q_i}(\alpha_i)}\right)
\end{aligned}$$

$$
\begin{aligned}
&\quad \times \left\{ N_1 \frac{(t_{j+1}-t_j)^{\alpha_j}}{\Gamma_{q_j}(\alpha_j+1)} + N_2 \right\} \Bigg] + \frac{|b|}{|\Omega|} N_1 \frac{(T-t_m)^{\alpha_m}}{\Gamma_{q_m}(\alpha_m+1)} + \sum_{l=0}^{m} \frac{|c_l|(t_{l+1}-t_l)^{\alpha_l+\gamma_l-1}}{|\Omega|\Gamma_{q_l}(\alpha_l+\gamma_l)} \\
&\quad \times \Bigg[\sum_{j=0}^{l-1} \Bigg(\prod_{j<i\le l-1} \frac{(t_{i+1}-t_i)^{\alpha_i-1}}{\Gamma_{q_i}(\alpha_i)} \Bigg) \left\{ N_1 \frac{(t_{j+1}-t_j)^{\alpha_j}}{\Gamma_{q_j}(\alpha_j+1)} + N_2 \right\} \Bigg] \\
&\quad + \sum_{l=0}^{m} \frac{|c_l|}{|\Omega|} N_1 \frac{(t_{l+1}-t_l)^{\alpha_l+\gamma_l}}{\Gamma_{q_l}(\alpha_l+\gamma_l+1)} \Bigg\} + \Bigg[\sum_{j=0}^{m-1} \Bigg(\prod_{j<i\le m} \frac{(t_{i+1}-t_i)^{\alpha_i-1}}{\Gamma_{q_i}(\alpha_i)} \Bigg) \\
&\quad \times \left\{ N_1 \frac{(t_{j+1}-t_j)^{\alpha_j}}{\Gamma_{q_j}(\alpha_j+1)} + N_2 \right\} \Bigg] + N_1 \frac{(T-t_m)^{\alpha_m}}{\Gamma_{q_m}(\alpha_m+1)} \\
&\le \Psi_1 N_1 + \Psi_2 N_2.
\end{aligned}
$$

This completes the proof. □

3 Main results

In view of Lemma 2.4, we define the operator $\mathcal{L} : PC(J,\mathbb{R}) \to PC(J,\mathbb{R})$ by

$$
\begin{aligned}
\mathcal{L}x(t) &= \frac{(t-t_k)^{\alpha_k-1}}{\Gamma_{q_k}(\alpha_k)} \Bigg(\prod_{j=0}^{k-1} \frac{(t_{j+1}-t_j)^{\alpha_j-1}}{\Gamma_{q_j}(\alpha_j)} \Bigg) \Bigg\{ \frac{b}{\Omega} \Bigg[\sum_{j=0}^{m-1} \Bigg(\prod_{j<i\le m} \frac{(t_{i+1}-t_i)^{\alpha_i-1}}{\Gamma_{q_i}(\alpha_i)} \Bigg) \\
&\quad \times \left\{ {}_{t_j}I_{q_j}^{\alpha_j} f\big(t_{j+1},x(t_{j+1})\big) + \varphi_{j+1}\big(x(t_{j+1})\big) \right\} \Bigg] + \frac{b}{\Omega} {}_{t_m}I_{q_m}^{\alpha_m} f\big(T,x(T)\big) \\
&\quad + \sum_{l=0}^{m} \frac{c_l(t_{l+1}-t_l)^{\alpha_l+\gamma_l-1}}{\Omega\Gamma_{q_l}(\alpha_l+\gamma_l)} \Bigg[\sum_{j=0}^{l-1} \Bigg(\prod_{j<i\le l-1} \frac{(t_{i+1}-t_i)^{\alpha_i-1}}{\Gamma_{q_i}(\alpha_i)} \Bigg) \\
&\quad \times \left\{ {}_{t_j}I_{q_j}^{\alpha_j} f\big(t_{j+1},x(t_{j+1})\big) + \varphi_{j+1}\big(x(t_{j+1})\big) \right\} \Bigg] + \sum_{l=0}^{m} \frac{c_l}{\Omega} {}_{t_l}I_{q_l}^{\alpha_l+\gamma_l} f\big(t_{l+1},x(t_{l+1})\big) \Bigg\} \\
&\quad + \frac{(t-t_k)^{\alpha_k-1}}{\Gamma_{q_k}(\alpha_k)} \Bigg[\sum_{j=0}^{k-1} \Bigg(\prod_{j<i\le k-1} \frac{(t_{i+1}-t_i)^{\alpha_i-1}}{\Gamma_{q_i}(\alpha_i)} \Bigg) \\
&\quad \times \left\{ {}_{t_j}I_{q_j}^{\alpha_j} f\big(t_{j+1},x(t_{j+1})\big) + \varphi_{j+1}\big(x(t_{j+1})\big) \right\} \Bigg] + {}_{t_k}I_{q_k}^{\alpha_k} f\big(t,x(t)\big),
\end{aligned}
\tag{3.1}
$$

where

$$
{}_aI_q^p f\big(u,x(u)\big) = \frac{1}{\Gamma_q(p)} \int_a^u {}_a\big(u - {}_a\Phi_q(s)\big)_q^{(p-1)} f\big(s,x(s)\big)_a \, d_q s,
$$

