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Regular approximations of isolated eigenvalues of singular second-order symmetric linear difference equations

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Abstract

This paper is concerned with regular approximations of isolated eigenvalues of singular second-order symmetric linear difference equations. It is shown that the kth eigenvalue of any given self-adjoint subspace extension is exactly the limit of the kth eigenvalues of the induced regular self-adjoint subspace extensions in the case that each endpoint is either regular or in the limit circle case. Furthermore, error estimates for the approximations of eigenvalues are given in this case. In addition, it is shown that isolated eigenvalues in every gap of the essential spectrum of any self-adjoint subspace extension are exactly the limits of eigenvalues of suitably chosen induced regular self-adjoint subspace extensions in the case that at least one endpoint is in the limit point case.

MSC: 39A10; 41A99; 47A06; 47A10

Keywords: symmetric linear difference equation; self-adjoint subspace extension; regular approximation; spectral exactness; error estimate

1 Introduction

Consider the following second-order symmetric linear difference equation:

$$-\nabla\big(p(t)\triangle x(t)\big) + q(t)x(t) = \lambda w(t)x(t), \quad t \in I, \tag{1.1$_\lambda$}$$

where I is the integer set $\{t\}_{t=a}^{b}$, a is a finite integer or $-\infty$ and b is a finite integer or $+\infty$; $\triangle$ and ∇ are the forward and backward difference operators, respectively, *i.e.*, $\triangle x(t) = x(t+1) - x(t)$, $\nabla x(t) = x(t) - x(t-1)$; $p(t)$ and $q(t)$ are all real-valued with $p(t) \neq 0$ for $t \in I$, $p(a-1) \neq 0$ if a is finite and $p(b+1) \neq 0$ if b is finite; $w(t) > 0$ for $t \in I$; and λ is a complex spectral parameter.

Spectral problems can be divided into two classifications. Those defined over finite closed intervals with well-behaved coefficients are called regular; otherwise they are called singular. Regular approximations of spectra of singular differential equations have been investigated widely and deeply, and some good results have been obtained, including spectral inclusion in general cases and spectral exactness in the case that each endpoint is either regular or in the limit circle case (briefly, l.c.c.) [1–9]. In particular, Stolz, Weidmann, and Teschl [5–9] got spectral exactness for isolated eigenvalues in essential spectral gaps. In addition, Brown *et al.* [3] constructed a sequence of regular problems for a given fourth-order singular symmetric differential operator and showed that the eigenvalues of the singular problem are exactly the limits of eigenvalues of this sequence in the case that each endpoint is either regular or in l.c.c.

In the present paper, we are wondering whether there are analogous results for singular symmetric difference equations. We shall study a similar problem for singular second-order symmetric linear difference equations. Note that for a symmetric linear difference equation, its minimal operator may not be densely defined, and its minimal and maximal operators may be multi-valued (*cf.* [10–12]). So it cannot be treated by the methods described in [3, 5–9], which are based on self-adjoint extensions of densely defined Hermitian operators.

This major difficulty can be overcome by using the theory of self-adjoint subspace extensions of Hermitian subspaces. This theory was developed by Coddington, Dijksma, de Snoo, and others (*cf.* [13–19]). The second author of the present paper extended the classical Glazman-Krein-Naimark (briefly, GKN) theory to Hermitian subspaces [11], and based on this, she with her coauthor Sun presented complete characterizations of self-adjoint extensions for second-order symmetric linear difference equation in both regular and singular cases [12]. Later, she studied some spectral properties of self-adjoint subspaces together with her coauthors Shao and Ren [20]. Recently, based on the above results, we studied the resolvent convergence and spectral approximations of sequences of self-adjoint subspaces [21].

Applying the results given in [12, 21], we studied regular approximations of spectra of singular second-order symmetric linear difference equations [22]. We constructed suitable induced regular self-adjoint subspace extensions and proved that the sequence of induced regular self-adjoint subspace extensions is both spectrally inclusive and exact for a given self-adjoint subspace extension in the case that each endpoint is either regular or in l.c.c., while, in general, it is only spectrally inclusive in the case that at least one endpoint is in the limit point case (briefly, l.p.c.). Here, we shall further investigate how to approximate the spectrum of singular second-order symmetric linear difference equations with eigenvalues of regular problems in the case that each endpoint is either regular or in l.c.c. Furthermore, we shall also give their error estimates. In addition, enlightened by Stolz, Weidmann, and Teschl's work [5–9], we shall show the spectral exactness in an open interval laking essential spectral points in the case that at least one endpoint is in l.p.c.

In the study of regular approximation problems, the related induced regular self-adjoint extensions should be extended to the whole interval referred for the singular problems. This problem can easily be dealt with by 'zero extension' in the continuous case. But it is somewhat difficult in the discrete case. This difficulty was overcome in Section 3.2 in [22] and recalled in Section 2.3 in the present manuscript. So the method used in the present manuscript is not a trivial and direct generalization of that used for ODEs [3, 5–9]. Further, we shall remark that although the minimal operator is densely defined in the case that $a = -\infty$ and $b = +\infty$, the minimal operators of the induce regular problems that will be used to approximate the singular one are not densely defined, and so their self-adjoint extensions have to be characterized by the theory of subspaces. These self-adjoint extensions are multi-valued in general. Therefore, it is better for us to uniformly apply the theory of subspaces to study regular approximations in all the cases in the present paper.

The rest of this paper is organized as follows. In Section 2, some basic concepts and fundamental results about subspaces in Hilbert spaces and second-order symmetric linear difference equations are introduced. In addition, the induced regular self-adjoint subspace extensions for any given self-adjoint subspace extension are introduced. In particular, a sufficient condition is given for spectral exactness of a sequence of self-adjoint subspaces in an open interval laking essential spectral points. It will play an important role in the study of regular approximations in the case that at least one endpoint is in l.p.c. In Section 3, regular approximations of isolated eigenvalues of equation (1.1) are studied in the case that each endpoints is either regular or in l.c.c. It is shown that the kth eigenvalue of the given self-adjoint subspace extension is exactly the limit of the kth eigenvalues of the induced regular self-adjoint subspace extensions. In addition, their error estimates are given. Spectral exactness in every gap of the essential spectrum of any self-adjoint subspace extension is obtained in the other three cases in Sections 4-6, separately.

2 Preliminaries

This section is divided into three parts. In Section 2.1, some basic concepts and fundamental results about subspaces are listed. In Section 2.2, the maximal, pre-minimal, and minimal subspaces corresponding to equation (1.1) are introduced. In Section 2.3, some results about self-adjoint subspace extensions of the minimal subspace and their induced self-adjoint restrictions given in [22] are recalled.

2.1 Some basic concepts and fundamental results about subspaces

By **C**, **R**, and **N** denote the sets of the complex numbers, real numbers, and positive integer numbers, respectively. Let X be a complex Hilbert space with inner product $\langle\cdot,\cdot\rangle$, and T a linear subspace (briefly, subspace) in the product space X^2 with the following induced inner product, still denoted by $\langle\cdot,\cdot\rangle$ without any confusion:

$$\big\langle(x,f),(y,g)\big\rangle = \langle x,y\rangle + \langle f,g\rangle, \quad (x,f),(y,g)\in X^2.$$

Denote the domain, range, and null space of T by $D(T)$, $R(T)$, and $N(T)$, respectively. Its adjoint subspace T^* is defined by

$$T^* = \big\{(y,g)\in X^2 : \langle g,x\rangle = \langle y,f\rangle \text{ for all } (x,f)\in T\big\}.$$

Further, denote

$$T(x) := \big\{f\in X : (x,f)\in T\big\}, \qquad T^{-1} := \big\{(f,x) : (x,f)\in T\big\}.$$

It is evident that $T(0)=\{0\}$ if and only if T can uniquely determine a (singled-valued) linear operator from $D(T)$ into X whose graph is T. For convenience, a linear operator in X will always be identified with a subspace in X^2 via its graph.

Let T and S be two subspaces in X^2 and $\lambda\in\mathbf{C}$. Define

$$\lambda T := \big\{(x,\lambda f) : (x,f)\in T\big\},$$

$$T+S := \big\{(x,f+g) : (x,f)\in T, (x,g)\in S\big\},$$

$$ST := \{(x,g) \in X^2 : (x,f) \in T, (f,g) \in S \text{ for some } f \in X\}.$$

It is evident that if T is closed, then $T - \lambda I_{id}$ is closed and $(T - \lambda I_{id})^* = T^* - \bar{\lambda} I_{id}$, where $I_{id} := \{(x,x) : x \in X\}$, briefly denoted by I without any confusion between it and the interval I.

Throughout the whole paper, denote the resolvent set, spectrum, point spectrum, essential spectrum, and discrete spectrum of T by $\rho(T)$, $\sigma(T)$, $\sigma_p(T)$, $\sigma_e(T)$, and $\sigma_d(T)$, respectively.

Definition 2.1 ([21], Definition 5.1) Let $\{T_n\}_{n=1}^\infty$ and T be subspaces in X^2.

(1) The sequence $\{T_n\}_{n=1}^\infty$ is said to be spectrally inclusive for T if for any $\lambda \in \sigma(T)$, there exists a sequence $\{\lambda_n\}_{n=1}^\infty$, $\lambda_n \in \sigma(T_n)$, such that $\lim_{n\to\infty} \lambda_n = \lambda$.
(2) The sequence $\{T_n\}_{n=1}^\infty$ is said to be spectrally exact for T if it is spectrally inclusive and every limit point of any sequence $\{\lambda_n\}_{n=1}^\infty$ with $\lambda_n \in \sigma(T_n)$ belongs to $\sigma(T)$.
(3) The sequence $\{T_n\}_{n=1}^\infty$ is said to be spectrally exact for T in some set $\Omega \subset \mathbf{R}$ if the condition in (2) holds in Ω.

Lemma 2.1 ([21], Lemma 2.1) *Let T be a closed subspace in X^2. Then*

$$\rho(T^{-1}) \setminus \{0\} = \{\lambda^{-1} : \lambda \in \rho(T) \textit{ with } \lambda \neq 0\},$$
$$\sigma(T^{-1}) \setminus \{0\} = \{\lambda^{-1} : \lambda \in \sigma(T) \textit{ with } \lambda \neq 0\}.$$

Consequently, if $\rho(T) \neq \emptyset$, then

$$\sigma((\lambda_0 I - T)^{-1}) \setminus \{0\} = \{(\lambda_0 - \lambda)^{-1} : \lambda \in \sigma(T)\}, \quad \lambda_0 \in \rho(T).$$

Let T and S be two subspaces in X^2. If $T \cap S = \{(0,0)\}$, denote

$$T \dot{+} S := \{(x+y, f+g) : (x,f) \in T, (y,g) \in S\}.$$

Further, if T and S are orthogonal, denoted by $T \perp S$; that is, $\langle (x,f), (y,g)\rangle = 0$ for any $(x,f) \in T$, $(y,g) \in S$, we denote

$$T \oplus S := T \dot{+} S.$$

In addition, we introduce the following notation for convenience:

$$T \ominus S := \{(x,f) \in T : \langle x,y\rangle + \langle f,g\rangle = 0 \text{ for all } (y,g) \in S\}.$$

In 1961, Arens [23] introduced the following important decomposition for a closed subspace T in X^2:

$$T = T_s \oplus T_\infty,$$

where

$$T_\infty := \{(0,f) \in X^2 : (0,f) \in T\}, \qquad T_s := T \ominus T_\infty.$$

It can easily be verified that T_s is an operator, and T is an operator if and only if $T = T_s$. T_s and T_∞ are called the operator and pure multi-valued parts of T, respectively. In addition,

$$R(T_\infty) = T(0), \qquad T_\infty = \{0\} \times T(0), \qquad R(T_s) \subset T(0)^\perp, \qquad D(T_s) = D(T), \tag{2.1}$$

and $D(T_s) = D(T)$ is dense in $T^*(0)^\perp$.

Throughout the present paper, the resolvent set and spectrum of T_s and T_∞ mean those of T_s and T_∞ restricted to $(T(0)^\perp)^2$ and $T(0)^2$, respectively.

Lemma 2.2 ([20], Proposition 2.1 and Theorems 2.1, 2.2, and 3.4) *Let T be a closed Hermitian subspace in X^2. Then*

$$T_\infty = T \cap T(0)^2, \qquad T_s = T \cap \bigl(T(0)^\perp\bigr)^2,$$

T_s is a closed Hermitian operator in $T(0)^\perp$, T_∞ is a closed Hermitian subspace in $T(0)^2$,

$$\rho(T) = \rho(T_s), \qquad \sigma(T) = \sigma(T_s), \qquad \sigma(T_\infty) = \emptyset,$$
$$\sigma_p(T) = \sigma_p(T_s), \qquad \sigma_e(T) = \sigma_e(T_s), \qquad \sigma_d(T) = \sigma_d(T_s),$$

and $N(T - \lambda I) = N(T_s - \lambda I)$ for every $\lambda \in \sigma_p(T)$.

