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On a fractional-order delay Mackey-Glass equation

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Abstract

In this paper, a fractional-order Mackey-Glass equation with constant delay is considered. The local stability of the fixed points is analyzed. Moreover, a discretization process is applied to convert the fractional-order delay equation to its discrete analog. A numerical simulation including Lyapunov exponent, phase diagrams, bifurcation, and chaos is carried out using Matlab to ensure theoretical results and to reveal more complex dynamics of the equation after discretization.

Keywords: fractional-order delay Mackey-Glass equations; fixed points; local stability; discretization; Lyapunov exponent; bifurcation; chaos

1 Introduction

Delay differential equations (DDEs) arise in the mathematical description of systems whose time evolution depends explicitly on a past state of the system, as for example in the case of delayed feedback. Neural systems [1], respiration regulation [2], agricultural commodity markets [3], nonlinear optics, and neutrophil populations in the blood [2] are but a few systems in which delayed feedback leads naturally to a description in terms of a delay differential equation. We will restrict our attention to systems modeled by evolutionary delay equations that can be expressed in the form

$$x'(t) = f\big(x(t), x(t-\tau)\big), \quad x(t) \in \Re^n, t \geq 0. \tag{1.1}$$

Here the 'state' of the system at time t is $x(t)$, whose rate of change depends explicitly, via the function f, on the past state $x(t-\tau)$ where τ is a fixed time delay. More general delay equations might be considered: multiple time delays, variable time delays, continuously distributed delays, and higher derivatives all arise in applications and lead to more complicated evolution equations. Nevertheless, equations of the form (1.1) constitute a sufficiently broad class of systems to be of practical importance, and they will provide adequate fodder for the types of problems we wish to consider.

DDEs arise in many areas of mathematical modeling: for example, population dynamics (taking into account the gestation times), infectious diseases (accounting for the incubation periods), physiological and pharmaceutical kinetics (modeling, for example, the body's reaction to CO_2, *etc.* in circulating blood), chemical kinetics (such as mixing reactants), the navigational control of ships and aircraft (with, respectively, large and short lags), and more general control problems (see for example [4–6]).

On the other hand, fractional calculus is a generalization of classical differentiation and integration to arbitrary (non-integer) order [7–9]. Many mathematicians and applied researchers have tried to model real processes using fractional calculus [10–16]. In recent years differential equations with fractional-order have attracted many researchers because of their applications in many areas of science and engineering. Analytical and numerical techniques have been implemented to study such equations. The fractional calculus has allowed the operations of integration and differentiation to be applied for any fractional order [17–21].

We recall the basic definitions (Caputo) and properties of fractional-order differentiation and integration.

Definition 1 The fractional integral of order $\beta \in \mathbb{R}^+$ of the function $f(t)$, $t > 0$ is defined by

$$I_a^\beta f(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f(s)\,ds,$$

and the fractional derivative of order $\alpha \in (n-1, n)$ of $f(t)$, $t > 0$ is defined by

$$D_a^\alpha f(t) = I^{n-\alpha} D^n f(t), \qquad D = \frac{d}{dt}.$$

In addition, the following results are the main features in fractional calculus. Let $\beta, \gamma \in \mathbb{R}^+$, $\alpha \in (0,1)$:

- $I_a^\beta : L^1 \to L^1$, and if $f(x) \in L^1$, then $I_a^\gamma I_a^\beta f(x) = I_a^{\gamma+\beta} f(x)$.
- $\lim_{\beta\to n} I_a^\beta f(x) = I_a^n f(x)$ uniformly on $[a,b]$, $n = 1, 2, 3, \ldots$, where $I_a^1 f(x) = \int_a^x f(s)\,ds$.
- $\lim_{\beta\to 0} I_a^\beta f(x) = f(x)$ weakly.
- If $f(x)$ is absolutely continuous on $[a,b]$, then $\lim_{\alpha\to 1} D_a^\alpha f(x) = \frac{df(x)}{dx}$.