$a \in \{t_0,t_1,\ldots,t_m\}$, $q \in \{q_0,q_1,\ldots,q_m\}$, $p \in \{\alpha_0,\alpha_1,\ldots,\alpha_m,\alpha_0+\gamma_0,\alpha_1+\gamma_1,\ldots,\alpha_m+\gamma_m\}$, $u \in \{t,t_1,t_2,\ldots,t_m,T\}$.

Now we present our first result, which deals with the existence and uniqueness of solutions for problem (1.1) and is based on the Banach contraction mapping principle.

Theorem 3.1 *Assume that there exist a function $\mathcal{M} \in C(J,\mathbb{R}^+)$ and a positive constant M_2 such that*

(H_1) $|f(t,x) - f(t,y)| \le \mathcal{M}(t)|x-y|$ *and* $|\varphi_k(x) - \varphi_k(y)| \le M_2|x-y|$ *for* $t \in J$, $x,y \in \mathbb{R}$ *and* $k = 1,2,\ldots,m$.

Then problem (1.1) *has a unique solution on* J *if*

$$(M_1\Psi_1 + M_2\Psi_2)T^\beta < 1, \tag{3.2}$$

where $M_1 = \sup_{t\in J} |\mathcal{M}(t)|$, *the constants* Ψ_1, Ψ_2 *are defined in Lemma* 2.5, *and* $\beta > 0$.

Proof Consider the operator $\mathcal{L} : PC(J,\mathbb{R}) \to PC(J,\mathbb{R})$ defined by (3.1) and show that $\mathcal{L} \in PC_\beta$. For this, let $\tau_1, \tau_2 \in J_k$. Then we have

$$\begin{aligned}
&\bigl|(\tau_1 - t_k)^\beta \mathcal{L}x(\tau_1) - (\tau_2 - t_k)^\beta \mathcal{L}x(\tau_2)\bigr| \\
&\quad\le \left|\frac{(\tau_1 - t_k)^{\beta+\alpha_k-1} - (\tau_2 - t_k)^{\beta+\alpha_k-1}}{\Gamma_{q_k}(\alpha_k)}\right| K_x \\
&\qquad + \bigl|(\tau_1 - t_k)^\beta {}_{t_k}I_{q_k}^{\alpha_k} f\bigl(\tau_1, x(\tau_1)\bigr) - (\tau_2 - t_k)^\beta {}_{t_k}I_{q_k}^{\alpha_k} f\bigl(\tau_2, x(\tau_2)\bigr)\bigr|,
\end{aligned}$$

where

$$\begin{aligned}
K_x := {}& \left(\prod_{j=0}^{k-1} \frac{(t_{j+1} - t_j)^{\alpha_j - 1}}{\Gamma_{q_j}(\alpha_j)}\right) \Biggl\{ \frac{|b|}{|\Omega|} \Biggl[\sum_{j=0}^{m-1} \left(\prod_{j<i\le m} \frac{(t_{i+1} - t_i)^{\alpha_i - 1}}{\Gamma_{q_i}(\alpha_i)}\right) \\
& \times \Bigl\{ {}_{t_j}I_{q_j}^{\alpha_j} \bigl|f\bigl(t_{j+1}, x(t_{j+1})\bigr)\bigr| + \bigl|\varphi_{j+1}\bigl(x(t_{j+1})\bigr)\bigr| \Bigr\}\Biggr] + \frac{|b|}{|\Omega|} {}_{t_m}I_{q_m}^{\alpha_m} \bigl|f\bigl(T, x(T)\bigr)\bigr| \\
& + \sum_{l=0}^{m} \frac{|c_l|(t_{l+1} - t_l)^{\alpha_l + \gamma_l - 1}}{|\Omega|\Gamma_{q_l}(\alpha_l + \gamma_l)} \Biggl[\sum_{j=0}^{l-1} \left(\prod_{j<i\le l-1} \frac{(t_{i+1} - t_i)^{\alpha_i - 1}}{\Gamma_{q_i}(\alpha_i)}\right) \\
& \times \Bigl\{ {}_{t_j}I_{q_j}^{\alpha_j} \bigl|f\bigl(t_{j+1}, x(t_{j+1})\bigr)\bigr| + \bigl|\varphi_{j+1}\bigl(x(t_{j+1})\bigr)\bigr| \Bigr\}\Biggr] \\
& + \sum_{l=0}^{m} \frac{|c_l|}{|\Omega|} {}_{t_l}I_{q_l}^{\alpha_l + \gamma_l} \bigl|f\bigl(t_{l+1}, x(t_{l+1})\bigr)\bigr| \Biggr\} \\
& + \Biggl[\sum_{j=0}^{k-1} \left(\prod_{j<i\le k-1} \frac{(t_{i+1} - t_i)^{\alpha_i - 1}}{\Gamma_{q_i}(\alpha_i)}\right) \\
& \times \Bigl\{ {}_{t_j}I_{q_j}^{\alpha_j} \bigl|f\bigl(t_{j+1}, x(t_{j+1})\bigr)\bigr| + \bigl|\varphi_{j+1}\bigl(x(t_{j+1})\bigr)\bigr| \Bigr\}\Biggr].
\end{aligned} \tag{3.3}$$