Lemma 2.3 ([17], p.26) *If T is a self-adjoint subspace in X^2, then T_∞ and T_s are self-adjoint subspaces in $T(0)^2$ and $(T(0)^\perp)^2$, respectively.*

To end this subsection, we shall briefly recall the concept of the spectral family of a self-adjoint subspace, which was introduced by Coddington and Dijksma in [15].

Let T be a self-adjoint subspace in X^2. By Lemma 2.3, T_s is a self-adjoint operator in $T(0)^\perp$. Then T_s has the following spectral resolution:

$$T_s = \int t\, dE_s(t),$$

where $\{E_s(t)\}_{t\in\mathbf{R}}$ is the spectral family of T_s in $T(0)^\perp$. The spectral family of the subspace T is defined by

$$E(t) = E_s(t) \oplus O, \quad t \in \mathbf{R},$$

where O is the zero operator defined on $T(0)$. It is obvious that for any $t \in \mathbf{R}$ and any $f \in X$,

$$E(t)f = E_s(t)f_1, \tag{2.2}$$

where $f = f_1 + f_2$ with $f_1 \in T(0)^\perp$ and $f_2 \in T(0)$.

The following result weakens the condition (5.7) of Theorem 5.3 in [21]. It will be useful in studying spectral exactness in every gap of the essential spectrum of any self-adjoint subspace extension in Sections 4-6.

Lemma 2.4 *Let $\{T_n\}_{n=1}^{\infty}$ and T be self-adjoint subspaces in X^2, and let $E(T_n,\lambda)$ and $E(T,\lambda)$ be spectral families of T_n and T, respectively. Assume that $I_0 \subset \mathbf{R}$ is an open interval and satisfies*

$$I_0 \cap \sigma_e(T) = \emptyset, \qquad I_0 \cap \sigma_d(T) \neq \emptyset. \tag{2.3}$$

Let $\gamma \in I_0$. If for any given $\alpha, \beta \in I_0 \cap \rho(T)$ with $\alpha < \gamma \leq \beta$, there exists an integer $n_0 \geq 1$ such that for all $n \geq n_0$,

$$\dim R\{E(T_n,\beta) - E(T_n,\alpha)\} = \dim R\{E(T,\beta) - E(T,\alpha)\}, \tag{2.4}$$

then $\{T_n\}_{n=1}^{\infty}$ is spectrally exact for T in I_0.

Proof By Theorem 5.3 in [21], it suffices to show that (2.4) holds for all $\alpha, \beta \in I_0 \cap \rho(T)$ with $\alpha < \beta$. Fix any $\alpha, \beta \in I_0 \cap \rho(T)$ with $\alpha < \beta$. The following discussions are divided into three cases.

Case 1. $\gamma \in (\alpha, \beta]$. Obviously, (2.4) holds in this case.

Case 2. $\gamma \leq \alpha$. By (2.3), there exists $\epsilon > 0$ such that $\gamma - \epsilon \in I_0 \cap \rho(T)$. By (2.4), there exists $n_0 \geq 1$ such that for all $n \geq n_0$,

$$\begin{aligned} &\dim R\{E(T_n,\beta) - E(T_n,\gamma-\epsilon)\} = \dim R\{E(T,\beta) - E(T,\gamma-\epsilon)\}, \\ &\dim R\{E(T_n,\alpha) - E(T_n,\gamma-\epsilon)\} = \dim R\{E(T,\alpha) - E(T,\gamma-\epsilon)\}. \end{aligned} \tag{2.5}$$

Note that

$$\begin{aligned} &E(T_n,\beta) - E(T_n,\alpha) = \{E(T_n,\beta) - E(T_n,\gamma-\epsilon)\} - \{E(T_n,\alpha) - E(T_n,\gamma-\epsilon)\}, \\ &E(T,\beta) - E(T,\alpha) = \{E(T,\beta) - E(T,\gamma-\epsilon)\} - \{E(T,\alpha) - E(T,\gamma-\epsilon)\}; \end{aligned}$$

that is,

$$\begin{aligned} &E(T_n,(\alpha,\beta]) = E(T_n,(\gamma-\epsilon,\beta]) - E(T_n,(\gamma-\epsilon,\alpha]), \\ &E(T,(\alpha,\beta]) = E(T,(\gamma-\epsilon,\beta]) - E(T,(\gamma-\epsilon,\alpha]). \end{aligned} \tag{2.6}$$

Since $E(T_n,(\alpha,\beta])$ and $E(T,(\alpha,\beta])$ are orthogonal projections and

$$\begin{aligned} &R\{E(T_n,(\gamma-\epsilon,\alpha])\} \subset R\{E(T_n,(\gamma-\epsilon,\beta])\}, \\ &R\{E(T,(\gamma-\epsilon,\alpha])\} \subset R\{E(T,(\gamma-\epsilon,\beta])\}, \end{aligned}$$

by (c) of Theorem 4.30 in [24] we have

$$\begin{aligned} &R\{E(T_n,(\alpha,\beta])\} = R\{E(T_n,(\gamma-\epsilon,\beta])\} \ominus R\{E(T_n,(\gamma-\epsilon,\alpha])\}, \\ &R\{E(T,(\alpha,\beta])\} = R\{E(T,(\gamma-\epsilon,\beta])\} \ominus R\{E(T,(\gamma-\epsilon,\alpha])\}. \end{aligned}$$

It follows that

$$\dim R\{E(T_n,(\alpha,\beta])\} = \dim R\{E(T_n,(\gamma-\epsilon,\beta])\} - \dim R\{E(T_n,(\gamma-\epsilon,\alpha])\},$$

$$\dim R\{E(T,(\alpha,\beta])\} = \dim R\{E(T,(\gamma-\epsilon,\beta])\} - \dim R\{E(T,(\gamma-\epsilon,\alpha])\}.$$

This, together with (2.5), shows that (2.4) holds in this case.

Case 3. $\gamma > \beta$. With a similar argument for Case 2, one can easily show that (2.4) holds in Case 3. This completes the proof. □

2.2 Maximal, pre-minimal and minimal subspaces

In this subsection, we first introduce the concepts of maximal, pre-minimal and minimal subspaces corresponding to (1.1) and then briefly recall their properties.

Since a, b may be finite or infinite, we give the following convention for briefness in the sequent expressions: $a-1$ means $-\infty$ in the case of $a=-\infty$; $b+1$ means $+\infty$ in the case of $b=+\infty$.

Denote

$$l_w^2(I) := \left\{ x = \{x(t)\}_{t=a-1}^{b+1} \subset \mathbf{C} : \sum_{t=a}^{b} w(t)|x(t)|^2 < +\infty \right\}.$$

Then $l_w^2(I)$ is a Hilbert space with the inner product

$$\langle x,y\rangle := \sum_{t=a}^{b} w(t)\bar{y}(t)x(t),$$

where $x=y$ in $l_w^2(I)$ if and only if $\|x-y\|=0$, *i.e.*, $x(t)=y(t)$, $t\in I$, while $\|\cdot\|$ is the induced norm.

The natural difference operator corresponding to (1.1) is denoted by

$$\mathcal{L}(x)(t) := -\nabla\big(p(t)\triangle x(t)\big) + q(t)x(t), \quad t\in I.$$

Now, we introduce the corresponding maximal, pre-minimal, and minimal subspaces corresponding to (1.1) in the interval I. Let

$$\begin{aligned} H &:= \{(x,f)\in (l_w^2(I))^2 : \mathcal{L}(x)(t) = w(t)f(t), t\in I\},\\ H_{00} &:= \{(x,f)\in H : \text{there exist two integers } t_0,t_1\in I \text{ with } t_0<t_1\\ &\qquad \text{such that } x(t)=0 \text{ for } t\le t_0 \text{ and } t\ge t_1\}, \end{aligned}$$

where H and H_{00} are called the maximal and pre-minimal subspaces corresponding to $\mathcal{L}$ or (1.1), respectively. The subspace $H_0 := \bar{H}_{00}$ is called the minimal subspace corresponding to $\mathcal{L}$ or (1.1). By Corollary 3.1 and Theorem 3.3 in [12], H_0 is a closed densely defined Hermitian operator in $l_w^2(I)$ in the case that $a=-\infty$ and $b=+\infty$, and a closed non-densely defined Hermitian operator in $l_w^2(I)$ in the other case that at least one of a and b is finite. In addition, $H\subset {H_0}^*$ and $H={H_0}^*$ in the sense of the norm $\|\cdot\|$.

In addition, we take the notation for convenience:

$$(x,y)(t) = p(t)\big[\big(\triangle\bar{y}(t)\big)x(t) - \bar{y}(t)\triangle x(t)\big], \quad t=\{t\}_{t=a-1}^{b}. \tag{2.7}$$

2.3 Self-adjoint subspace extensions and their induced self-adjoint restrictions

In this subsection, we recall the results about self-adjoint subspace extensions of H_0 and their induced regular self-adjoint subspace extensions, *i.e.*, induced self-adjoint restrictions constructed in [22].

Let $I_r = \{t\}_{t=a_r}^{b_r}$, where $-\infty < a_r + 1 < b_r - 1 < +\infty$, $a_{r+1} \le a_r < b_r \le b_{r+1}$, $r \in \mathbf{N}$, and $a_r \to a$, $b_r \to b$ as $r \to \infty$. That is, $\lim_{r\to\infty} I_r = I$. If a (resp. b) is finite, take $a_r = a$ (resp. $b_r = b$). For convenience, by H^r and H_0^r denote the corresponding maximal and minimal subspaces to equation (1.1) or $\mathcal{L}$ on I_r, respectively. Noting that all the coefficient functions p and q and weight function w in (1.1) are real-valued, one has $d_+(H_0) = d_-(H_0)$, where $d_\pm(H_0)$ are the positive and negative defect indices of H_0. Consequently, H_0 has self-adjoint subspace extensions by [14].

Let $\varphi_1(\cdot,\lambda)$ and $\varphi_2(\cdot,\lambda)$ be two linearly independent solutions of (1.1_λ) with $\lambda \in \mathbf{R}$ satisfying the following initial conditions:

$$\begin{aligned} &\varphi_1(d_0-1,\lambda) = 0, \qquad p(d_0-1)\triangle\varphi_1(d_0-1,\lambda) = -1,\\ &\varphi_2(d_0-1,\lambda) = 1, \qquad p(d_0-1)\triangle\varphi_2(d_0-1,\lambda) = 0, \end{aligned} \tag{2.8}$$

where $d_0 \in I$ is any fixed.

In the case that $I = [a,+\infty)$ (resp. $I = (-\infty,b]$), $\mathcal{L}$ is regular at a (resp. b) and either in l.c.c. or l.p.c. at $t = +\infty$ (resp. $t = -\infty$). In the case that $I = (-\infty,+\infty)$, $\mathcal{L}$ is either in l.c.c. or l.p.c. at each endpoint. Consequently, the following discussions are divided into the five cases due to different expressions of their self-adjoint subspace extensions.

Case 1. *One endpoint is regular and the other in l.c.c.*

Without loss of generality, we only consider the case that $\mathcal{L}$ is regular at a and in l.c.c. at $t = +\infty$. Take $d_0 = a$ in (2.8) in this case.

Suppose that H_1 is any fixed self-adjoint subspace extension of H_0. Then, by (3.5) in [22], we have

$$H_1 = \bigl\{(x,f) \in H : (x,\hat{y}_j)(a-1) - (x,\hat{y}_j)(+\infty) = 0, j = 1,2\bigr\}, \tag{2.9}$$

where

$$\hat{y}_j(a-1) = -m_{2j}, \qquad p(a-1)\triangle\hat{y}_j(a-1) = m_{1j}, \qquad \hat{y}_j(t) = u_j(t), \quad t \ge c_0, \tag{2.10}$$

$$JM^* = \begin{pmatrix} -m_{21} & -m_{22} \\ m_{11} & m_{12} \end{pmatrix}, \qquad N = -(n_{ij})_{2\times 2}, \qquad u_j := \sum_{k=1}^{2} \bar{n}_{jk}\varphi_k, \quad j = 1,2,$$

while

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

matrices $M,N \in \mathbf{C}^{2\times 2}$ satisfying $\operatorname{rank}(M,N) = 2$ and $MJM^* = NJN^*$, and $c_0 > a + 1$ is any fixed integer.

Let $a_r = a$ and $b_r > c_0$. According to (3.7) in [22], an induced self-adjoint restriction of H_1 on I_r can be given by

$$H_{1,r} = \bigl\{(x,f) \in H^r : (x,\hat{y}_j)(a-1) - (x,\hat{y}_j)(b_r) = 0, j = 1,2\bigr\}. \tag{2.11}$$

Case 2. *One endpoint is regular and the other in l.p.c.*

Without loss of generality, we only consider the case that $\mathcal{L}$ is regular at a and in l.p.c. at $t = +\infty$. Still take $d_0 = a$ in (2.8) in this case.