The Mackey-Glass equation is a nonlinear time delay differential equation, which was proposed as a model of hematopoiesis, given by

$$\frac{dx}{dt} = \frac{\rho x_\tau}{1 + x_\tau^c} - \gamma x, \tag{1.2}$$

where γ, c, ρ, τ are real parameters, and x_τ represents the value of the variable x at time $(t-\tau)$. Depending on the values of the parameters, this equation displays a range of periodic and chaotic dynamics.

In this work, we will show that considering a fractional-order derivative with delay in equation (1.2) will exhibit more complex and richer dynamics.

Consider the fractional-order delay Mackey-Glass equation given in the form

$$D^\alpha x(t) = \frac{\rho x(t-\tau)}{1 + x(t-\tau)^c} - x(t-\tau), \qquad t \in (0, T], \tag{1.3}$$

with the initial condition

$$x(0) = x_0, \tag{1.4}$$

where $\alpha \in (0,1]$, $\rho \in R^+$, and $c > 0$. In equation (1.3), we consider delay in the last term.

2 Discretization process

In this part, we apply the discretization process represented in [22, 23], and [24] for discretizing the delay fractional-order Mackey Glass equation with piecewise constant arguments given by

$$D^{\alpha}x(t)=\frac{\rho x([\frac{t}{r}]r-\tau)}{1+x^{c}([\frac{t}{r}]r-\tau)}-x\left(\left[\frac{t}{r}\right]r-\tau\right), \tag{2.1}$$

with initial condition (1.4).

The steps of the discretization process are as follows.

Let $t\in[0,r)$, then $\frac{t}{r}\in[0,1)$. That is,

$$D^{\alpha}x(t)=\frac{\rho x_0(-\tau)}{1+x_0^{c}(-\tau)}-x_0(-\tau), \tag{2.2}$$

and the solution of (2.2) is given by

$$\begin{aligned}x(t)&=x_0+I^{\alpha}\left(\frac{\rho x_0(-\tau)}{1+x_0^{c}(-\tau)}-x_0(-\tau)\right)\\&=x_0+\left(\frac{\rho x_0(-\tau)}{1+x_0^{c}(-\tau)}-x_0(-\tau)\right)\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\,ds\\&=x_0+\frac{t^{\alpha}}{\Gamma(1+\alpha)}\left(\frac{\rho x_0(-\tau)}{1+x_0^{c}(-\tau)}-x_0(-\tau)\right).\end{aligned}$$

Let $t\in[r,2r)$, then $\frac{t}{r}\in[1,2)$. That is,

$$D^{\alpha}x(t)=\frac{\rho x_1(r-\tau)}{1+x_1^{c}(r-\tau)}-x_1(r-\tau),\quad t\in[r,2r), \tag{2.3}$$

and the solution of (2.3) is given by

$$\begin{aligned}x_2(t)&=x_1(r)+I_r^{\alpha}\left(\frac{\rho x_1(r-\tau)}{1+x_1^{c}(r-\tau)}-x_1(r-\tau)\right)\\&=x_1(r)+\left(\frac{\rho x_1(r-\tau)}{1+x_1^{c}(r-\tau)}-x_1(r-\tau)\right)\int_r^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\,ds\\&=x_1(r)+\frac{(t-r)^{\alpha}}{\Gamma(1+\alpha)}\left(\frac{\rho x_1(r-\tau)}{1+x_1^{c}(r-\tau)}-x_1(r-\tau)\right).\end{aligned}$$

Repeating the process we can easily get

$$x(t)=x_n(nr)+\frac{(t-nr)^{\alpha}}{\Gamma(1+\alpha)}\left(\frac{\rho x_n(nr-\tau)}{1+x_n^{c}(nr-\tau)}-x_n(nr-\tau)\right),\quad t\in\bigl[nr,(n+1)r\bigr).$$

Let $t\to(n+1)r$, we obtain the discretization

$$x_{n+1}(n+1)=x_n+\frac{(r)^{\alpha}}{\Gamma(1+\alpha)}\left(\frac{\rho x_n(r-\tau)}{1+x_n^{c}(r-\tau)}-x_n(r-\tau)\right). \tag{2.4}$$

It is worth to pay attention here that Euler's discretization method is an approximation for the derivative while the predictor-corrector method is an approximation for the integral.