As $\tau_1 \to \tau_2$, we get $|(\tau_1 - t_k)^\beta \mathcal{L}x(\tau_1) - (\tau_2 - t_k)^\beta \mathcal{L}x(\tau_2)| \to 0$ for each $k = 0, 1, \ldots, m$. Thus, $\mathcal{L}x(t) \in PC_\beta$.

Now we define the ball $B_r = \{x \in PC_\beta(J,\mathbb{R}) : \|x\|_{PC_\beta} \le r\}$. We will show that $\mathcal{L}B_r \subset B_r$. Let $\sup_{t\in J} |f(t,0)| = A_1$, $\max\{|\varphi(0)| : k = 1, \ldots, m\} = A_2$ and choose a constant r such that

$$r \ge \frac{(A_1\Psi_1 + A_2\Psi_2)T^\beta}{1 - (M_1\Psi_1 + M_2\Psi_2)T^\beta}.$$

Then, for any $x \in B_r$ and $t \in J$, we have

$$(t - t_k)^\beta \bigl|\mathcal{L}x(t)\bigr| \le \frac{(t - t_k)^{\beta+\alpha_k-1}}{\Gamma_{q_k}(\alpha_k)} K_x + (t - t_k)^\beta {}_{t_k}I_{q_k}^{\alpha_k} \bigl|f\bigl(t, x(t)\bigr)\bigr|, \tag{3.4}$$

where K_x is given by (3.3). Using the inequalities

$$\begin{aligned} \big|f(s,x)\big| &\le \big|f(s,x)-f(s,0)\big| + \big|f(s,0)\big| \le M_1 r + A_1, \\ \big|\varphi(x)\big| &\le \big|\varphi(x)-\varphi(0)\big| + \big|\varphi(0)\big| \le M_2 r + A_2 \end{aligned}$$

in (3.4) for $x \in B_r$ and $s \in J$ and the computational details of Lemma 2.5, together with

$$\begin{aligned} K_x \le{}& \left(\prod_{j=0}^{k-1} \frac{(t_{j+1}-t_j)^{\alpha_j-1}}{\Gamma_{q_j}(\alpha_j)}\right)\left\{\frac{|b|}{|\Omega|}\left[\sum_{j=0}^{m-1}\left(\prod_{j<i\le m}\frac{(t_{i+1}-t_i)^{\alpha_i-1}}{\Gamma_{q_i}(\alpha_i)}\right)\right.\right. \\ &\times \left.\left\{(M_1 r + A_1)\left(\frac{(t_{j+1}-t_j)^{\alpha_j}}{\Gamma(\alpha_j+1)}\right) + (M_2 r + A_2)\right\}\right] \\ &+ (M_1 r + A_1)\frac{|b|}{|\Omega|}\left(\frac{(T-t_m)^{\alpha_m}}{\Gamma(\alpha_m+1)}\right) \\ &+ \sum_{l=0}^{m}\frac{|c_l|(t_{l+1}-t_l)^{\alpha_l+\gamma_l-1}}{|\Omega|\Gamma_{q_l}(\alpha_l+\gamma_l)}\left[\sum_{j=0}^{l-1}\left(\prod_{j<i\le l-1}\frac{(t_{i+1}-t_i)^{\alpha_i-1}}{\Gamma_{q_i}(\alpha_i)}\right)\right. \\ &\times \left.\left.\left\{(M_1 r + A_1)\left(\frac{(t_{j+1}-t_j)^{\alpha_j}}{\Gamma(\alpha_j+1)}\right) + (M_2 r + A_2)\right\}\right] + \sum_{l=0}^{m}\frac{|c_l|}{|\Omega|}\frac{(t_{l+1}-t_l)^{\alpha_l+\gamma_l}}{\Gamma_{q_l}(\alpha_l+\gamma_l+1)}\right\} \\ &+ \left[\sum_{j=0}^{k-1}\left(\prod_{j<i\le k-1}\frac{(t_{i+1}-t_i)^{\alpha_i-1}}{\Gamma_{q_i}(\alpha_i)}\right)\left\{(M_1 r + A_1)\left(\frac{(t_{j+1}-t_j)^{\alpha_j}}{\Gamma(\alpha_j+1)}\right) + (M_2 r + A_2)\right\}\right], \end{aligned}$$

we obtain

$$\begin{aligned} (t-t_k)^\beta\big|\mathcal{L}x(t)\big| &\le (t-t_k)^\beta\big(\Psi_1(M_1 r + A_1) + \Psi_2(M_2 r + A_2)\big) \\ &\le r(\Psi_1 + M_1)T^\beta + (\Psi_1 A_1 + \Psi_2 A_2)T^\beta \\ &\le r. \end{aligned}$$

This implies that $\|\mathcal{L}x\|_{PC_\beta} \le r$ and, consequently, $\mathcal{L}B_r \subset B_r$.