Suppose that H_1 is any fixed self-adjoint subspace extension of H_0. Then, by (3.9) in [22], we have

$$H_1 = \{(x,f) \in H : (x,\hat{y})(a-1) = 0\}, \tag{2.12}$$

where

$$\hat{y}(a-1) = -m_2, \qquad p(a-1)\triangle\hat{y}(a-1) = m_1, \qquad \hat{y}(t) = 0, \quad t \geq c_0, \tag{2.13}$$

with $M = (m_1, m_2) \in \mathbf{R}^{1\times 2}$ and $M \neq 0$, and $c_0 > a + 1$ is any fixed integer.

Let $a_r = a$ and $b_r > c_0$. According to the discussion for (3.12) in [22], an induced self-adjoint restriction of H_1 on I_r can be given by

$$H_{1,r} = \{(x,f) \in H^r : (x,\hat{y})(a-1) = 0, (x,u)(b_r) = 0\}, \tag{2.14}$$

where

$$u = n_1\varphi_1(\cdot,\lambda) + n_2\varphi_2(\cdot,\lambda), \tag{2.15}$$

with $N = (n_1, n_2) \in \mathbf{R}^{1\times 2}$ and $N \neq 0$.

Case 3. *Both endpoints are in l.c.c.*

Suppose that H_1 is any fixed self-adjoint subspace extension of H_0. Then, by (4.4) in [22], we have

$$H_1 = \{(x,f) \in H : (x,\hat{y}_j)(-\infty) - (x,\hat{y}_j)(+\infty) = 0, j = 1,2\}, \tag{2.16}$$

where

$$\hat{y}_j = \begin{cases} u_j, & t \leq d_0 - 1, \\ v_j, & t \geq d_0 + 1, \end{cases} \qquad u_j := \sum_{k=1}^{2} \bar{m}_{jk}\varphi_k(\cdot,\lambda), \quad v_j := \sum_{k=1}^{2} \bar{n}_{jk}\varphi_k(\cdot,\lambda), \quad j = 1,2, \tag{2.17}$$

with matrices $M = (m_{jk})_{2\times 2}$ and $N = (n_{jk})_{2\times 2}$ satisfying $\operatorname{rank}(M,N) = 2$ and $MJM^* = NJN^*$.

Let $a_r < d_0 - 1$, $b_r > d_0$. Based on the discussion for (4.6) in [22], an induced self-adjoint restriction of H_1 on I_r can be given by

$$H_{1,r} = \{(x,f) \in H^r : (x,\hat{y}_j)(a_r - 1) - (x,\hat{y}_j)(b_r) = 0, j = 1,2\}. \tag{2.18}$$

Case 4. *One endpoint is in l.c.c. and the other in l.p.c.*

Without loss of generality, we only consider the case that $\mathcal{L}$ is in l.c.c. at $t = -\infty$ and l.p.c. at $t = +\infty$.

Suppose that H_1 is any fixed self-adjoint subspace extension of H_0. According to the discussion for (4.8) in [22], we have

$$H_1 = \{(x,f) \in H : (x,\hat{y})(-\infty) = 0\}, \tag{2.19}$$

where

$$\hat{y}(t) = \begin{cases} m_1\varphi_1(t,\lambda) + m_2\varphi_2(t,\lambda), & t \le d_0 - 1, \\ 0, & t \ge d_0 + 1, \end{cases} \tag{2.20}$$

while $M = (m_1, m_2) \in \mathbf{R}^{1\times 2}$ with $M \neq 0$.

Let $a_r < d_0 - 1$, $b_r > d_0$. By the discussion for (4.11) in [22], an induced self-adjoint restriction of H_1 on I_r can be given by

$$H_{1,r} = \{(x,f) \in H^r : (x,\hat{y})(a_r - 1) = 0, (x,u)(b_r) = 0\}, \tag{2.21}$$

where

$$u = n_1\varphi_1(\cdot,\lambda) + n_2\varphi_2(\cdot,\lambda), \tag{2.22}$$

while $N = (n_1, n_2) \in \mathbf{R}^{1\times 2}$ with $N \neq 0$.

Case 5. *Both endpoints are in l.p.c.*

In this case that $\mathcal{L}$ is in l.p.c. at both endpoints $t = \pm\infty$, $H_1 = H_0$ is the unique self-adjoint subspace extension of H_0.

Let $a_r < d_0 - 1$, $b_r > d_0$. By the discussion for (4.12) in [22], an induced self-adjoint restriction of H_1 on I_r can be given by

$$H_{1,r} = \{(x,f) \in H^r : (x,u)(a_r - 1) = 0, (x,u)(b_r) = 0\}, \tag{2.23}$$

where

$$u = \begin{cases} m_1\varphi_1(\cdot,\lambda) + m_2\varphi_2(\cdot,\lambda), & t \le d_0 - 1, \\ n_1\varphi_1(\cdot,\lambda) + n_2\varphi_2(\cdot,\lambda), & t \ge d_0 + 1, \end{cases} \tag{2.24}$$

while $M = (m_1, m_2) \in \mathbf{R}^{1\times 2}$ with $M \neq 0$ and $N = (n_1, n_2) \in \mathbf{R}^{1\times 2}$ with $N \neq 0$.

Remark 2.1 By Theorem 6.1 in [12], each self-adjoint subspace extension H_1 of H_0 is a self-adjoint operator in the case that $I = (-\infty, +\infty)$; that is, H_1 can define a single-valued self-adjoint operator in $l_w^2(-\infty, +\infty)$ whose graph is H_1.

To end this section, we consider extensions of the induced self-adjoint restrictions from I_r to I.

Note that H_1, $H_{1,r}$ are self-adjoint subspaces in $(l_w^2(I))^2$ and $(l_w^2(I_r))^2$, respectively. It is difficult to study the convergence of $H_{1,r}$ to H_1 in some sense since $l_w^2(I)$ and $l_w^2(I_r)$ are different spaces. In order to overcome this problem, we extended $l_w^2(I_r)$ and $H_{1,r}$ to $\tilde{l}_w^2(I_r)$ and $\tilde{H}_{1,r}$, separately, in [22]. Now, we recall them for convenience.

In the case that $I = [a, +\infty)$,

$$\begin{aligned} &\tilde{l}_w^2(I_r) := \{f \in l_w^2(I) : f(t) = 0, t \ge b_r + 1\}, \\ &\tilde{H}_{1,r} := \{(\tilde{x},\tilde{f}) \in (\tilde{l}_w^2(I_r))^2 : \text{there exists } (x,f) \in H_{1,r} \text{ such that} \\ &\qquad \tilde{x}(t) = x(t), \tilde{f}(t) = f(t), a - 1 \le t \le b_r\}. \end{aligned} \tag{2.25}$$

In the case that $I = (-\infty, +\infty)$,

$$\begin{aligned}
&\tilde{l}_w^2(I_r) := \{f \in l_w^2(I) : f(t) = 0, t \le a_r - 1 \text{ and } t \ge b_r + 1\},\\
&\tilde{H}_{1,r} := \{(\tilde{x}, \tilde{f}) \in (\tilde{l}_w^2(I_r))^2 : \text{there exists } (x,f) \in H_{1,r} \text{ such that}\\
&\qquad \tilde{x}(t) = x(t), \tilde{f}(t) = f(t), a_r \le t \le b_r\}.
\end{aligned} \tag{2.26}$$

Let P_r be the orthogonal projection from $l_w^2(I)$ onto $\tilde{l}_w^2(I_r)$. Define

$$H'_{1,r} := \tilde{H}_{1,r} G(P_r). \tag{2.27}$$

Lemma 2.5 ([22], Lemmas 3.1, 3.2, 3.3 and 4.1) *$\tilde{H}_{1,r}$ and $H'_{1,r}$ are self-adjoint subspaces in $(\tilde{l}_w^2(I_r))^2$ and $(l_w^2(I))^2$, respectively, $D(H'_{1,r}) = D(\tilde{H}_{1,r}) \oplus (\tilde{l}_w^2(I_r))^{\perp}$, $\sigma(\tilde{H}_{1,r}) = \sigma(H_{1,r})$, and $\sigma(H'_{1,r}) = \sigma(\tilde{H}_{1,r}) \cup \{0\} = \sigma(H_{1,r}) \cup \{0\}$.*

The following result can be directly derived from (2.25)-(2.27).

Lemma 2.6 *$H'_{1,r}(0) = \tilde{H}_{1,r}(0) = \{\tilde{f} \in \tilde{l}_w^2(I_r)$: there exists $f \in H_{1,r}(0)$ such that $\tilde{f}(t) = f(t)$ for $t \in I_r\}$.*

3 One endpoint is regular or in l.c.c. and the other in l.c.c

In this section, we study regular approximations of isolated eigenvalues of (1.1) in Cases 1 and 3. Without loss of generality, we only consider the case that $\mathcal{L}$ is regular or in l.c.c. at a and l.c.c. at $t = +\infty$.

We showed that the induced self-adjoint restrictions $\{H_{1,r}\}_{r=1}^{\infty}$ is spectrally exact for the given self-adjoint subspace extension H_1 in Cases 1 and 3 in [22]. Now, we shall further study how the spectrum $\sigma(H_1)$ of H_1 is approximated by the eigenvalues of $H_{1,r}$. In addition, we also give their error estimates.

Lemma 3.1 *Each self-adjoint subspace extension of H_0 has a pure discrete spectrum in Cases* 1 *and* 3.

Proof According to Theorems 6.7 and 6.10 in [24] and Lemma 2.1, it suffices to prove that $(zI - H_1)^{-1}$ is a Hilbert-Schmidt operator for any $z \in \rho(H_1)$.

We only prove that $(zI - H_1)^{-1}$ is a Hilbert-Schmidt operator for any $z \in \rho(H_1)$ in Case 1 with $I = [a, +\infty)$. For the other cases, it can be proved similarly.

By Proposition 3.1 in [22], for any $z \in \rho(H_1)$ and any $f \in l_w^2(I)$,

$$(zI - H_1)^{-1}(f)(t) = \sum_{j=a}^{+\infty} G(t,j,z)w(j)f(j), \quad t \in I,$$

where

$$G(t,j,z) = \begin{cases} \sum_{k,l=1}^{2} m_{kl}\phi_k(t)\phi_l(j), & a \le j \le t < +\infty, \\ \sum_{k,l=1}^{2} n_{kl}\phi_k(t)\phi_l(j), & a \le t < j < +\infty, \end{cases}$$

while m_{kl}, n_{kl} $(1 \le k, l \le 2)$ are constants, and ϕ_1, ϕ_2 are two linearly independent solutions of (1.1_z) satisfying certain initial conditions. It is evident that $\phi_1, \phi_2 \in l_w^2(I)$. Denote

$$m_0 := \max_{k,l=1,2}\{|m_{kl}|\}, \qquad \alpha_0 := \max_{k=1,2}\{\|\phi_k\|\}.$$

Define

$$\mathcal{F}_1(f)(t) := \sum_{j=a}^{+\infty} F_1(t,j,z)w(j)f(j),$$

$$\mathcal{F}_2(f)(t) := \sum_{j=a}^{+\infty} F_2(t,j,z)w(j)f(j), \quad t \in I,$$

where

$$F_1(t,j,z) = \begin{cases} \sum_{k,l=1}^{2} m_{kl}\phi_k(t)\phi_l(j), & a \le j \le t < +\infty, \\ 0, & a \le t < j < +\infty, \end{cases}$$

$$F_2(t,j,z) = \begin{cases} 0, & a \le j \le t < +\infty, \\ \sum_{k,l=1}^{2} n_{kl}\phi_k(t)\phi_l(j), & a \le t < j < +\infty. \end{cases}$$

Obviously, $(zI - H_1)^{-1} = \mathcal{F}_1 + \mathcal{F}_2$. Thus, it is sufficient to prove that $\mathcal{F}_1$ and $\mathcal{F}_2$ are both Hilbert-Schmidt operators.