However, our proposed discretization method here is an approximation for the right-hand side of the system under consideration as is pretty clear from (2.4). Moreover, we have noticed that when $\alpha \to 1$, the discretization will be Euler's discretization.

In the following, we will discuss two cases of the delay: Case I: $\tau = r$, and Case II: $\tau = 2r$.

3 Case I: $\tau = r$

In this case we have a second-order difference equation given by

$$x_{n+1} = x_n + \frac{r^\alpha}{\Gamma(1+\alpha)}\left(\frac{\rho x_{n-1}}{1+x_{n-1}^c} - x_{n-1}\right). \tag{3.1}$$

Existence and stability of fixed points

To find the fixed points of system (3.1), we first split it into two first-order difference equations as follows:

$$\begin{aligned} x_{n+1} &= x_n + \frac{r^\alpha}{\Gamma(1+\alpha)}\left(\frac{\rho y_n}{1+y_n^c} - y_n\right), \\ y_{n+1} &= x_n. \end{aligned} \tag{3.2}$$

- For all values of the parameter ρ, system (3.2) has one fixed point, namely, $\mathrm{fix}_1 = (0,0)$.
- For $\rho > 1$, we have an additional fixed point, which is $\mathrm{fix}_2 = (\sqrt[c]{\rho-1}, \sqrt[c]{\rho-1})$.

In order to study the local stability of these fixed points, we need the moduli of the eigenvalues of the Jacobian matrix evaluated at each of the fixed points [25]. The Jacobian matrix of system (3.2) evaluated at any fixed point (x,y) is obtained by

$$J = \begin{pmatrix} 1 & \frac{r^\alpha}{\Gamma(1+\alpha)}\left(\frac{\rho+\rho y^c(1-c)}{(1+y^c)^2} - 1\right) \\ 1 & 0 \end{pmatrix}.$$

The eigenvalues associated to the Jacobian matrix are

$$\lambda_{1,2} = 0.5 \pm 0.5\sqrt{1+4R(s-1)},$$

where

$$R = \frac{r^\alpha}{\Gamma(1+\alpha)}, \qquad S = \frac{\rho + \rho y^c(1-c)}{(1+y^c)^2}.$$

The fixed points fix_1, fix_2 of the system equation (3.1) are stable if $|\lambda_i| < 1$, $i = 1,2$. In order to study the qualitative behavior of the solution of system (3.2) we rely on the Jury criteria given generally by

1. $F := 1 + T + D > 0$,
2. $TC := 1 - T + D > 0$,
3. $H := 1 - D > 0$,

where the trace and determinant of the Jacobian matrix are given, respectively, by

$$\begin{aligned} &T\colon\ \mathrm{Tr}(J) = J_{11} + J_{22}, \\ &D\colon\ \mathrm{Det}(J) = J_{11}J_{22} - J_{12}J_{21}. \end{aligned}$$

Proposition 1 *The fixed point* fix_1 *is locally asymptotically stable if* $\rho < (1 + 2/R)$, *and losses stability via a flip bifurcation when* $\rho > 1$ *and via a Neimark-Sacker bifurcation when* $\rho > \frac{r^\alpha - \Gamma(1+\alpha)}{r^\alpha}$.

Proof The Jacobian matrix at the first fixed point fix_1 is obtained by

$$J = \begin{pmatrix} 1 & \frac{r^\alpha}{\Gamma(1+\alpha)}(\rho - 1) \\ 1 & 0 \end{pmatrix},$$

which has two eigenvalues,

$$\lambda_{1,2} = 0.5 \pm 0.5\sqrt{1 + 4R(\rho - 1)}.$$

According to the Jury criteria [26, 27], where $T = 1$, $D = \frac{r^\alpha(1-\rho)}{\Gamma(1+\alpha)}$, the first condition is always satisfied, while the second and third may be violated. That is, the fixed point fix_1 loses stability via a flip bifurcation when $\rho > 1$, and via a Neimark-Sacker bifurcation when $\rho > \frac{r^\alpha - \Gamma(1+\alpha)}{r^\alpha}$. □

Proposition 2 *The fixed point* fix_2 *of system* (3.2) *is stable if* $\rho < \frac{cR}{2+cR}$, *and it loses stability via a pitchfork bifurcation if* $\rho > \frac{cR}{2+cR}$, *via a flip bifurcation if* $\rho > 1$, *and via a Neimark-Sacker bifurcation if* $\rho < \frac{cR}{cR-1}$.