For all $x, y \in PC_\beta(J,\mathbb{R})$ and $t \in J$, as in Lemma 2.5, we get

$$\big|\mathcal{L}x(t) - \mathcal{L}y(t)\big| \le (M_1\Psi_1 + M_2\Psi_2)\|x-y\|_{PC_\beta}.$$

Multiplying both sides of this inequality by $(t-t_k)^\beta$ for each $t \in J_k$, we have

$$\begin{aligned} (t-t_k)^\beta\big|\mathcal{L}x(t) - \mathcal{L}y(t)\big| &\le (t-t_k)^\beta(M_1\Psi_1 + M_2\Psi_2)\|x-y\|_{PC_\beta} \\ &\le T^\beta(M_1\Psi_1 + M_2\Psi_2)\|x-y\|_{PC_\beta}, \end{aligned}$$

which leads to $\|\mathcal{L}x - \mathcal{L}y\|_{PC_\beta} \le T^\beta(M_1\Psi_1 + M_2\Psi_2)\|x-y\|_{PC_\beta}$. In view of condition (3.2), it follows by the Banach contraction mapping principle that the operator $\mathcal{L}$ is a contraction. Hence, $\mathcal{L}$ has a fixed point, which is a unique solution of problem (1.1) on J. □

The next existence result is based on Leray-Schauder's nonlinear alternative.

Lemma 3.1 (Nonlinear alternative for single valued maps [17]) *Let E be a Banach space, C a closed, convex subset of E, U an open subset of C, and $0 \in U$. Suppose that $F : \overline{U} \to C$ is continuous and compact (that is, $F(\overline{U})$ is a relatively compact subset of C) map. Then either*

(i) *F has a fixed point in $\overline{U}$, or*
(ii) *there are $u \in \partial U$ (the boundary of U in C) and $\theta \in (0,1)$ with $u = \theta F(u)$.*

Theorem 3.2 *Assume that*

(H_2) *there exist continuous nondecreasing functions $Q, V : [0,\infty) \to (0,\infty)$ and a continuous function $p : J \to \mathbb{R}^+$ such that*

$$\bigl|f(t,x)\bigr| \le p(t)Q\bigl(|x|\bigr) \quad \textit{and} \quad \bigl|\varphi_k(x)\bigr| \le V\bigl(|x|\bigr) \tag{3.5}$$

for all $(t,x) \in (J \times \mathbb{R})$ and $k = 1,2,\dots,m$;

(H_3) *there exists a constant $M^* > 0$ such that such that*

$$\frac{M^*}{(p^*Q(M^*)\Psi_1 + V(M^*)\Psi_2)T^\beta} > 1, \tag{3.6}$$

where $p^* = \sup_{t\in J} |p(t)|$, $\beta > 0$, and the constants Ψ_1, Ψ_2 are defined in Lemma 2.5.

Then problem (1.1) has at least one solution on J.

Proof First, we show that the operator $\mathcal{L}$ *defined by* (3.1) *maps bounded sets* (*balls*) *into bounded sets in* PC_β. To accomplish this, for a positive number ρ, let $B_\rho = \{x \in PC_\beta : \|x\|_{PC_\beta} \le \rho\}$ be a ball in PC_β. Then, for $x \in B_\rho$ and $t \in J$, using the method of proof used in Lemma 2.5, we obtain

$$\bigl|\mathcal{L}x(t)\bigr| \le \frac{(t-t_k)^{\alpha_k-1}}{\Gamma_{q_k}(\alpha_k)} K_x + {}_{t_k}I_{q_k}^{\alpha_k}\bigl|f\bigl(t,x(t)\bigr)\bigr|,$$