We first prove that $\mathcal{F}_1$ is a Hilbert-Schmidt operator. Let $\{e_n : n \in \mathbf{N}\}$ be an orthonormal basis of $l_w^2(I)$. Then

$$\begin{aligned} \sum_{n=1}^{\infty} \|\mathcal{F}_1(e_n)\|^2 &= \sum_{n=1}^{\infty} \left\| \sum_{j=a}^{t} \sum_{k,l=1}^{2} m_{kl}\phi_k(t)\phi_l(j)w(j)e_n(j) \right\|^2 \\ &\le 8m_0^2\alpha_0^2 \sum_{l=1}^{2}\sum_{n=1}^{\infty} |\langle \phi_l, e_n\rangle|^2 \\ &\le 16m_0^2\alpha_0^4 < \infty, \end{aligned}$$

in which Parseval's identity have been used. Therefore, $\mathcal{F}_1$ is a Hilbert-Schmidt operator. Similarly, one can show that $\mathcal{F}_2$ is a Hilbert-Schmidt operator and thus $(zI - H_1)^{-1}$ is a Hilbert-Schmidt operator. The proof is complete. □

Remark 3.1

(1) In Lemma 2.22 in [25], Teschl showed that each self-adjoint operator extension H_1 with separated boundary conditions has a pure discrete spectrum, and its resolvent is a Hilbert-Schmidt operator in Case 3.

(2) By applying the Green functions of resolvents of $H_{1,r}$ given in Propositions 3.2 and 4.2 in [22], which still hold for $z \in \rho(H_{1,r})$, it can easily be verified that the resolvents of $H_{1,r}$ are Hilbert-Schmidt operators in Cases 1 and 3. In addition, by (2.25)-(2.26), it is evident that the resolvent of $\tilde{H}_{1,r}$ is also a Hilbert-Schmidt operator in Cases 1 and 3. Moreover, we point out that the results given in Propositions 3.1 and 4.1 in [22] still hold for $z \in \rho(H_1)$.

The following useful lemma can be directly derived from (i)-(ii) of Theorem 3.6 in [21].

Lemma 3.2 *Assume that X_1 is a proper closed subspace in X, $P : X \to X_1$ the orthogonal projection, and T a self-adjoint operator on X_1. Then*

(i) *$T' = TP$ is a self-adjoint operator on X with $D(T') = D(T) \oplus X_1^{\perp}$;*
(ii) *$\sigma(T') = \sigma(T) \cup \{0\}$.*

By Lemma 3.1, H_1 has a discrete spectrum. Via translating it if necessary, we may suppose that 0 is not an eigenvalue of H_1. The eigenvalues of H_1 may be ordered as (multiplicity included):

$$\cdots \le \lambda_{-3} \le \lambda_{-2} \le \lambda_{-1} < 0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \le \cdots. \tag{3.1}$$

For convenience, we briefly denote it by $\sigma(H_1) = \{\lambda_n : n \in \Lambda \subset \mathbf{Z}\backslash\{0\}\}$, where $\mathbf{Z}$ denotes the set of all integer numbers. Recall that $\{H_{1,r}\}$ is spectrally exact for H_1 if $0 \notin \sigma(H_1)$ (see Theorems 3.2 and 4.2 in [22]). Since $0 \notin \sigma(H_1)$, there exists r_0 such that $0 \notin \sigma(H_{1,r})$ for all $r \ge r_0$. Therefore, for $r \ge r_0$, the eigenvalues of $H_{1,r}$ may be ordered as (multiplicity included):

$$\lambda_{-m(r)}^{(r)} \le \cdots \le \lambda_{-2}^{(r)} \le \lambda_{-1}^{(r)} < 0 < \lambda_1^{(r)} \le \lambda_2^{(r)} \le \cdots \le \lambda_{n(r)}^{(r)}, \tag{3.2}$$

where $m(r)$ and $n(r)$ are the numbers of negative and positive eigenvalues of $H_{1,r}$, respectively. For convenience, we briefly denote it by $\sigma(H_{1,r}) = \{\lambda_n^{(r)} : n \in \Lambda_r \subset \mathbf{Z}\backslash\{0\}\}$. Let

$$S = (-H_1)^{-1}, \qquad S_r = (-\tilde{H}_{1,r})^{-1}, \quad r \ge r_0.$$

Then, according to (2) of Remark 3.1 and Lemma 3.2, it follows that S_rP_r and S are both self-adjoint and Hilbert-Schmidt operators. Note that the results of Theorems 3.2 and 4.2 in [22] still hold for every $z \in \rho(H_1) \cap \rho(\tilde{H}_{1,r})$. By Lemma 2.5, $\sigma(H_{1,r}) = \sigma(\tilde{H}_{1,r})$, which implies that $0 \in \rho(\tilde{H}_{1,r})$ as $r \ge r_0$, and thus $0 \in \rho(H_1) \cap \rho(\tilde{H}_{1,r})$ as $r \ge r_0$. Therefore, $S_rP_r \to S$ in norm as $r \to \infty$ by Theorems 3.2 and 4.2 in [22].

Theorem 3.1 *In Cases* 1 *and* 3, *for each $n \in \Lambda$, there exists an $r_n \ge r_0$ such that for $r \ge r_n$, $n \in \Lambda_r$ and $\lambda_n^{(r)} \to \lambda_n$ as $r \to \infty$.*

Proof Based on the above discussion, S and S_rP_r are self-adjoint and Hilbert-Schmidt operators for $r \ge r_0$, and $S_rP_r \to S$ in norm as $r \to \infty$. Thus they are self-adjoint and compact operators with eigenvalues $\mu_n = -1/\lambda_n$ for $n \in \Lambda$ and $\mu_n^{(r)} = -1/\lambda_n^{(r)}$ for $n \in \Lambda_r$, separately, by Lemma 2.1. (S_rP_r also has 0 as an eigenvalue of infinite multiplicity. But it is not related to $H_{1,r}$ or H_1, and so can be ignored.) Furthermore, since $S_rP_r \to S$ in norm as $r \to \infty$, we can get $S_rP_r \to S$ in the norm resolvent sense as $r \to \infty$ according to the proof of Theorem 8.18 in [26] (for the concept of convergence of self-adjoint operators in the norm resolvent sense, please see [24, 26]). Let $E(S_rP_r, \lambda)$ and $E(S, \lambda)$ be spectral families of S_rP_r and S, respectively. Then, by (b) of Theorem 8.23 in [26], it follows that for any $\alpha, \beta \in \mathbf{R} \cap \rho(S)$ with $\alpha < \beta$,

$$\left\| \left\{E(S_rP_r, \beta) - E(S_rP_r, \alpha)\right\} - \left\{E(S, \beta) - E(S, \alpha)\right\} \right\| \to 0 \quad \text{as } r \to \infty,$$

which, together with Theorem 4.35 in [24], shows that

$$\dim R\big\{E(S_rP_r,\beta)-E(S_rP_r,\alpha)\big\}=\dim R\big\{E(S,\beta)-E(S,\alpha)\big\}$$

for all sufficiently large r. Hence, for each $n\in\Lambda$, there exists an $r_n\geq r_0$ such that for $r\geq r_n$, $\mu_n^{(r)}$ exists. This implies that $H_{1,r}$ has an eigenvalue $\lambda_n^{(r)}$ for all $r\geq r_n$; namely, $n\in\Lambda_r$ for all $r\geq r_n$.

Next, we show that $\lambda_n^{(r)}\to\lambda_n$ as $r\to\infty$. To do so, it suffices to prove that $\mu_n^{(r)}\to\mu_n$ as $r\to\infty$. The negative eigenvalues are described by a min-max principle, and the positive eigenvalues by a max-min principle according to Section 12.1 in [27]; that is,

$$\mu_n=\begin{cases}\min_{V_n}\max_{x\in V_n,\,\|x\|=1}\langle Sx,x\rangle, & n\in\Lambda \text{ with } n>0,\\ \max_{V_n}\min_{x\in V_n,\,\|x\|=1}\langle Sx,x\rangle, & n\in\Lambda \text{ with } n<0,\end{cases}\tag{3.3}$$

where V_n runs through all the $|n|$-dimensional subspaces of $l_w^2(I)$. For $r\geq r_n$, $\mu_n^{(r)}$ is similarly expressed in terms of $\langle S_rP_rx,x\rangle$; that is,

$$\mu_n^{(r)}=\begin{cases}\min_{V_n}\max_{x\in V_n,\,\|x\|=1}\langle S_rP_rx,x\rangle, & n\in\Lambda \text{ with } n>0,\\ \max_{V_n}\min_{x\in V_n,\,\|x\|=1}\langle S_rP_rx,x\rangle, & n\in\Lambda \text{ with } n<0.\end{cases}\tag{3.4}$$

We first consider the case that $n\in\Lambda$ with $n>0$. Let $r\geq r_n$. It follows from (3.3)-(3.4) that there exist two n-dimensional subspaces V_n and $\tilde{V}_n$ of $l_w^2(I)$ such that

$$\mu_n=\max_{\substack{x\in V_n,\\ \|x\|=1}}\langle Sx,x\rangle,\qquad \mu_n^{(r)}=\max_{\substack{x\in\tilde{V}_n,\\ \|x\|=1}}\langle S_rP_rx,x\rangle.\tag{3.5}$$

In addition, there exist $x_1\in\tilde{V}_n$ with $\|x_1\|=1$ and $x_2\in V_n$ with $\|x_2\|=1$ such that

$$\max_{\substack{x\in\tilde{V}_n,\\ \|x\|=1}}\langle Sx,x\rangle=\langle Sx_1,x_1\rangle,\qquad \max_{\substack{x\in V_n,\\ \|x\|=1}}\langle S_rP_rx,x\rangle=\langle S_rP_rx_2,x_2\rangle.\tag{3.6}$$

From (3.3)-(3.6), we have

$$\mu_n-\mu_n^{(r)}\leq\max_{\substack{x\in\tilde{V}_n,\\ \|x\|=1}}\langle Sx,x\rangle-\max_{\substack{x\in\tilde{V}_n,\\ \|x\|=1}}\langle S_rP_rx,x\rangle\leq\big\langle(S-S_rP_r)x_1,x_1\big\rangle,$$

$$\mu_n-\mu_n^{(r)}\geq\max_{\substack{x\in V_n,\\ \|x\|=1}}\langle Sx,x\rangle-\max_{\substack{x\in V_n,\\ \|x\|=1}}\langle S_rP_rx,x\rangle\geq\big\langle(S-S_rP_r)x_2,x_2\big\rangle.$$

Therefore, it follows that

$$\begin{aligned}\big|\mu_n-\mu_n^{(r)}\big|&\leq\max\big\{\big|\big\langle(S-S_rP_r)x_1,x_1\big\rangle\big|,\big|\big\langle(S-S_rP_r)x_2,x_2\big\rangle\big|\big\}\\ &\leq\|S-S_rP_r\|\to0\quad\text{as } r\to\infty.\end{aligned}\tag{3.7}$$

Thus, $\mu_n^{(r)}\to\mu_n$ as $r\to\infty$ for $n\in\Lambda$ with $n>0$.

Similarly, we can get $\mu_n^{(r)}\to\mu_n$ as $r\to\infty$ for $n\in\Lambda$ with $n<0$. This completes the proof. □

At the end of this section, we shall try to give an error estimate for the approximation of λ_n by $\lambda_n^{(r)}$ for each $n \in \Lambda$. Obviously, it is very important in numerical analysis and applications. In order to give error estimates of $\lambda_n^{(r)}$ to λ_n, in view of $\lambda_n = -1/\mu_n$ and $\lambda_n^{(r)} = -1/\mu_n^{(r)}$, we shall first investigate the error estimates of $\mu_n^{(r)}$ to μ_n for $n \in \Lambda$ instead.

In view of the arbitrariness of $\lambda \in \mathbf{R}$ in (2.8), we might as well take $\lambda = 0$ in (2.8) in the following discussions.