Proof Calculating the Jacobian matrix at the second fixed point fix_2 of system (3.2) we obtain

$$J = \begin{pmatrix} 1 & \frac{r^\alpha c(1-\rho)}{\rho\Gamma(1+\alpha)} \\ 1 & 0 \end{pmatrix},$$

which has two eigenvalues,

$$\lambda_{1,2} = 0.5 \pm 0.5\sqrt{1 + \frac{4Rc(1-\rho)}{\rho}},$$

where the trace and determinant of $J(\mathrm{fix}_2)$ are given, respectively, by

$$T = 1, \qquad D = -Rc\left(\frac{1-\rho}{\rho}\right).$$

According to the Jury criteria, the three conditions may be all violated. That is, fix_2 loses stability via a pitchfork bifurcation if $\rho > \frac{cR}{2+cR}$, via a flip bifurcation if $\rho > 1$, and via a Neimark-Sacker bifurcation if $\rho < \frac{cR}{cR-1}$. □

4 Case II: $\tau = 2r$

In this section, we take the delay to be $\tau = 2r$ in equation (1.3). Applying the discretization process we end up with a system of third-order difference equations given by

$$x_{n+1} = x_n + \frac{r^\alpha}{\Gamma(1+\alpha)}\left(\frac{\rho x_{n-2}}{1 + x_{n-2}^c} - x_{n-2}\right). \tag{4.1}$$

To study the fixed points of system (4.1) we first split it into three first-order difference equations as follows:

$$x_{n+1} = x_n + \frac{r^\alpha}{\Gamma(1+\alpha)}\left(\frac{\rho y_n}{1+z_n^c} - z_n\right),$$
$$y_{n+1} = x_n, \tag{4.2}$$
$$z_{n+1} = y_n.$$

In the following, we study the local stability of the fixed points of the system (4.2).

Existence and stability of fixed points

System (4.2) has the following fixed points:

- For all parameter values, there is only one fixed point $\operatorname{fix} x_1 = (0,0,0)$.
- For $\rho > 1$, there is an additional fixed point $\operatorname{fix} x_2 = (\sqrt[c]{\rho-1}, \sqrt[c]{\rho-1}, \sqrt[c]{\rho-1})$.

By considering a Jacobian matrix for one of these fixed points and calculating their eigenvalues, we can investigate the stability of each fixed point based on the roots of the system characteristic equation. The Jacobian matrix is given by

$$J = \begin{pmatrix} 1 & 0 & R(S-1) \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

where $R = \frac{r^\alpha}{\Gamma(1+\alpha)}$, $S = \frac{\rho+\rho z^c(1-c)}{(1+z^c)^2}$.

Linearizing the system (4.2) about $\operatorname{fix} x_1$ yields the following characteristic equation:

$$P(\lambda) = \lambda^3 - \lambda^2 - R(\rho - 1). \tag{4.3}$$

Let

$$a_0 = 1, \qquad a_1 = -1, \qquad a_2 = 0, \qquad a_3 = -R(\rho-1).$$

From the Jury test, if $P(1) > 0$, $P(-1) < 0$, and $a_3 < 1$, $|b_3| > b_1$, $c_3 > |c_2|$, where $b_3 = 1 - a_3^2$, $b_2 = a_1 - a_3 a_2$, $b_1 = a_2 - a_3 a_1$, $c_3 = b_3^2 - b_1^2$, and $c_2 = b_3 b_2 - b_1 b_2$, then the roots of $P(\lambda)$ satisfy $\lambda < 1$ and thus fix_1 is asymptotically stable.