where K_x is defined by (3.3). From (H_2) we have

$$\begin{aligned}
K_x \le{}& \left(\prod_{j=0}^{k-1} \frac{(t_{j+1}-t_j)^{\alpha_j-1}}{\Gamma_{q_j}(\alpha_j)}\right)\left\{\frac{|b|}{|\Omega|}\left[\sum_{j=0}^{m-1}\left(\prod_{j<i\le m} \frac{(t_{i+1}-t_i)^{\alpha_i-1}}{\Gamma_{q_i}(\alpha_i)}\right)\right.\right.\\
&\left.\times\left\{p^*Q(\rho)\left(\frac{(t_{j+1}-t_j)^{\alpha_j}}{\Gamma(\alpha_j+1)}\right)+V(\rho)\right\}\right] + p^*Q(\rho)\frac{|b|}{|\Omega|}\left(\frac{(T-t_m)^{\alpha_m}}{\Gamma(\alpha_m+1)}\right)\\
&+\sum_{l=0}^{m}\frac{|c_l|(t_{l+1}-t_l)^{\alpha_l+\gamma_l-1}}{|\Omega|\Gamma_{q_l}(\alpha_l+\gamma_l)}\left[\sum_{j=0}^{l-1}\left(\prod_{j<i\le l-1}\frac{(t_{i+1}-t_i)^{\alpha_i-1}}{\Gamma_{q_i}(\alpha_i)}\right)\right.\\
&\left.\left.\times\left\{p^*Q(\rho)\left(\frac{(t_{j+1}-t_j)^{\alpha_j}}{\Gamma(\alpha_j+1)}\right)+V(\rho)\right\}\right] + \sum_{l=0}^{m}\frac{|c_l|}{|\Omega|}\frac{(t_{l+1}-t_l)^{\alpha_l+\gamma_l}}{\Gamma_{q_l}(\alpha_l+\gamma_l+1)}\right\}\\
&+\left[\sum_{j=0}^{k-1}\left(\prod_{j<i\le k-1}\frac{(t_{i+1}-t_i)^{\alpha_i-1}}{\Gamma_{q_i}(\alpha_i)}\right)\left\{p^*Q(\rho)\left(\frac{(t_{j+1}-t_j)^{\alpha_j}}{\Gamma(\alpha_j+1)}\right)+V(\rho)\right\}\right],
\end{aligned}$$

and thus

$$\left|\mathcal{L}x(t)\right| \leq p^* Q(\rho)\Psi_1 + V(\rho)\Psi_2.$$

Therefore, $(t-t_k)^\beta|\mathcal{L}x(t)| \leq (t-t_k)^\beta(p^*Q(\rho)\Psi_1 + V(\rho)\Psi_2)$, which means that $\|\mathcal{L}x\|_{PC_\beta} \leq T^\beta(p^*Q(\rho)\Psi_1 + V(\rho)\Psi_2)$.

Next we show that $\mathcal{L}$ *maps bounded sets into equicontinuous sets of* PC_β.

Letting $\tau_1, \tau_2 \in J_k$ for some $k \in \{0,1,2,\dots,m\}$ with $\tau_1 < \tau_2$ and $x \in B_\rho$, where B_ρ is a ball in PC_β, we have

$$\begin{aligned}
\left|\mathcal{L}x(\tau_2) - \mathcal{L}x(\tau_1)\right| &\leq \left|\frac{(\tau_2-t_k)^{\alpha_k-1} - (\tau_1-t_k)^{\alpha_k-1}}{\Gamma_{q_k}(\alpha_k)}\right| K_x \\
&\quad + \left|{}_{t_k}I_{q_k}^{\alpha_k} f\bigl(\tau_2, x(\tau_2)\bigr) - {}_{t_k}I_{q_k}^{\alpha_k} f\bigl(\tau_1, x(\tau_1)\bigr)\right| \\
&\leq \left|\frac{(\tau_2-t_k)^{\alpha_k-1} - (\tau_1-t_k)^{\alpha_k-1}}{\Gamma_{q_k}(\alpha_k)}\right| K_x \\
&\quad + p^*Q(\rho)\left|\frac{(\tau_2-t_k)^{\alpha_k} - (\tau_1-t_k)^{\alpha_k}}{\Gamma_{q_k}(\alpha_k+1)}\right|.
\end{aligned} \tag{3.7}$$

As $\tau_1 \to \tau_2$, the right-hand side of inequality (3.7) tends to zero independently of x, that is,

$$\left|(\tau_2-t_k)^\beta \mathcal{L}x(\tau_2) - (\tau_1-t_k)^\beta \mathcal{L}x(\tau_1)\right| \to 0 \quad \text{as } |\tau_2-\tau_1| \to 0.$$

Therefore, by the Arzelà-Ascoli theorem, $\mathcal{L}: PC_\beta \to PC_\beta$ is completely continuous.