Proposition 3.1 *Assume that $\mathcal{L}$ is regular at $t = a$ and in l.c.c. at $t = +\infty$. Then, for each $n \in \Lambda$ and $r \geq r_n$, where r_n is specified in Theorem* 3.1,

$$\left|\mu_n^{(r)} - \mu_n\right| \leq 4\alpha_0\left(2m_0^2 + 5n_0^2\right)^{\frac{1}{2}}\left(2 + \frac{1}{p^2(a-1)}\right)\varepsilon_r, \tag{3.8}$$

where α_0, m_0, and n_0 are constants and determined by (3.10)-(3.16), *and ε_r is completely determined by the coefficients of* (1.1), *more precisely, it is determined by* (3.17)-(3.18), (3.21), (3.23)-(3.24). *In addition, $\varepsilon_r \to 0$ as $r \to \infty$.*

Proof Note that the results of Propositions 3.1 and 3.2 and Theorem 3.2 in [22] still hold for every $z \in \rho(H_1) \cap \rho(\tilde{H}_{1,r})$. By Lemma 2.5, $\sigma(H_{1,r}) = \sigma(\tilde{H}_{1,r})$, which implies that $0 \in \rho(\tilde{H}_{1,r})$ as $r \geq r_0$, and thus $0 \in \rho(H_1) \cap \rho(\tilde{H}_{1,r})$ as $r \geq r_0$. Consequently, by (3.43)-(3.44) in [22], one has

$$\|S - S_rP_r\| \leq 4\alpha_0\left[3\alpha_0^2\left(m_r^2 + n_r^2\right) + \left(2m_0^2 + 5n_0^2\right)\alpha_r^2\right]^{\frac{1}{2}}, \quad r \geq r_0, \tag{3.9}$$

where

$$\begin{aligned}
&m_0 := \max_{k,l=1,2}\left\{\left|m_{kl}^0\right|\right\}, \qquad n_0 := \max_{k,l=1,2}\left\{\left|n_{kl}^0\right|\right\}, \qquad m_r := \max_{k,l=1,2}\left\{\left|m_{kl}^0 - m_{kl}^r\right|\right\}, \\
&n_r := \max_{k,l=1,2}\left\{\left|n_{kl}^0 - n_{kl}^r\right|\right\}, \qquad \alpha_0 := \max_{i=1,2}\left\{\|\phi_i\|\right\}, \\
&\alpha_r := \max_{i=1,2}\left\{\sum_{t=b_r+1}^{\infty}\left|\phi_i(t)\right|^2 w(t)\right\},
\end{aligned} \tag{3.10}$$

while ϕ_1 and ϕ_2 are two linearly independent solutions of (1.1_λ) with $\lambda = 0$ satisfying the following initial conditions:

$$\begin{aligned}
&\phi_1(a-1) = 0, \qquad \phi_2(a-1) = -1, \\
&p(a-1)\Delta\phi_1(a-1) = 1, \qquad p(a-1)\Delta\phi_2(a-1) = 0,
\end{aligned} \tag{3.11}$$

$$M_0 = \begin{pmatrix} -m_{12}^0 & m_{11}^0 \\ -m_{22}^0 & m_{21}^0 \end{pmatrix} = I + N_0, \tag{3.12}$$

$$N_0 = \begin{pmatrix} -n_{12}^0 & n_{11}^0 \\ -n_{22}^0 & n_{21}^0 \end{pmatrix} = -\left(\hat{Y}_0^*(a-1) + K\right)^{-1}K, \qquad K = \lim_{t\to+\infty}\hat{Y}_0^*(t)J\Phi(t), \tag{3.13}$$

$$M_r = \begin{pmatrix} -m_{12}^r & m_{11}^r \\ -m_{22}^r & m_{21}^r \end{pmatrix} = I + N_r, \tag{3.14}$$

$$N_r = \begin{pmatrix} -n_{12}^r & n_{11}^r \\ -n_{22}^r & n_{21}^r \end{pmatrix} = -\left(\hat{Y}_0^*(a-1) + K_r\right)^{-1}K_r, \qquad K_r = \hat{Y}_0^*(b_r)J\Phi(b_r), \tag{3.15}$$

$$\Phi(t) = \begin{pmatrix} \phi_1(t) & \phi_2(t) \\ p(t)\triangle\phi_1(t) & p(t)\triangle\phi_2(t) \end{pmatrix}, \qquad \hat{Y}_0(t) = \begin{pmatrix} \hat{y}_1(t) & \hat{y}_2(t) \\ p(t)\triangle\hat{y}_1(t) & p(t)\triangle\hat{y}_2(t) \end{pmatrix}, \tag{3.16}$$

where $\hat{y}_1(t)$ and $\hat{y}_2(t)$ are defined by (2.10). Noting that $\hat{y}_1$, $\hat{y}_2$, ϕ_1, and ϕ_2 are both solutions of (1.1_λ) with $\lambda = 0$ in $[c_0, +\infty)$, where c_0 is the same as in (2.10). Since $b_r > c_0$, by Lemma 2.3 in [12], we get $K = K_r$, which shows that $M_r = M_0$, $N_r = N_0$, and thus $m_r = n_r = 0$.

Now, it remains to estimate α_r. Let

$$W(t) := \begin{pmatrix} w(t+1) & 0 \\ 0 & w(t) \end{pmatrix}, \qquad U(t) := \begin{pmatrix} c(t) & -d(t) \\ 1 & 0 \end{pmatrix}, \quad t \geq a, \tag{3.17}$$

where

$$c(t) := 1 + \frac{q(t)}{p(t)} + \frac{p(t-1)}{p(t)}, \qquad d(t) := \frac{p(t-1)}{p(t)}, \quad t \geq a. \tag{3.18}$$

From (1.1_λ) with $\lambda = 0$, we get

$$x(t+1) = c(t)x(t) - d(t)x(t-1), \quad t \geq a.$$

It follows that

$$\begin{pmatrix} x(t+1) \\ x(t) \end{pmatrix} = U(t) \begin{pmatrix} x(t) \\ x(t-1) \end{pmatrix}, \quad t \geq a. \tag{3.19}$$

By (3.17) and (3.19), we get

$$\begin{aligned} & w(t+1)x^2(t+1) + w(t)x^2(t) \\ &\quad = \bigl(x(t+1), x(t)\bigr) W(t) \bigl(x(t+1), x(t)\bigr)^\top \\ &\quad = \bigl(x(t), x(t-1)\bigr) U^\top(t) W(t) U(t) \bigl(x(t), x(t-1)\bigr)^\top \\ &\quad = \bigl(x(a), x(a-1)\bigr) V^\top(t) W(t) V(t) \bigl(x(a), x(a-1)\bigr)^\top, \quad t \geq a, \end{aligned} \tag{3.20}$$

where

$$V(t) := U(t)U(t-1)\cdots U(a+1)U(a), \quad t \geq a. \tag{3.21}$$

Let $x(a)$ and $x(a-1)$ be any real numbers. Since $\mathcal{L}$ is regular at $t = a$ and in l.c.c. at $t = +\infty$, it follows from (3.20) that

$$\begin{aligned} & \sum_{t=b_r+1}^{\infty} w(t)x^2(t) \\ &\quad = \sum_{i=0}^{\infty} \bigl[w(b_r+2i+2)x^2(b_r+2i+2) + w(b_r+2i+1)x^2(b_r+2i+1)\bigr] \\ &\quad = \bigl(x(a), x(a-1)\bigr) D_r \bigl(x(a), x(a-1)\bigr)^\top \to 0 \quad \text{as } r \to \infty, \end{aligned} \tag{3.22}$$

where

$$D_r := \sum_{i=0}^{\infty} V^\top(b_r + 2i + 1) W(b_r + 2i + 1) V(b_r + 2i + 1). \tag{3.23}$$

Denote

$$\varepsilon_r := \|D_r\|, \tag{3.24}$$

where the norm of the matrix $D_r = (d_{ij}^r)_{2\times 2}$ is defined as $\|D_r\| = \sum_{i,j=1}^2 |d_{ij}^r|$. Since D_r is symmetric and $(x(a), x(a-1))$ is arbitrary, it follows from (3.22) that $\varepsilon_r \to 0$ as $r \to \infty$. In addition, since ϕ_1 and ϕ_2 satisfy (3.22), it follows that

$$\begin{aligned} \alpha_r &= \max_{i=1,2}\bigl\{\bigl(\phi_i(a), \phi_i(a-1)\bigr) D_r \bigl(\phi_i(a), \phi_i(a-1)\bigr)^\top\bigr\} \\ &\le \varepsilon_r \max_{i=1,2}\bigl\{\phi_i^2(a) + \phi_i^2(a-1)\bigr\} \\ &\le \varepsilon_r \left(2 + \frac{1}{p^2(a-1)}\right), \end{aligned} \tag{3.25}$$

in which (3.11) has been used. This, together with $m_r = n_r = 0$, (3.9), and (3.7), shows that (3.8) holds. This completes the proof. □

Theorem 3.2 *Assume that $\mathcal{L}$ is regular at $t = a$ and in l.c.c. at $t = +\infty$. Then, for each $n \in \Lambda$, there exists an $r_n' \ge r_n$ such that for all $r \ge r_n'$,*

$$\bigl|\lambda_n^{(r)} - \lambda_n\bigr| \le \frac{|\lambda_n^{(r)}|^2 e_r}{1 - |\lambda_n^{(r)}| e_r}, \tag{3.26}$$

$$\bigl|\lambda_n^{(r)} - \lambda_n\bigr| \le \frac{|\lambda_n|^2 e_r}{1 - |\lambda_n| e_r}, \tag{3.27}$$

where e_r denotes the number on the right-hand side in (3.8).

Proof For each $n \in \Lambda$, λ_n and $\lambda_n^{(r)}$ have the same sign for sufficiently large r. In view of $\lambda_k = -1/\mu_k$ and $\lambda_k^{(r)} = -1/\mu_k^{(r)}$, it follows from (3.8) that for each $n \in \Lambda$,

$$\left|\frac{1}{\lambda_n^{(r)}} - \frac{1}{\lambda_n}\right| \le e_r, \quad r \ge r_n,$$

which shows that

$$\bigl|\lambda_n^{(r)} - \lambda_n\bigr| \le e_r \bigl|\lambda_n^{(r)}\bigr| |\lambda_n|, \quad r \ge r_n. \tag{3.28}$$

Thus,

$$|\lambda_n| = \bigl|\lambda_n + \lambda_n^{(r)} - \lambda_n^{(r)}\bigr| \le \bigl|\lambda_n - \lambda_n^{(r)}\bigr| + \bigl|\lambda_n^{(r)}\bigr| \le e_r \bigl|\lambda_n^{(r)}\bigr| |\lambda_n| + \bigl|\lambda_n^{(r)}\bigr|, \quad r \ge r_n,$$

which implies that

$$|\lambda_n| \bigl(1 - \bigl|\lambda_n^{(r)}\bigr| e_r\bigr) \le \bigl|\lambda_n^{(r)}\bigr|. \tag{3.29}$$

By Theorem 3.1 and Proposition 3.1, there exists an $r_n' \geq r_n$ such that $1 - |\lambda_n^{(r)}| e_r > 0$. Hence, it follows from (3.28) and (3.29) that (3.26) holds. With a similar argument, one can show that (3.27) holds. This completes the proof. □

We now turn to error estimates of approximations of $\lambda_n^{(r)}$ to λ_n in Case 3.

Proposition 3.2 *Assume that $\mathcal{L}$ is in l.c.c. at $t = \pm\infty$. Then, for each $n \in \Lambda$ and $r \geq r_n$, where r_n is specified in Theorem* 3.1,

$$\left|\mu_n^{(r)} - \mu_n\right| \leq 8\sqrt{2}\tilde{\alpha}_0 \left(\tilde{m}_0^2 + \tilde{n}_0^2\right)^{\frac{1}{2}} \left(2 + \frac{1}{p^2(d_0 - 1)}\right) \tilde{\varepsilon}_r, \tag{3.30}$$

where $\tilde{\alpha}_0$, $\tilde{m}_0$, and $\tilde{n}_0$ are constants and given by (3.32), *d_0 is the same as in* (2.8), *and $\tilde{\varepsilon}_r$ is completely determined by the coefficients of* (1.1), *more precisely, it is determined by* (3.17)-(3.18), (3.36), (3.39)-(3.41). *In addition, $\tilde{\varepsilon}_r \to 0$ as $r \to \infty$.*

Proof The main idea of the proof is similar to that of Proposition 3.1, where the interval $I = [a, +\infty)$ is replaced by $I = (-\infty, +\infty)$. For completeness, we now give its detailed proof.

Note that the results of Propositions 4.1 and 4.2 and Theorem 4.2 in [22] still hold for every $z \in \rho(H_1) \cap \rho(\tilde{H}_{1,r})$. By Lemma 2.5, $\sigma(H_{1,r}) = \sigma(\tilde{H}_{1,r})$, which implies that $0 \in \rho(\tilde{H}_{1,r})$ as $r \geq r_0$, and thus $0 \in \rho(H_1) \cap \rho(\tilde{H}_{1,r})$ as $r \geq r_0$. Consequently, by Theorem 4.2 in [22] one has

$$\|S - S_r P_r\| \leq 8\tilde{\alpha}_0 \left[\tilde{\alpha}_0^2 \left(\tilde{m}_r^2 + \tilde{n}_r^2\right) + 2\left(\tilde{m}_0^2 + \tilde{n}_0^2\right)\tilde{\alpha}_r^2\right]^{\frac{1}{2}}, \quad r \geq r_0, \tag{3.31}$$

where

$$\begin{aligned}
&\tilde{m}_0 := \max_{k,l=1,2}\left\{\left|m_{kl}^0\right|\right\}, \qquad \tilde{n}_0 := \max_{k,l=1,2}\left\{\left|n_{kl}^0\right|\right\}, \\
&\tilde{m}_r := \max_{k,l=1,2}\left\{\left|m_{kl}^0 - m_{kl}^r\right|\right\}, \qquad \tilde{n}_r := \max_{k,l=1,2}\left\{\left|n_{kl}^0 - n_{kl}^r\right|\right\}, \\
&\tilde{\alpha}_0 := \max_{i=1,2}\left\{\|\phi_i\|\right\}, \qquad \tilde{\alpha}_r := \max_{i=1,2}\left\{\sum_{t=-\infty}^{a_r-1}\left|\phi_i(t)\right|^2 w(t) + \sum_{t=b_r+1}^{\infty}\left|\phi_i(t)\right|^2 w(t)\right\},
\end{aligned} \tag{3.32}$$

while m_{kl}^0, n_{kl}^0 and m_{kl}^r, n_{kl}^r are given by (4.19) and (4.21) in [22], separately; ϕ_1 and ϕ_2 are two linearly independent solutions of (1.1_λ) with $\lambda = 0$ satisfying the initial conditions (3.11), in which a is replaced by d_0 given in (2.8). With a similar discussion to that in the proof of Proposition 3.1, we can prove that $\tilde{m}_r = \tilde{n}_r = 0$. Therefore, it remains to estimate $\tilde{\alpha}_r$.