The first condition gives $\rho < 1$, while the second condition gives

$$\rho > \frac{r^\alpha - 2\Gamma(\alpha+1)}{r^\alpha}.$$

The remaining conditions give the following inequalities:

- $a_3 < 1 \Rightarrow \rho > 1 - \frac{1}{R}$,
- $|b_3| > b_1 \Rightarrow (1 - (R - \rho R)^2) > (R - \rho R)$,
- $c_3 > |c_2| \Rightarrow (1 - (R-\rho R)^2)^2 - (R - \rho R)^2 > (R - \rho R)^2 - 1 + (R - \rho R)$.

Linearizing the system (4.2) about $\operatorname{fix} x_2$ yields the following characteristic equation:

$$P(\lambda) = \lambda^3 - \lambda^2 - \frac{cR(1-\rho)}{\rho}. \tag{4.4}$$

Let

$$a_0 = 1, \qquad a_1 = -1, \qquad a_2 = 0, \qquad a_3 = -\frac{cR(1-\rho)}{\rho}.$$

From the Jury test, if $P(1) > 0$, $P(-1) < 0$, and $a_3 < 1$, $|b_3| > b_1$, $c_3 > |c_2|$, where $b_3 = 1 - a_3^2$, $b_2 = a_1 - a_3 a_2$, $b_1 = a_2 - a_3 a_1$, $c_3 = b_3^2 - b_1^2$, and $c_2 = b_3 b_2 - b_1 b_2$, then the roots of $P(\lambda)$ satisfy $\lambda < 1$ and thus fix_1 is asymptotically stable.

We are going to check these conditions at $\mathrm{fix}\, x_2$:

- $p(1) > 0 \Rightarrow \rho > 1$,
- $p(-1) < 0 \Rightarrow \rho < \frac{cr^\alpha}{cr^\alpha - 2\Gamma(\alpha+1)}$,
- $a_3 < 1 \Rightarrow \rho < \frac{cr^\alpha}{cr^\alpha - \Gamma(\alpha+1)}$,
- $|b_3| > b_1 \Rightarrow (1 + \frac{c^2R^2}{\rho^2}(1-\rho)^2) > \frac{cR(\rho-1)}{\rho}$.

Thus, any condition may be violated resulting in instability of $\mathrm{fix}\, x_2$.

5 Numerical simulation

In this section, a numerical simulation is carried out with the aid of Matlab to illustrate our theoretical results and to reveal the more complex dynamics of equation (1.2) in the two cases $\tau = r$ and $\tau = 2r$. In all numerical simulations, we take $c = 6$, and $r = 0.5$. First of all, let us consider system (3.2). Indeed, if one is interested in determining whether a dynamical system is chaotic or not, often just a few of the largest Lyapunov characteristic exponents (LCEs) may provide the answer. This actually is so because a positive LCE is a good indicator for chaos. Since for non-chaotic systems all LCEs are non-positive, the presence of a positive LCE has often been used to help determine if a system is chaotic or not. In this paper, we compute the LCEs via the Householder QR-based methods described in [28]. For system (3.2), we get when $\alpha = 0.95$, $c = 6$, and $r = 0.5$, LCE1 = 0.3397, and LCE2 = 0.185 as shown in Figure 1. We vary the parameter ρ and fix the other parameters, c, r, and α. Bifurcation diagrams of system (3.2) are also shown in Figure 1 for different values of the fractional-order parameter α. If we consider $\alpha = 0.95$, it is shown that the fixed point $\mathrm{fix}_1(0,0)$ is stable if $\rho < 1$, and at $\rho > 1$ it losses stability via a flip bifurcation. Afterwards, a stable periodic solution of period 2 appears, then the periodic solution of period 4 becomes unstable, and a periodic solution of period 8 appears and chaos happens. Figure 2 shows the different phase plane for system (3.2) for $\alpha = 0.95$. For $\rho = 1.55$, Figure 2(a) shows an invariant closed curve bifurcating from $\mathrm{fix}_1(0,0)$, while for $\rho = 3.6$, Figure 2(b) shows a chaotic attractor. Now we vary the parameter α from 0 to 1 and fix ρ to plot the bifurcation diagram for system (3.2) as a function of α as shown in Figure 3.