Our result will follow from the Leray-Schauder nonlinear alternative once we show the boundedness of the set of all solutions to the equation $x(t) = \lambda\mathcal{L}x(t)$ for $0 < \lambda < 1$. Let x be a solution. For any $t \in J$ and $x \in PC_\beta$, following the method of proof used in the first step together with condition (H_2), we get

$$\|x\|_{PC_\beta} \leq \bigl(p^*Q\bigl(\|x\|_{PC_\beta}\bigr)\Psi_1 + V\bigl(\|x\|_{PC_\beta}\bigr)\Psi_2\bigr)T^\beta.$$

In consequence, we have

$$\frac{\|x\|_{PC_\beta}}{(p^*Q(\|x\|_{PC_\beta})\Psi_1 + V(\|x\|_{PC_\beta})\Psi_2)T^\beta} \leq 1.$$

By condition (H_3) there exists M^* such that $\|x\|_{PC_\beta} \neq M^*$. We define $U = \{x \in PC_\beta : \|x\|_{PC_\beta} < M^*\}$. Note that the operator $\mathcal{L}: \overline{U} \to PC_\beta$ is continuous and completely continuous. By the choice of U there is no $x \in \partial U$ such that $x = \lambda\mathcal{L}x$ for some $\lambda \in (0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type (Lemma 3.1) we deduce that $\mathcal{L}$ has a fixed point $x \in \overline{U}$, which is a solution of problem (1.1) on J. This completes the proof. □

A key to prove the final result is based on the following fixed point theorem.

Lemma 3.2 [18] *Suppose that* $A : \bar{\Omega} \to E$ *is a completely continuous operator. Suppose that one of the following condition is satisfied*:

(i) (*Altman*) $\|Ax - x\|^2 \geq \|Ax\|^2 - \|x\|^2$ *for all* $x \in \partial\Omega$,
(ii) (*Rothe*) $\|Ax\| \leq \|x\|$ *for all* $x \in \partial\Omega$,
(iii) (*Petryshyn*) $\|Ax\| \leq \|Ax - x\|$ *for all* $x \in \partial\Omega$.

Then $\deg(I - A, \Omega, \theta) = 1$, *and hence* A *has at least one fixed point in* Ω.

Theorem 3.3 *Assume that*

(H_4) *the continuous functions* $f : J \times \mathbb{R} \to \mathbb{R}$ *and* $\varphi_k : \mathbb{R} \to \mathbb{R}$, $k = 1, 2, \ldots, m$, *satisfy*

$$\lim_{x\to 0} \frac{f(t,x)}{x} = 0 \quad \text{and} \quad \lim_{x\to 0} \frac{\varphi_k(x)}{x} = 0, \quad k = 1, 2, \ldots, m. \tag{3.8}$$

Then problem (1.1) *has at least one solution on* J.

Proof Let $x \in PC_\beta$. Taking ε sufficiently small, we can choose two positive constants δ_1 and δ_2 such that $|f(t,x)| < \varepsilon|x|$ whenever $\|x\|_{PC_\beta} < \delta_1$ and $\varphi_k(x) < \varepsilon|x|$ whenever $\|x\|_{PC_\beta} < \delta_2$ for $k = 1, 2, \ldots, m$. Setting $\delta = \min\{\delta_1, \delta_2\}$, we define the open ball $B_\delta = \{x \in PC_\beta : \|x\|_{PC_\beta} < \delta\}$. As in Theorem 3.2, it is clear that the operator $\mathcal{L} : PC \to PC$ is completely continuous. For any $x \in \partial B_\delta$, we have

$$\begin{aligned}
\bigl|\mathcal{L}x(t)\bigr| = {} & \frac{(t-t_k)^{\alpha_k-1}}{\Gamma_{q_k}(\alpha_k)} \left(\prod_{j=0}^{k-1} \frac{(t_{j+1}-t_j)^{\alpha_j-1}}{\Gamma_{q_j}(\alpha_j)} \right) \left\{ \frac{|b|}{|\Omega|} \left[\sum_{j=0}^{m-1} \left(\prod_{j<i\leq m} \frac{(t_{i+1}-t_i)^{\alpha_i-1}}{\Gamma_{q_i}(\alpha_i)} \right) \right.\right. \\
& \left. \times \left\{ {}_{t_j}I_{q_j}^{\alpha_j} \bigl|f\bigl(t_{j+1}, x(t_{j+1})\bigr)\bigr| + \bigl|\varphi_{j+1}\bigl(x(t_{j+1})\bigr)\bigr| \right\} \right] + \frac{|b|}{|\Omega|} {}_{t_m}I_{q_m}^{\alpha_m} \bigl|f\bigl(T, x(T)\bigr)\bigr| \\
& + \sum_{l=0}^{m} \frac{|c_l|(t_{l+1}-t_l)^{\alpha_l+\gamma_l-1}}{|\Omega|\Gamma_{q_l}(\alpha_l+\gamma_l)} \left[\sum_{j=0}^{l-1} \left(\prod_{j<i\leq l-1} \frac{(t_{i+1}-t_i)^{\alpha_i-1}}{\Gamma_{q_i}(\alpha_i)} \right) \right. \\
& \left.\left. \times \left\{ {}_{t_j}I_{q_j}^{\alpha_j} \bigl|f\bigl(t_{j+1}, x(t_{j+1})\bigr)\bigr| + \bigl|\varphi_{j+1}\bigl(x(t_{j+1})\bigr)\bigr| \right\} \right] + \sum_{l=0}^{m} \frac{|c_l|}{|\Omega|} {}_{t_l}I_{q_l}^{\alpha_l+\gamma_l} \bigl|f\bigl(t_{l+1}, x(t_{l+1})\bigr)\bigr| \right\} \\
& + \frac{(t-t_k)^{\alpha_k-1}}{\Gamma_{q_k}(\alpha_k)} \left[\sum_{j=0}^{k-1} \left(\prod_{j<i\leq k-1} \frac{(t_{i+1}-t_i)^{\alpha_i-1}}{\Gamma_{q_i}(\alpha_i)} \right) \right. \\
& \left. \times \left\{ {}_{t_j}I_{q_j}^{\alpha_j} \bigl|f\bigl(t_{j+1}, x(t_{j+1})\bigr)\bigr| + \bigl|\varphi_{j+1}\bigl(x(t_{j+1})\bigr)\bigr| \right\} \right] + {}_{t_k}I_{q_k}^{\alpha_k} \bigl|f\bigl(t, x(t)\bigr)\bigr| \\
\leq {} & (\Psi_1 + \Psi_2)\varepsilon|x|.
\end{aligned}$$