Let $W(t)$, $U(t)$, $c(t)$, $d(t)$ be defined as these in (3.17)-(3.18) for every $t \in I$. It is evident that $U(t)$ is invertible for any $t \in I$. From (1.1_λ) with $\lambda = 0$, we get

$$x(t+1) = c(t)x(t) - d(t)x(t-1), \quad t \in I,$$

which shows that

$$\begin{pmatrix} x(t+1) \\ x(t) \end{pmatrix} = U(t) \begin{pmatrix} x(t) \\ x(t-1) \end{pmatrix}, \quad t \in I. \tag{3.33}$$

By (3.17) and (3.33), we get

$$\begin{aligned}
& w(t+1)x^2(t+1) + w(t)x^2(t) \\
&\quad = \big(x(t+1),\, x(t)\big) W(t) \big(x(t+1),\, x(t)\big)^\top \\
&\quad = \big(x(t),\, x(t-1)\big) U^\top(t) W(t) U(t) \big(x(t),\, x(t-1)\big)^\top \\
&\quad = \big(x(d_0),\, x(d_0-1)\big) V_+^\top(t) W(t) V_+(t) \big(x(d_0),\, x(d_0-1)\big)^\top, \quad t \geq d_0, \qquad (3.34) \\
& w(t)x^2(t) + w(t-1)x^2(t-1) \\
&\quad = \big(x(t),\, x(t-1)\big) W(t-1) \big(x(t),\, x(t-1)\big)^\top \\
&\quad = \big(x(t+1),\, x(t)\big) \big(U^{-1}\big)^\top(t) W(t-1) U^{-1}(t) \big(x(t+1),\, x(t)\big)^\top \\
&\quad = \big(x(d_0),\, x(d_0-1)\big) V_-^\top(t) W(t-1) V_-(t) \big(x(d_0),\, x(d_0-1)\big)^\top, \quad t \leq d_0 - 1, \qquad (3.35)
\end{aligned}$$

where

$$\begin{aligned}
& V_+(t) = U(t)U(t-1)\cdots U(d_0+1)U(d_0), \quad t \geq d_0, \\
& V_-(t) = U^{-1}(t)U^{-1}(t+1)\cdots U^{-1}(d_0-2)U^{-1}(d_0-1), \quad t \leq d_0 - 1.
\end{aligned} \qquad (3.36)$$

Let $x(d_0)$ and $x(d_0-1)$ be any real numbers. Recall that $a_r < d_0 - 1$, $b_r > d_0$. Since $\mathcal{L}$ is in l.c.c. at $t = \pm\infty$, it follows from (3.34)-(3.35) that

$$\begin{aligned}
& \sum_{t=b_r+1}^{\infty} w(t)x^2(t) \\
&\quad = \sum_{i=0}^{\infty} \big[w(b_r+2i+2)x^2(b_r+2i+2) + w(b_r+2i+1)x^2(b_r+2i+1)\big] \\
&\quad = \big(x(d_0),\, x(d_0-1)\big) D_{r_+} \big(x(d_0),\, x(d_0-1)\big)^\top \to 0 \quad \text{as } r \to \infty, \qquad (3.37) \\
& \sum_{t=-\infty}^{a_r-1} w(t)x^2(t) \\
&\quad = \sum_{i=0}^{\infty} \big[w(a_r-2i-1)x^2(a_r-2i-1) + w(a_r-2i-2)x^2(a_r-2i-2)\big] \\
&\quad = \big(x(d_0),\, x(d_0-1)\big) D_{r_-} \big(x(d_0),\, x(d_0-1)\big)^\top \to 0 \quad \text{as } r \to \infty, \qquad (3.38)
\end{aligned}$$

where

$$\begin{aligned}
& D_{r_+} := \sum_{i=0}^{\infty} V_+^\top(b_r+2i+1) W(b_r+2i+1) V_+(b_r+2i+1), \\
& D_{r_-} := \sum_{i=0}^{\infty} V_-^\top(a_r-2i-1) W(a_r-2i-2) V_-(a_r-2i-1).
\end{aligned} \qquad (3.39)$$

Denote

$$\varepsilon_{r_+} := \|D_{r_+}\|, \qquad \varepsilon_{r_-} := \|D_{r_-}\|, \qquad (3.40)$$

where the norm of the matrix $D = (d_{ij})_{2\times 2}$ is defined as $\|D\| = \sum_{i,j=1}^{2} |d_{ij}|$. Since D_{r_+} and D_{r_+} are both symmetric and $(x(d_0), x(d_0-1))$ is arbitrary, it follows from (3.37)-(3.38) that

$$\tilde{\varepsilon}_r := \varepsilon_{r_+} + \varepsilon_{r_-} \to 0 \quad \text{as } r \to \infty. \tag{3.41}$$

In addition, since ϕ_1 and ϕ_2 satisfy (3.37)-(3.38), one gets

$$\begin{aligned}
\alpha_r &= \max_{i=1,2} \left\{ \big(\phi_i(d_0),\ \phi_i(d_0-1)\big)(D_{r_+} + D_{r_-})\big(\phi_i(d_0),\ \phi_i(d_0-1)\big)^{\top} \right\} \\
&\leq \tilde{\varepsilon}_r \max_{i=1,2} \left\{ \phi_i^2(d_0) + \phi_i^2(d_0-1) \right\} \\
&\leq \tilde{\varepsilon}_r \left(2 + \frac{1}{p^2(d_0-1)} \right),
\end{aligned} \tag{3.42}$$

in which (3.11) with $a = d_0$ has been used. This, together with $\tilde{m}_r = \tilde{n}_r = 0$, (3.31), and (3.7), shows that (3.30) holds. This completes the proof. □

The proof of the following result is similar to that of Theorem 3.2 and so its details are omitted.

Theorem 3.3 *Assume that $\mathcal{L}$ is in l.c.c. at $t = \pm\infty$. Then, for each $n \in \Lambda$, there exists an $r_n' \geq r_n$ such that for all $r \geq r_n'$,*

$$\left|\lambda_n^{(r)} - \lambda_n\right| \leq \frac{|\lambda_n^{(r)}|^2 \tilde{e}_r}{1 - |\lambda_n^{(r)}|\tilde{e}_r}, \qquad \left|\lambda_n^{(r)} - \lambda_n\right| \leq \frac{|\lambda_n|^2 \tilde{e}_r}{1 - |\lambda_n|\tilde{e}_r},$$

where $\tilde{e}_r$ denotes the number on the right-hand side in (3.30).

Remark 3.2 The authors in [1, 4] and [3] gave similar results to Theorem 3.1 for singular second-order and fourth-order differential Sturm-Liouville problems, respectively, where the results in [1, 4] hold under the assumption that each endpoint is regular or in l.c.c. and non-oscillatory. However, they did not give any error estimate for the approximations of isolated eigenvalues. To the best of our knowledge, there have been no results about error estimates for approximations of isolated eigenvalues of singular differential and difference equations in the existing literature.

4 One endpoint is regular and the other in l.p.c

In this section, we shall study spectral exactness in an open interval laking essential spectral points in Case 2. Without loss of generality, we only consider the case that $\mathcal{L}$ is regular at a and in l.p.c. at $t = +\infty$.

In [22], we proved that the sequence of induced self-adjoint restrictions $\{H_{1,r}\}_{r=1}^{\infty}$ is spectrally inclusive for a given self-adjoint subspace extension H_1 in Case 2 and pointed out that it is not spectrally exact in general. In this section, we will choose a sequence of special induced self-adjoint restrictions, still denoted by $\{H_{1,r}\}_{r=1}^{\infty}$ without any confusion, such that it is spectrally exact for H_1 in an interval laking essential spectral points.

The following are some useful lemmas.

Lemma 4.1 ([24], Exercise 7.37) *Let T be a self-adjoint operator with spectral family E, and S a subspace in $D(T)$ such that $\|(\lambda - T)f\| \leq c\|f\|$ for all $f \in S$. Then* $\dim R\{E(\lambda + c) - E(\lambda - c-)\} \geq \dim S$.

Lemma 4.2 ([28], Lemma 8.1.23) *If P and P_n are orthogonal projections on X with* $\dim R(P_n) \leq \dim R(P) < \infty$ *for $n \geq 1$ and P_n is strongly convergent to P as $n \to \infty$, denoted by $P_n \xrightarrow{s} P$, then* $\dim R(P_n) = \dim R(P)$ *for sufficiently large n.*

Lemma 4.3 *Let $\mathcal{L}$ be regular or in l.c.c. at one endpoint and in l.p.c. at the other endpoint, i.e., in Case* 2 *or Case* 4. *If for some $\lambda \in \mathbf{R}$ the equation* (1.1_λ) *has no nontrivial square summable solutions, then λ belongs to the essential spectrum of every self-adjoint subspace extension H_1 of H_0.*

Proof By the assumption, λ is not an eigenvalue of H_1. Denote

$$M_\lambda := \{(x, \lambda x) \in H\}.$$

Then $M_\lambda = M_{\bar{\lambda}} = \{(0,0)\}$. Hence, by Lemma 2.2 in [11] we have $R(H_0 - \lambda I)^\perp = D(M_{\bar{\lambda}}) = \{0\}$. In addition, since the deficiency indices of H_0 are (1,1), λ is not in the regularity domain of H_0 by Theorem 2.3 in [11], and therefore it is not in the resolvent set of H_1. Hence, λ is in the essential spectrum of H_1, *i.e.*, $\lambda \in \sigma_e(H_1)$. This completes the proof. □

Remark 4.1 Teschl in Lemma 2.2 in [25] showed the same statement as Lemma 4.3 when one endpoint is finite and the other endpoint is in l.p.c. and H_1 is an operator. The authors in Corollary 6.4 in [29] showed a similar result to Lemma 4.3 when it is regular at one endpoint and in l.p.c. at the other endpoint. Since our proof of Lemma 4.3 is more simple, we list it here.

Let $E_s(H_1, \lambda)$, $E_s(H_{1,r}, \lambda)$, $E_s(\tilde{H}_{1,r}, \lambda)$, and $E_s(H'_{1,r}, \lambda)$ be spectral families of $H_{1,s}$, $H_{1,r,s}$, $\tilde{H}_{1,r,s}$, and $H'_{1,r,s}$, respectively, which denote the operator parts of H_1, $H_{1,r}$, $\tilde{H}_{1,r}$, and $H'_{1,r}$, respectively.

Theorem 4.1 *Assume that $\mathcal{L}$ is regular at $t = a$ and in l.p.c. at $t = +\infty$. Let H_1 be any fixed self-adjoint subspace extension of H_0 given by* (2.12). *Assume that $0 \notin I_0 \subset \mathbf{R}$ is an open interval with $I_0 \cap \sigma_e(H_1) = \emptyset$ and $I_0 \cap \sigma_d(H_1) \neq \emptyset$. Let ν be a nontrivial real square summable solution of* (1.1_γ) *with any fixed $\gamma \in I_0$, $H_{1,r}$ the induced self-adjoint restriction of H_1 on I_r defined by*

$$H_{1,r} = \{(x,f) \in H^r : (x, \hat{y})(a-1) = 0, (x, \nu)(b_r) = 0\}, \tag{4.1}$$

where $\hat{y}$ is defined by (2.13) *and $\{b_r\}_{r=1}^\infty$ specified in Section* 2.3 *satisfies $\nu(b_r) \neq 0$ for $r \in \mathbf{N}$. Then $\{H_{1,r}\}_{r=1}^\infty$ is spectrally exact for H_1 in I_0.*

By Lemma 4.3, there exists at least one nontrivial square summable solution ν of (1.1_γ) for any $\gamma \in I_0$, where I_0 is specified in Theorem 4.1. Consequently, there are infinite $t \in I = \{t\}_{t=a}^{+\infty}$ such that $\nu(t) \neq 0$, and so we can choose $\{b_r\}_{r=1}^\infty$ specified in Section 2.3 such that $\nu(b_r) \neq 0$ for $r \in \mathbf{N}$ in (4.1). Hence, $H_{1,r}$ given by (4.1) is well defined.