Next, we turn to the second case, when $\tau = 2r$. Figure 4 shows the bifurcation diagram for system (4.2) as a function of ρ. If $\rho = 0.95$, the figure shows that the fixed point $\mathrm{fix}\, x_1$ becomes unstable when $\rho > 1$ as it loses stability via a flip bifurcation. Then the stable period-2 orbit appears at $\rho = 1.2$, which in turn loses stability; then chaos appears. Finally, Figure 5 shows the phase plane for system (4.2) for $\alpha = 0.95$. Figure 5(a) shows a stable fixed point $\mathrm{fix}\, x_1$ for $\rho = 0.6$, Figure 5(b) shows a double scroll for $\rho = 1.2$, and Figure 5(c), (d) shows chaotic attractors for $\rho = 2$, and $\rho = 2.7$, respectively.

6 Conclusion

In this paper, the dynamic behavior of a fractional-order delay Mackey-Glass equation is investigated after applying a discretization process to it. We have considered two different

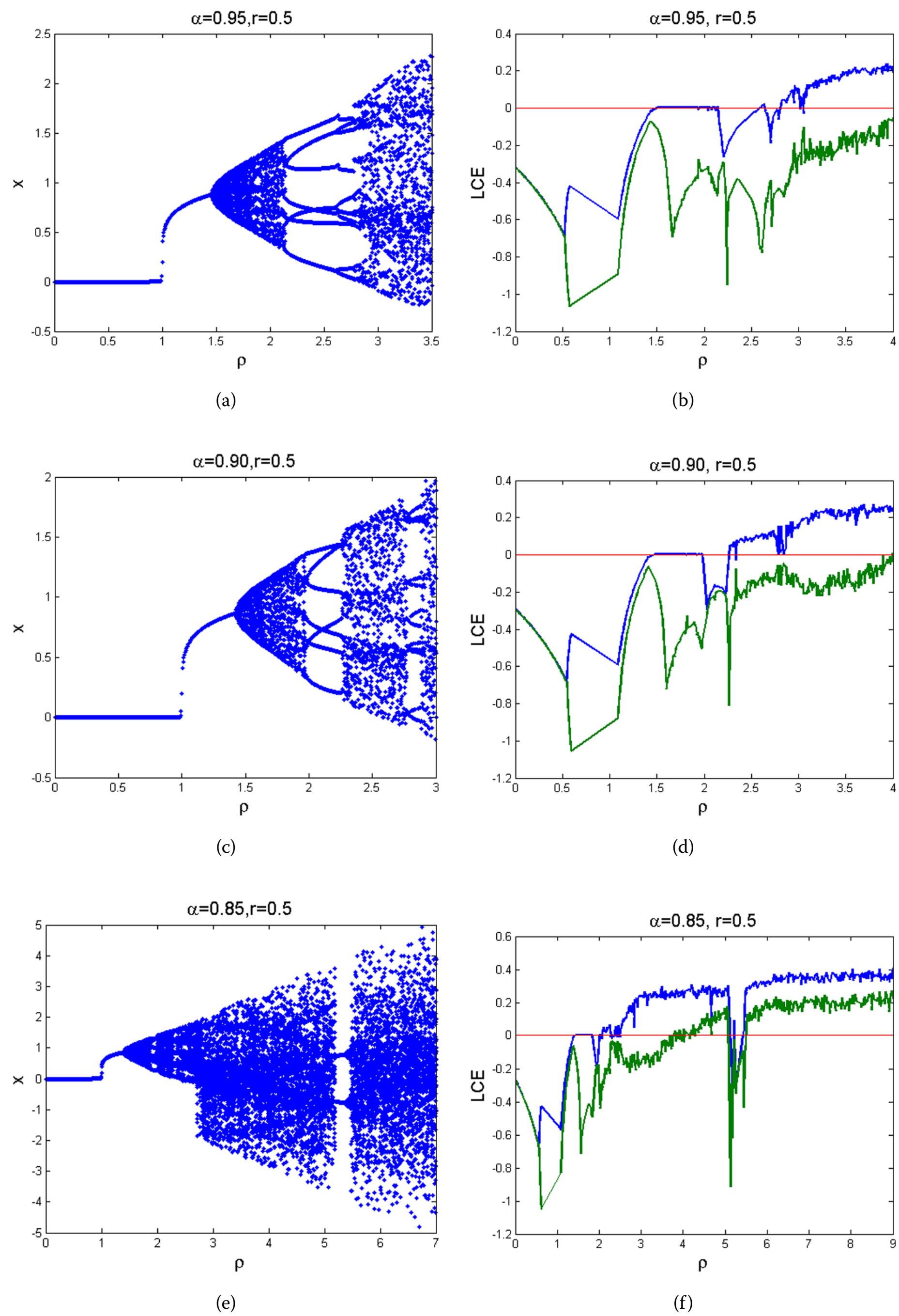

Figure 1 Bifurcation diagram and Lyapunov exponent for system (3.2) with different values of the fractional-order parameter α, $c = 6$ and $r = 0.5$.

cases for the delay τ, the first is when $\tau = r$, and the second is when $\tau = 2r$, where r is the discretization parameter. Stability of the fixed points and local bifurcations of fixed points of the discretized systems in the two cases was are analyzed. A numerical simulation was carried out to ensure our theoretical analysis and to reveal the more complex dynamics of the system.

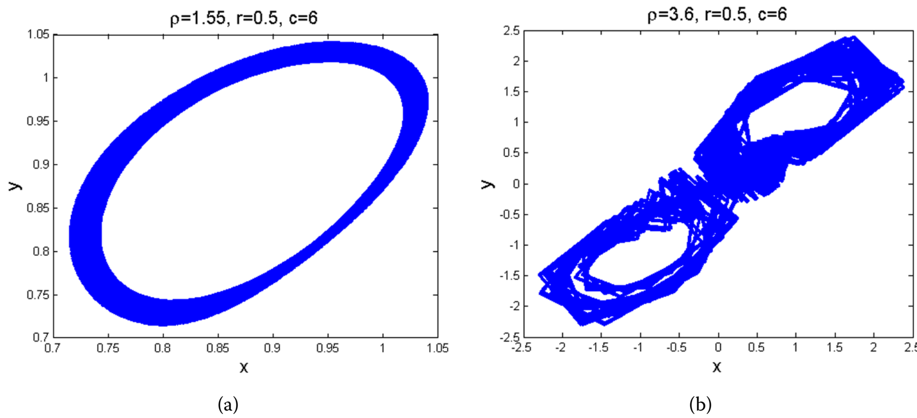

Figure 2 Phase plane for system (3.2) with $\alpha = 0.95$, $c = 6$ and $r = 0.5$.

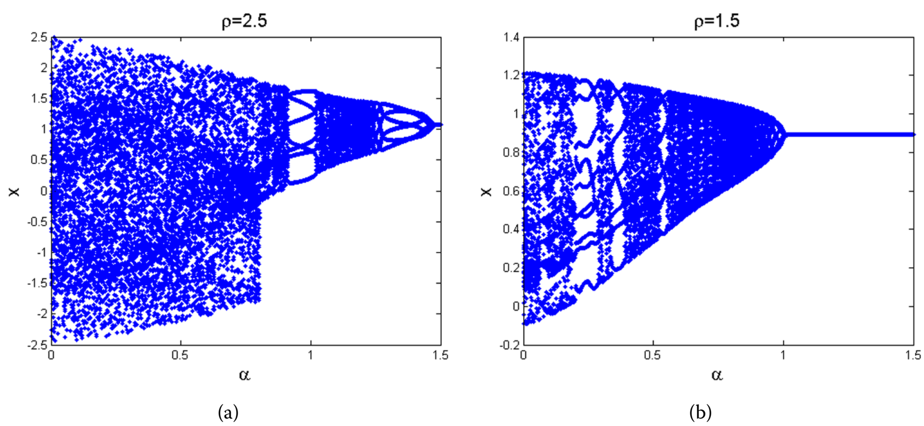

Figure 3 Bifurcation diagrams for system (3.2) as a function of α.