Setting $\varepsilon \leq (\Psi_1 + \Psi_2)^{-1}$, we deduce that

$$|\mathcal{L}x| \leq |x|.$$

Multiplying both sides of this inequality by $(t - t_k)^\beta$, we have $\|\mathcal{L}x\|_{PC_\beta} \leq \|x\|_{PC_\beta}$. It follows from Lemma 3.2(ii) that problem (1.1) has at least one solution on J. □

4 Examples

In this section, we present three examples to illustrate our results.

Example 4.1 Consider the following nonlocal boundary value problem for impulsive fractional q-difference equations:

$$\begin{cases} {}_{t_k}D^{(\frac{k+1}{k+2})}_{(\frac{k^2+2}{k^2+3})}x(t) = (\frac{\cos^2 t+e^{-t}}{60})(\frac{x^2(t)+|x(t)|}{|x(t)|+1}) + \frac{3}{4}, \quad t \in [0,4/3] \setminus t_k, \\ {}_{t_k}I^{(\frac{1}{k+2})}_{(\frac{k^2+2}{k^2+3})}x(t_k^+) - x(t_k) = \frac{1}{16\pi k}\sin(|\pi x(t_k)|), \quad t_k = \frac{k}{3}, k = 1,2,3, \\ \frac{1}{2}{}_0I^{\frac{1}{2}}_{\frac{2}{3}}x(0) = \frac{2}{3}x(\frac{4}{3}) + \sum_{l=0}^{3}(\frac{l^2+l+1}{l^2+2l+2}){}_{t_l}I^{(\frac{2l+1}{l+3})}_{(\frac{l^2+2}{l^2+3})}x(t_{l+1}). \end{cases} \tag{4.1}$$

Here $\alpha_k = (k+1)/(k+2)$, $q_k = (k^2+2)/(k^2+3)$, $\gamma_k = (2k+1)/(k+3)$, $c_k = (k^2+k+1)/(k^2+2k+2)$, $k = 0,1,2,3$, $a = 1/2$, $b = 2/3$, $T = 4/3$, $t_k = k/3$, $k = 1,2,3$. With the given values, we find that $\Omega = -2.102954268$, $\Psi_1 = 4.421252518$, and $\Psi_2 = 6.317984153$. Also, we have

$$\big|f(t,x) - f(t,y)\big| \le \frac{\cos^2 t + e^{-t^2}}{30}|x-y| \le \frac{1}{15}|x-y|$$

and

$$\big|\varphi_k(x) - \varphi_k(y)\big| \le \frac{1}{16}|x-y|, \quad k = 1,2,3,$$

which suggests that (H_1) is satisfied with $M_1 = 1/15$ and $M_2 = 1/16$. Further, there exists $\beta = 1$ such that $(M_1\Psi_1 + M_2\Psi_2)T^\beta = 0.9194989033 < 1$. Thus, all the conditions of Theorem 3.1 hold. Therefore, by the conclusion of Theorem 3.1, problem (4.1) has a unique solution on $[0,4/3]$.