Proof of Theorem 4.1 By (2.2) and Lemmas 2.4 and 2.5, it suffices to show that

$$H'_{1,r}(0) = H_1(0), \tag{4.2}$$

and for any given $\alpha, \beta \in I_0 \cap \rho(H_1)$ with $\alpha < \gamma \leq \beta$, there exists an integer $\tilde{r} \geq 1$ such that for all $r \geq \tilde{r}$,

$$\dim R\{E_s(H'_{1,r}, \beta) - E_s(H'_{1,r}, \alpha)\} = \dim R\{E_s(H_1, \beta) - E_s(H_1, \alpha)\}. \tag{4.3}$$

We first prove (4.2). By (2.7) and (2.12), it can be deduced that

$$H_1(0) = \{f \in l^2_w(I) : \hat{y}(a)f(a) = 0, f(t) = 0, t \geq a+1\}, \tag{4.4}$$

By (4.1) and $\nu(b_r) \neq 0$ we get

$$H_{1,r}(0) = \{f \in l^2_w(I_r) : \hat{y}(a)f(a) = 0, f(t) = 0, a+1 \leq t \leq b_r\},$$

which, together with Lemma 2.6, shows that

$$H'_{1,r}(0) = \tilde{H}_{1,r}(0) = \{f \in l^2_w(I) : \hat{y}(a)f(a) = 0, f(t) = 0, t \geq a+1\}.$$

Therefore, (4.2) holds.

Next, we show that (4.3) holds. Fix any $\alpha, \beta \in I_0 \cap \rho(H_1)$ with $\alpha < \gamma \leq \beta$. For any fixed r, let $\lambda_1, \ldots, \lambda_{k_r}$ be all k_r (counting multiplicity) eigenvalues of $H_{1,r}$ in $(\alpha, \beta]$, and $\eta_1, \ldots, \eta_{k_r}$ the corresponding orthonormal eigenfunctions. By Lemma 2.5 and the assumption that $0 \notin I_0$ we get

$$\sigma(H'_{1,r}) \cap (\alpha, \beta] = \sigma(\tilde{H}_{1,r}) \cap (\alpha, \beta] = \sigma(H_{1,r}) \cap (\alpha, \beta] = \{\lambda_1, \ldots, \lambda_{k_r}\}.$$

By (i) of Theorem 3.9 in [20] we have

$$\begin{aligned} &\dim R\{E_s(H'_{1,r}, \beta) - E_s(H'_{1,r}, \alpha)\} \\ &\quad = \dim R\{E_s(\tilde{H}_{1,r}, \beta) - E_s(\tilde{H}_{1,r}, \alpha)\} \\ &\quad = \dim R\{E_s(H_{1,r}, \beta) - E_s(H_{1,r}, \alpha)\} = k_r. \end{aligned} \tag{4.5}$$

In addition, it follows from Lemma 2.2 that $\lambda_1, \ldots, \lambda_{k_r}$ are all k_r (counting multiplicity) eigenvalues of $H_{1,r,s}$ in $(\alpha, \beta]$, and $\eta_1, \ldots, \eta_{k_r}$ are the corresponding orthonormal eigenfunctions. Since $\eta_j \in D(H_{1,r})$, by (4.1) we have $\eta_j(b_r+1) = \frac{\nu(b_r+1)}{\nu(b_r)}\eta_j(b_r)$, $1 \leq j \leq k_r$. Hence, there exist constants c_j, $1 \leq j \leq k_r$, such that

$$\begin{pmatrix} \eta_j(b_r) \\ \eta_j(b_r+1) \end{pmatrix} = c_j \begin{pmatrix} \nu(b_r) \\ \nu(b_r+1) \end{pmatrix}. \tag{4.6}$$

For every $j \in \{1, \ldots, k_r\}$, let

$$\psi_j(t) = \begin{cases} \eta_j(t), & a-1 \leq t \leq b_r, \\ c_j \nu(t), & t \geq b_r+1. \end{cases}$$

Then $\psi_j \in D(H)$. This, together with (4.1) and $\eta_j \in D(H_{1,r})$, shows that $\psi_j \in D(H_1) = D(H_{1,s})$, where (2.1) has been used. Let

$$S := L\{\psi_1, \dots, \psi_{k_r}\}.$$

Then S obviously is k_r-dimensional, and every $\psi \in S$ is of the form

$$\psi(t) = \begin{cases} \sum_{j=1}^{k_r} d_j \eta_j(t), & a-1 \le t \le b_r, \\ c\nu(t), & t \ge b_r + 1, \end{cases} \tag{4.7}$$

where d_j, $j = 1, \dots, k_r$, are constants, and $c = \sum_{j=1}^{k_r} d_j c_j$. It follows from (4.7) that

$$\|\psi\|^2 = \sum_{t=a}^{+\infty} w(t)\left|\psi(t)\right|^2 = \sum_{j=1}^{k_r} |d_j|^2 + |c|^2 \sum_{t=b_r+1}^{+\infty} w(t)\left|\nu(t)\right|^2.$$

Therefore,

$$\begin{aligned}
&\left\|\left(H_{1,s} - \frac{\alpha+\beta}{2}\right)\psi\right\|^2 \\
&\quad = \sum_{t=a}^{+\infty} w(t)\left|\left(H_{1,s} - \frac{\alpha+\beta}{2}\right)\psi(t)\right|^2 \\
&\quad = \sum_{j=1}^{k_r} |d_j|^2 \sum_{t=a}^{b_r} w(t)\left|\left(\lambda_j - \frac{\alpha+\beta}{2}\right)\eta_j(t)\right|^2 + |c|^2 \sum_{t=b_r+1}^{+\infty} w(t)\left|\left(\gamma - \frac{\alpha+\beta}{2}\right)\nu(t)\right|^2 \\
&\quad = \sum_{j=1}^{k_r} |d_j|^2 \left|\lambda_j - \frac{\alpha+\beta}{2}\right|^2 + |c|^2 \left|\gamma - \frac{\alpha+\beta}{2}\right|^2 \sum_{t=b_r+1}^{+\infty} w(t)\left|\nu(t)\right|^2 \\
&\quad \le \left(\frac{\beta-\alpha}{2}\right)^2 \|\psi\|^2.
\end{aligned}$$

This shows that

$$\left\|\left(H_{1,s} - \frac{\alpha+\beta}{2}\right)\psi\right\| \le \frac{\beta-\alpha}{2}\|\psi\|.$$

Consequently, by Lemma 4.1 one has

$$k_r = \dim S \le \dim R\left\{E_s(H_1, \beta) - E_s(H_1, \alpha-)\right\} < \infty. \tag{4.8}$$

Further, note that $\alpha \in \rho(H_1) = \rho(H_{1,s})$ by Lemma 2.2. Hence, by Theorem 7.22 in [24] we have

$$\dim R\left\{E_s(H_1, \beta) - E_s(H_1, \alpha-)\right\} = \dim R\left\{E_s(H_1, \beta) - E_s(H_1, \alpha)\right\},$$

which, together with (4.5) and (4.8), shows that

$$\dim R\left\{E_s\left(H_{1,r}', \beta\right) - E_s\left(H_{1,r}', \alpha\right)\right\} \le \dim R\left\{E_s(H_1, \beta) - E_s(H_1, \alpha)\right\}. \tag{4.9}$$

On the other hand, one can show that $\{H'_{1,r}\}$ converges to H_1 in the strong resolvent sense with a completely similar argument to that in the proof of Theorem 3.3 in [22] (for the concept of convergence of self-adjoint subspaces in the strong resolvent sense, please see Definition 4.1 in [21]). From (4.2), it follows that $\{H'_{1,r,s}\}$ converges to $H_{1,s}$ in the strong resolvent sense by Theorem 4.2 in [21]. Therefore, by Theorem 9.19 in [24] we get

$$E_s(H'_{1,r},\lambda) \xrightarrow{s} E_s(H_1,\lambda) \quad \text{for } \lambda \in \mathbf{R} \setminus \sigma_p(H_1), \tag{4.10}$$

where $\sigma_p(H_1) = \sigma_p(H_{1,s})$ has been used. By (4.9)-(4.10), (4.3) follows from Lemma 4.2. This completes the proof. □

The following result is a direct consequence of Theorem 4.1.

Corollary 4.1 *Assume that $\mathcal{L}$ is regular at $t = a$ and in l.p.c. at $t = +\infty$. Let H_1 be any fixed self-adjoint subspace extension of H_0 given by* (2.12). *If H_1 has a pure discrete spectrum, then the sequence $\{H_{1,r}\}_{r=1}^{\infty}$ defined by* (4.1) *is spectrally exact for H_1 if $0 \notin \sigma(H_1)$.*

5 One endpoint is in l.c.c. and the other in l.p.c

In this section, we shall study spectral exactness in an open interval laking essential spectral points in Case 4. Without loss of generality, we only consider the case that $\mathcal{L}$ is in l.c.c. at $t = -\infty$ and in l.p.c. at $t = +\infty$.

In this case, it was shown that the sequence of induced self-adjoint restrictions $\{H_{1,r}\}$ is spectrally inclusive for any given self-adjoint subspace extension H_1 but not spectrally exact for it in general in [22]. By Remark 2.1, every H_1 given by (2.19) is a self-adjoint operator extension of H_0. Now, we shall try to choose a sequence of induced regular self-adjoint operator extensions, still denoted by $\{H_{1,r}\}$ without any confusion, such that it is spectrally exact for H_1 in an open interval laking essential spectral points.

Let H_1 be any fixed self-adjoint operator extension of H_0 given by (2.19). Denote $I_1 := \{t\}_{t=-\infty}^{d_0-1}$, where d_0 is the same as in (2.8). Let H_a and $H_{a,0}$ be the left maximal and minimal subspaces corresponding to (1.1) or $\mathcal{L}$ on I_1, respectively. Let

$$\hat{H}_{a,0} = \{(x,f) \in H_a : x(d_0-1) = x(d_0) = 0 \text{ and } (x,y)(-\infty) = 0 \text{ for all } y \in D(H^*_{a,0})\}.$$

Assume that $0 \notin I_0 \subset \mathbf{R}$ is an open interval and $I_0 \cap \sigma_e(H_1) = \emptyset$, $I_0 \cap \sigma_d(H_1) \neq \emptyset$. Then for $\hat{y}$ defined by (2.20) and any fixed $\delta \in I_0$, by virtue of Theorem 3.8 in [12], there exist uniquely $y_0 \in D(\hat{H}_{a,0})$ and one solution h in $l^2_w(I_1)$ of (1.1_δ) such that

$$\hat{y}(t) = y_0(t) + h(t), \quad t \le d_0 - 2. \tag{5.1}$$

We assert that h is nontrivial. In fact, if the assertion would not hold, then $\hat{y}(t) = y_0(t)$ for $t \le d_0 - 2$. Hence, for any $(x,f) \in H$, one has $(x,\hat{y})(-\infty) = (x,y_0)(-\infty) = 0$, where the definition of $\hat{H}_{a,0}$ and $H_a \subset H^*_{a,0}$ have been used. So, it follows from (2.19) that $H_1 = H$. This leads to a contradiction. Thus, this assertion holds. For any $(x,f) \in H$, it follows from (5.1) that

$$(x,\hat{y})(-\infty) = (x,h)(-\infty). \tag{5.2}$$

Therefore, H_1 determined by (2.19) can be expressed as

$$H_1 = \{(x,f) \in H : (x,h)(-\infty) = 0\}.$$

In addition, by Lemma 4.3, there exists at least one nontrivial square summable solution ν of (1.1_γ) for any $\gamma \in I_0$. Consequently, we can choose $\{a_r\}_{r=1}^\infty$ and $\{b_r\}_{r=1}^\infty$ specified in Section 2.3 such that $h(a_r) \neq 0$ and $\nu(b_r) \neq 0$ for $r \in \mathbf{N}$.

Theorem 5.1 *Assume that $\mathcal{L}$ is in l.c.c. at $t = -\infty$ and in l.p.c. at $t = +\infty$. Let H_1 be any fixed self-adjoint operator extension of H_0 given by* (2.19). *Assume that $0 \notin I_0 \subset \mathbf{R}$ is an open interval with $I_0 \cap \sigma_e(H_1) = \emptyset$ and $I_0 \cap \sigma_d(H_1) \neq \emptyset$. Let ν be a nontrivial real square summable solution of* (1.1_γ) *with any fixed $\gamma \in I_0$, $H_{1,r}$ the induced self-adjoint restriction of H_1 on I_r defined by*

$$H_{1,r} = \{(x,f) \in H^r : (x,h)(a_r - 1) = 0, (x,\nu)(b_r) = 0\}, \tag{5.3}$$

where h is determined by (5.1) *with $\delta = \gamma$ and $\{a_r\}_{r=1}^\infty$ and $\{b_r\}_{r=1}^\infty$ specified in Section* 2.3 *satisfy $h(a_r) \neq 0$ and $\nu(b_r) \neq 0$ for $r \in \mathbf{N}$, respectively. Then $\{H_{1,r}\}_{r=1}^\infty$ is spectrally exact for H_1 in I_0.*

Proof The main idea of the proof is similar to that of the proof of Theorem 4.1. For completeness, we shall give its details.