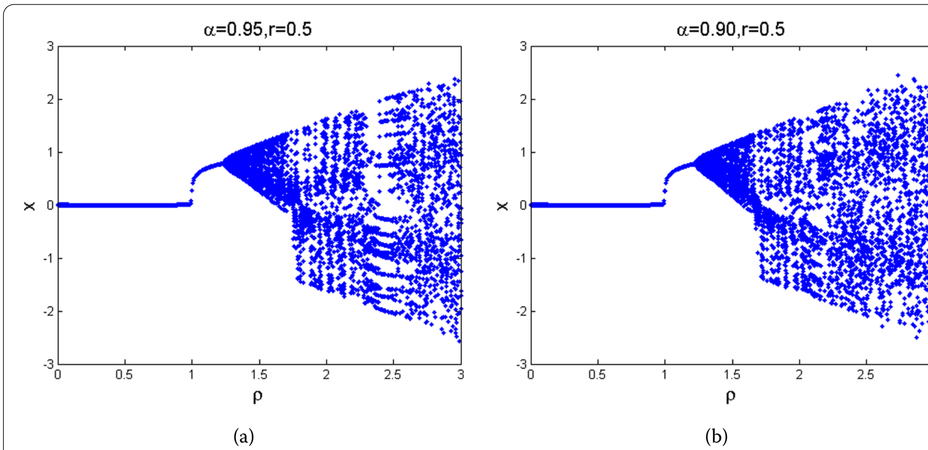

Figure 4 Bifurcation diagram and chaos for system (4.2) as a function of ρ.

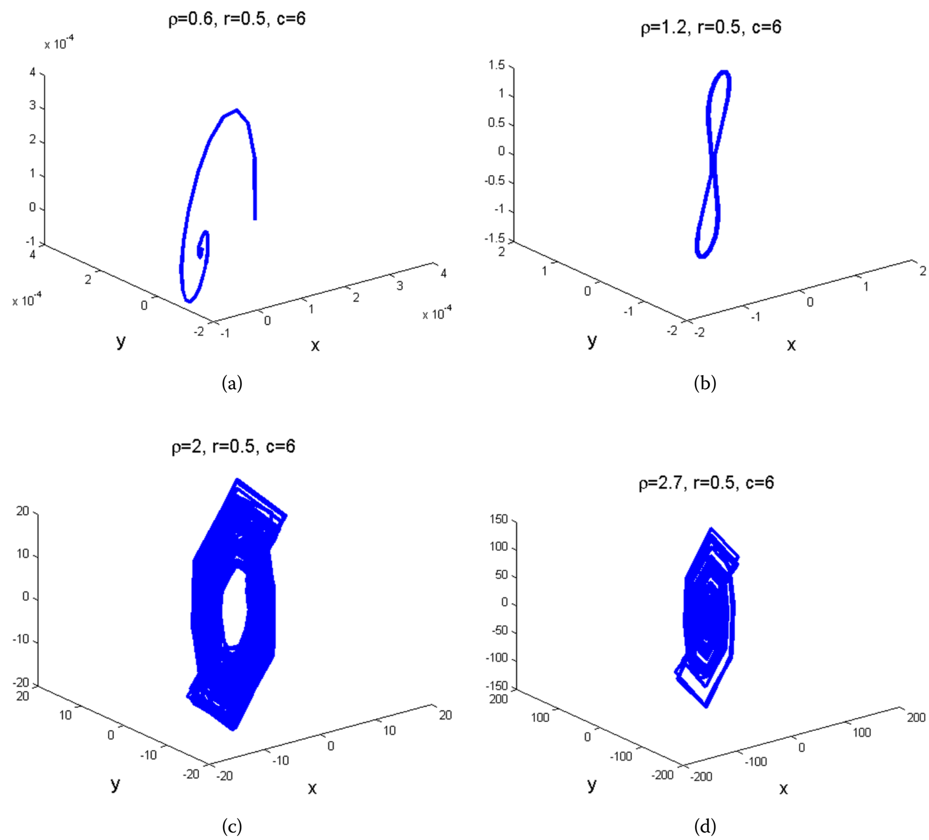

Figure 5 Phase plane for system (3.1) with $\alpha = 0.95$, $c = 6$, and $r = 0.5$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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