Example 4.2 Consider the problem of impulsive fractional q-difference equations given by

$$\begin{cases} {}_{t_k}D^{(\frac{k^2+2k+2}{k^2+3k+3})}_{(\frac{k^2+k+2}{k^2+k+3})}x(t) = \frac{e^{-3t^2}}{10+t^2}\log_e^2(\frac{|x(t)|}{10} + 2), \quad t \in [0,5] \setminus t_k, \\ {}_{t_k}I^{(\frac{k+1}{k^2+3k+3})}_{(\frac{k^2+k+2}{k^2+k+3})}x(t_k^+) - x(t_k) = \frac{x^2(t_k)}{50(|x(t_k)|+1)} + \frac{1}{5k}, \quad t_k = k, k = 1,2,3,4, \\ \frac{2}{3}{}_0I^{\frac{1}{3}}_{\frac{2}{3}}x(0) = \frac{3}{4}x(5) + \sum_{l=0}^{4}(\frac{l+3}{l^2+3l+4}){}_{t_l}I^{(\frac{l^2+2l+1}{l+2})}_{(\frac{l^2+l+2}{l^2+l+3})}x(t_{l+1}). \end{cases} \tag{4.2}$$

Here $\alpha_k = (k^2+2k+2)/(k^2+3k+3)$, $q_k = (k^2+k+2)/(k^2+k+3)$, $\gamma_k = (k^2+2k+1)/(k+2)$, $c_k = (k+3)/(k^2+3k+4)$, $k = 0,1,2,3,4$, $a = 2/3$, $b = 3/4$, $T = 5$, $t_k = k$, $k = 1,2,3,4$. With this data, we find that $\Omega = -0.8144800590$, $\Psi_1 = 6.521521011$, and $\Psi_2 = 4.376841316$. Further, we have

$$\big|f(t,x)\big| = \left|\frac{e^{-3t^2}}{10+t^2}\log_e^2\left(\frac{|x|}{10} + 2\right)\right| \le \frac{e^{-3t^2}}{10+t^2}\left(\frac{|x|}{10} + 2\right)$$

and

$$\big|\varphi_k(x)\big| = \frac{x^2}{50(|x|+1)} + \frac{1}{5k} \le \frac{|x|}{50} + \frac{1}{5}, \quad k = 1,2,3,4.$$

Setting $Q(x) = (x/10) + 2$, $V(x) = (x/50) + (1/5)$, $p^* = 1/10$, and $\beta = 1$, there exists a constant $M^* > 46.13262248$ satisfying (3.6). Thus, the hypothesis of Theorem 3.2 is satisfied. In consequence, the conclusion of Theorem 3.2 applies, and problem (4.2) has at least one solution on $[0,5]$.

Example 4.3 Consider the problem of impulsive fractional q-difference equations given by

$$\begin{cases} {}_{t_k}D^{(\frac{2k^2+k+3}{3k^2+2k+4})}_{(\frac{k^2+2k+2}{2k^2+2k+3})} x(t) = \frac{2t}{3t+1}(\sin x(t) - x(t))e^{x^2(t)\cos^4 x(t)}, \quad t \in [0,5/4] \setminus t_k, \\ {}_{t_k}I^{(\frac{k^2+k+1}{3k^2+2k+4})}_{(\frac{k^2+2k+2}{2k^2+2k+3})} x(t_k^+) - x(t_k) = \frac{kx^4(t_k)+2kx^2(t_k)}{\log(|x^3(t_k)|+2)}, \quad t_k = k, k = 1,2,3,4, \\ \frac{3}{4}{}_0I^{\frac{1}{4}}_{\frac{2}{3}} x(0) = \frac{4}{5}x(\frac{5}{4}) + \sum_{l=0}^{4}(\frac{2l^2+3l+1}{3l^2+2l+2})_{t_l}I^{(\frac{2l+1}{2})}_{(\frac{l^2+2l+2}{2l^2+2l+3})} x(t_{l+1}). \end{cases} \tag{4.3}$$

Here $\alpha_k = (2k^2 + k + 3)/(3k^2 + 2k + 4)$, $q_k = (k^2 + 2k + 2)/(2k^2 + 2k + 3)$, $\gamma_k = (2k + 1)/2$, $c_k = (2k^2 + 3k + 1)/(3k^2 + 2k + 2)$, $k = 0,1,2,3,4$, $a = 3/4$, $b = 4/5$, $T = 5/4$, $t_k = k/4$, $k = 1,2,3,4$. With this data, we find that $|\Omega| = 2.037343386 \neq 0$. The functions $f(t,x) = ((2t)/(3t + 1))(\sin x - x)e^{x^2 \cos^4 x}$ and $\varphi_k(x) = (kx^4 + 2kx^2)/(\log(|x^3| + 2))$, $k = 1,2,3,4$, satisfy

$$\lim_{x\to 0}\frac{f(t,x)}{x} = \lim_{x\to 0}\frac{2t}{3t+1}\left(\frac{\sin x}{x} - 1\right)e^{x^2\cos^4 x} = 0$$

and

$$\lim_{x\to 0}\frac{\varphi_k(x)}{x} = \lim_{x\to 0}\frac{kx^3 + 2kx}{\log(|x^3| + 2)} = 0, \quad k = 1,2,3,4.$$

Thus, condition (H_4) of Theorem 3.3 holds. Therefore, by applying Theorem 3.3 we conclude that problem (4.3) has at least one solution on $[0,5/4]$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Each of the authors, BA, AA, SKN, JT, and FA contributed to each part of this work equally and read and approved the final version of the manuscript.

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