Since $\nu(b_r) \neq 0$ and $h(a_r) \neq 0$, we have $H_{1,r}(0) = \{0\}$. Therefore, $H_{1,r}$ given by (5.3) is a self-adjoint operator extension of H_0^r. This, together with Lemma 2.6, shows that $\tilde{H}_{1,r}$ and $H'_{1,r}$ are self-adjoint operators in $\tilde{l}^2_w(I_r)$ and $l^2_w(I)$, respectively.

By Lemmas 2.4 and 2.5, in order to prove $\{H_{1,r}\}_{r=1}^\infty$ is spectrally exact for H_1 in I_0, it suffices to show that for any given $\alpha, \beta \in I_0 \cap \rho(H_1)$ with $\alpha < \gamma \leq \beta$ and sufficiently large r,

$$\dim R\{E(H'_{1,r}, \beta) - E(H'_{1,r}, \alpha)\} = \dim R\{E(H_1, \beta) - E(H_1, \alpha)\}. \tag{5.4}$$

Fix any $\alpha, \beta \in I_0 \cap \rho(H_1)$ with $\alpha < \gamma \leq \beta$. For any fixed r, let $\lambda_1, \ldots, \lambda_{k_r}$ be all k_r (counting multiplicity) eigenvalues of $H_{1,r}$ in $(\alpha, \beta]$, and $\eta_1, \ldots, \eta_{k_r}$ the corresponding orthonormal eigenfunctions. Then, by Lemma 2.5, the assumption that $0 \notin I_0$, and the first proposition in [24], p.204, we get

$$\begin{aligned} &\dim R\{E(H'_{1,r}, \beta) - E(H'_{1,r}, \alpha)\} \\ &\quad = \dim R\{E(\tilde{H}_{1,r}, \beta) - E(\tilde{H}_{1,r}, \alpha)\} \\ &\quad = \dim R\{E(H_{1,r}, \beta) - E(H_{1,r}, \alpha)\} = k_r. \end{aligned} \tag{5.5}$$

On the other hand, it is obvious that $\eta_j \in D(H_{1,r})$ and thus by (5.3) we have

$$\eta_j(a_r - 1) = \frac{h(a_r - 1)}{h(a_r)}\eta_j(a_r), \qquad \eta_j(b_r + 1) = \frac{\nu(b_r + 1)}{\nu(b_r)}\eta_j(b_r), \quad 1 \leq j \leq k_r.$$

Hence, there exist constants c_j and d_j, $1 \leq j \leq k_r$, such that

$$\begin{pmatrix} \eta_j(a_r - 1) \\ \eta_j(a_r) \end{pmatrix} = c_j \begin{pmatrix} h(a_r - 1) \\ h(a_r) \end{pmatrix}, \qquad \begin{pmatrix} \eta_j(b_r) \\ \eta_j(b_r + 1) \end{pmatrix} = d_j \begin{pmatrix} \nu(b_r) \\ \nu(b_r + 1) \end{pmatrix}. \tag{5.6}$$

For every $j \in \{1, \dots, k_r\}$, let

$$\psi_j(t) = \begin{cases} c_j h(t), & t \le a_r - 1, \\ \eta_j(t), & a_r \le t \le b_r, \\ d_j \nu(t), & t \ge b_r + 1. \end{cases} \tag{5.7}$$

Then $\psi_j \in D(H)$. Since $\hat{y} \in D(H_1)$, it follows from (5.2) that $(h, \hat{y})(-\infty) = (\hat{y}, \hat{y})(-\infty) = 0$ and thus $\psi_j \in D(H_1)$ by (2.19). Let

$$S := L\{\psi_1, \dots, \psi_{k_r}\}.$$

Then S obviously is k_r-dimensional, and every $\psi \in S$ is of the form

$$\psi(t) = \begin{cases} ch(t), & t \le a_r - 1, \\ \sum_{j=1}^{k_r} l_j \eta_j(t), & a_r \le t \le b_r, \\ d\nu(t), & t \ge b_r + 1, \end{cases} \tag{5.8}$$

where l_j, $1 \le j \le k_r$, are constants, $c = \sum_{j=1}^{k_r} l_j c_j$, and $d = \sum_{j=1}^{k_r} l_j d_j$. It follows from (5.8) that

$$\begin{aligned} \|\psi\|^2 &= \sum_{t=-\infty}^{+\infty} w(t) \left|\psi(t)\right|^2 \\ &= |c|^2 \sum_{t=-\infty}^{a_r-1} w(t) \left|h(t)\right|^2 + \sum_{j=1}^{k_r} |l_j|^2 + |d|^2 \sum_{t=b_r+1}^{+\infty} w(t) \left|\nu(t)\right|^2. \end{aligned}$$

Therefore,

$$\begin{aligned} &\left\| \left(H_1 - \frac{\alpha+\beta}{2}\right)\psi \right\|^2 \\ &\quad = \sum_{t=-\infty}^{+\infty} w(t) \left| \left(H_1 - \frac{\alpha+\beta}{2}\right)\psi(t) \right|^2 \\ &\quad = \sum_{t=-\infty}^{a_r-1} w(t) \left| \left(\gamma - \frac{\alpha+\beta}{2}\right) ch(t) \right|^2 + \sum_{t=a_r}^{b_r} w(t) \left| \left(\lambda_j - \frac{\alpha+\beta}{2}\right) \sum_{j=1}^{k_r} l_j \eta_j(t) \right|^2 \\ &\qquad + \sum_{t=b_r+1}^{+\infty} w(t) \left| \left(\gamma - \frac{\alpha+\beta}{2}\right) d\nu(t) \right|^2 \\ &\quad = |c|^2 \left|\gamma - \frac{\alpha+\beta}{2}\right|^2 \sum_{t=-\infty}^{a_r-1} w(t) \left|h(t)\right|^2 + \sum_{j=1}^{k_r} |l_j|^2 \left|\lambda_j - \frac{\alpha+\beta}{2}\right|^2 \\ &\qquad + |d|^2 \left|\gamma - \frac{\alpha+\beta}{2}\right|^2 \sum_{t=b_r+1}^{+\infty} w(t) \left|\nu(t)\right|^2 \\ &\quad \le \left(\frac{\beta-\alpha}{2}\right)^2 \|\psi\|^2. \end{aligned}$$

Thus,

$$\left\| \left(H_1 - \frac{\alpha + \beta}{2} \right) \psi \right\| \le \frac{\beta - \alpha}{2} \| \psi \|.$$

Consequently, by Lemma 4.1 one has

$$k_r = \dim S \le \dim R\{E(H_1, \beta) - E(H_1, \alpha-)\} < \infty. \tag{5.9}$$

Since $\alpha \in \rho(H_1)$, by Theorem 7.22 in [24] we have

$$\dim R\{E(H_1, \beta) - E(H_1, \alpha-)\} = \dim R\{E(H_1, \beta) - E(H_1, \alpha)\},$$

which, together with (5.5) and (5.9), shows that

$$\dim R\{E(H'_{1,r}, \beta) - E(H'_{1,r}, \alpha)\} \le \dim R\{E(H_1, \beta) - E(H_1, \alpha)\}. \tag{5.10}$$

On the other hand, one can show that $\{H'_{1,r}\}$ converges to H_1 in the strong resolvent sense with a completely similar argument to that in the proof of Theorem 4.3 in [22] and so we omit the details. Therefore, by Theorem 9.19 in [24], it follows that

$$E(H'_{1,r}, \lambda) \xrightarrow{s} E(H_1, \lambda) \quad \text{for } \lambda \in \mathbf{R} \setminus \sigma_p(H_1).$$

Together with (5.10), (5.4) follows from Lemma 4.2. This completes the proof. □

The following result is a direct consequence of Theorem 5.1.

Corollary 5.1 *Assume that $\mathcal{L}$ is in l.c.c. at $t = -\infty$ and in l.p.c. at $t = +\infty$. Let H_1 be any fixed self-adjoint subspace extension of H_0 given by* (2.19). *If H_1 has a pure discrete spectrum, then the sequence $\{H_{1,r}\}_{r=1}^{\infty}$ defined by* (5.3) *is spectrally exact for H_1 if $0 \notin \sigma(H_1)$.*

6 Both endpoints are in l.p.c

In this section, we shall study spectral exactness in an open interval laking essential spectral points in Case 5. In this case, $H_1 = H_0 = H$ is a self-adjoint operator. In [22], it was shown that the sequence of induced self-adjoint restrictions $\{H_{1,r}\}$ is spectrally inclusive for H_0 but not spectrally exact for H_0 in general. Now, we shall try to choose a sequence of induced regular self-adjoint operator extensions, denoted by $\{H_{0,r}\}$, which is spectrally exact for H_0 in an open interval laking essential spectral points.

Denote $I_2 := \{t\}_{t=d_0}^{+\infty}$, where d_0 is the same as in (2.8). Let H_b and $H_{b,0}$ be the right maximal and minimal subspaces corresponding to (1.1) or $\mathcal{L}$ on I_2, respectively. I_1, H_a, and $H_{a,0}$ are specified in Section 5. Let $H_{a,1}$ and $H_{b,1}$ be any self-adjoint subspace extensions of $H_{a,0}$ and $H_{b,0}$, separately. Then, by Theorem 3.3 in [29] and Corollary 2.1 in [30], one has

$$\sigma_e(H_1) = \sigma_e(H_0) = \sigma_e(H_{a,1}) \cup \sigma_e(H_{b,1}) \tag{6.1}$$

in Case 5. Assume that $0 \notin I_0 \subset \mathbf{R}$ is an open interval with $I_0 \cap \sigma_e(H_0) = \emptyset$ and $I_0 \cap \sigma_d(H_0) \neq \emptyset$. Then, by Lemma 4.3 and (6.1), there exist two nontrivial real solutions v_1 in $l_w^2(I_1)$ and v_2 in $l_w^2(I_2)$ of (1.1_γ) with any $\gamma \in I_0$. Consequently, we can choose $\{a_r\}_{r=1}^{\infty}$ and $\{b_r\}_{r=1}^{\infty}$ specified in Section 2.3 such that $v_1(a_r) \neq 0$ and $v_2(b_r) \neq 0$ for $r \in \mathbf{N}$.

Theorem 6.1 *Assume that $\mathcal{L}$ is in l.p.c. at $t = \pm\infty$, $0 \notin I_0 \subset \mathbf{R}$ is an open interval with $I_0 \cap \sigma_e(H_0) = \emptyset$ and $I_0 \cap \sigma_d(H_0) \neq \emptyset$. Let v_1 and v_2 be two nontrivial real solutions of (1.1_γ) with any fixed $\gamma \in I_0$, which are square summable near $\mp\infty$, respectively, $H_{0,r}$ the induced self-adjoint restriction of H_0 on I_r defined by*

$$H_{0,r} = \{(x,f) \in H^r : (x,v_1)(a_r - 1) = 0, (x,v_2)(b_r) = 0\}, \tag{6.2}$$

where $\{a_r\}_{r=1}^\infty$ and $\{b_r\}_{r=1}^\infty$ specified in Section 2.3 satisfy $v_1(a_r) \neq 0$ and $v_2(b_r) \neq 0$ for $r \in \mathbf{N}$, respectively. Then $\{H_{0,r}\}_{r=1}^\infty$ is spectrally exact for H_0 in I_0.

Proof The main idea of the proof is similar to that of the proof of Theorem 5.1. So we omit its details. This completes the proof. □

Corollary 6.1 *Assume that $\mathcal{L}$ is in l.p.c. at $t = \pm\infty$, and H_0 has a pure discrete spectrum. Then the sequence $\{H_{0,r}\}_{r=1}^\infty$ defined by (6.2) is spectrally exact for H_0 if $0 \notin \sigma(H_0)$.*

Remark 6.1 $H_{1,r}$ defined by (4.1), (5.3), and (6.2) can be viewed as special cases of those defined by (2.14), (2.18), and (2.21), respectively. For example, consider $H_{1,r}$ defined by (4.1). It can be obtained by taking $\lambda = \gamma$ in (2.8) and $u = v$ in (2.14) and choosing $\{b_r\}_{t=1}^\infty$ specified in Section 2.3 such that $v(b_r) \neq 0$ for $r \in \mathbf{N}$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
YL carried out the results of this article and drafted the manuscript. YS proposed this study and inspected the manuscript. All authors read and approved the final manuscript.

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