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Positive solution for singular fractional differential equations involving derivatives

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Abstract

By using the fixed point theorem for the mixed monotone operator, the existence of unique positive solutions for singular nonlocal boundary value problems of fractional differential equations is established. An example is provided to illustrate the main results.

MSC: 34A08; 34B18

Keywords: singular nonlocal BVP; positive solution; fractional differential equation; mixed monotone operator

1 Introduction

The purpose of this paper is to establish the uniqueness of a positive solution to the following higher order fractional differential equation:

$$\begin{cases} D_{0^+}^{\alpha}x(t) + q(t)f(t, x(t), D_{0^+}^{\mu_1}x(t), \ldots, D_{0^+}^{\mu_{n-2}}x(t)) = 0, & 0 < t < 1, n-1 < \alpha \leq n, \\ x(0) = D_{0^+}^{\mu_1}x(0) = D_{0^+}^{\mu_2}x(0) = \cdots = D_{0^+}^{\mu_{n-2}}x(0) = 0, \\ D_{0^+}^{\mu}x(1) = \sum_{j=1}^{p-2} a_j D_{0^+}^{\mu}x(\xi_j), \end{cases} \tag{1.1}$$

where $n \geq 3$, $n \in N$, $n - i - 1 < \alpha - \mu_i < n - i$ for $i = 1, 2, \ldots, n-2$, and $\mu - \mu_{n-2} > 0$, $\alpha - \mu > 1$, $a_j \in [0, +\infty)$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{p-2} < 1$, $0 < \sum_{j=1}^{p-2} a_j \xi_j^{\alpha-\mu-1} < 1$, $D_{0^+}^{\alpha}$ is the standard Riemann-Liouville derivative, $f : [0,1] \times (0, +\infty)^{n-1} \to [0, +\infty)$ is continuous, $q : (0,1) \to [0, +\infty)$ is continuous, $f(t, x_1, x_2, \ldots, x_{n-1})$ may be singular at $x_1 = 0$, $x_2 = 0$, $\ldots$, $x_{n-1} = 0$, and $q(t)$ may be singular at $t = 0$ and/or $t = 1$.

Recently, one has found numerous applications of fractional differential equations in viscoelasticity, electrochemistry, control, porous media, and electromagnetics; see [1–14] and the references therein. Particularly, the theory of boundary value problems (BVPs) for nonlinear fractional differential equations has received great attention, but many aspects of the theory still need to be explored.

In [6], Rehman and Khan studied the following multi-point boundary value problems for fractional differential equations:

$$\begin{cases} D_t^{\alpha}x(t) = f(t, x(t), D_t^{\beta}x(t)), & 0 < t < 1, \\ x(0) = 0, \quad D_t^{\beta}x(1) - \sum_{i=1}^{m-2} \zeta_i D_t^{\beta}x(\xi_i) = x_0, \end{cases} \tag{1.2}$$

where $1 < \alpha \leq 2$, $0 < \beta < 1$, $0 < \xi_i < 1$ ($i = 1, 2, \ldots, m-2$), $\zeta_i \geq 0$, with $\sum_{i=1}^{m-2} \zeta_i \xi_i^{\alpha-\beta-1} < 1$, D_t^α represents the standard Riemann-Liouville fractional derivative. The nonlinear function $f : [0,1] \times R \times R \to R$ is continuous and satisfies certain growth conditions. The existence and uniqueness of nontrivial solutions for BVP (1.2) are established by using the Schauder fixed point theorem and the Banach contraction mapping principle. In [7], Zhang investigated the existence of positive solutions of the following equation by a fixed point theorem for the mixed monotone operator:

$$\begin{cases} D_{0^+}^\alpha x(t) + q(t) f(x, x', \ldots, x^{(n-2)}) = 0, \quad 0 < t < 1, n-1 < \alpha \leq n, \\ x(0) = x'(0) = \cdots = x^{(n-2)}(0) = x^{(n-2)}(1) = 0, \end{cases}$$

where $f(u_1, u_2, \ldots, u_{n-1})$ may be singular at $u_1 = 0$, $u_2 = 0$, …, $u_{n-1} = 0$, $q(t)$ may be singular at $t = 0$, $D_{0^+}^\alpha$ is the Riemann-Liouville fractional derivative of order α. Xu and Fei [8] considered the properties of the Green's function for the nonlinear fractional differential equation three-point boundary value problem

$$\begin{cases} D_{0^+}^\alpha x(t) + f(t, x(t)) + e(t) = 0, \quad 0 < t < 1, \\ x(0) = 0, \qquad D_{0^+}^\beta x(1) = a D_{0^+}^\beta x(\xi), \end{cases}$$

where $1 < \alpha \leq 2$, $0 < \beta \leq 1$, $0 \leq a \leq 1$, $0 < \xi < 1$, $\alpha - \beta - 1 \geq 0$, $D_{0^+}^\alpha$ is the standard Riemann-Liouville derivative, $f : (0,1) \times (0, +\infty) \to (0, +\infty)$ satisfies the Caratheodory conditions. The authors obtained some multiple positive solutions by means of the Schauder fixed point theorem.

In [11], Zhang, Liu and Wu investigated the following singular eigenvalue problem for a higher order fractional differential equation:

$$\begin{cases} -D^\alpha x(t) = \lambda f(x(t), D^{\mu_1} x(t), \ldots, D^{\mu_{n-1}} x(t)), \quad 0 < t < 1, \\ x(0) = 0, \qquad D^{\mu_i} x(0) = 0, \qquad D^\mu x(1) = \sum_{j=1}^{p-2} a_j D^\mu x(\xi_j), \quad 1 \leq i \leq n-1, \end{cases}$$

where D^α is the standard Riemann-Liouville derivative. The eigenvalue interval for the existence of positive solutions is obtained by the Schauder fixed point theorem and the upper and lower solutions method.

Motivated by the work mentioned above, we consider the fractional order singular nonlocal BVP (1.1). In this paper, we establish the existence of a unique positive solution for BVP (1.1). The main tool used in the proofs of the existence results is a fixed point theorem for the mixed monotone operator. The present paper has the following features. First of all, the nonlinear f involves fractional derivatives of an unknown function. Second, BVP (1.1) possesses a singularity, that is, $f(t, x_1, \ldots, x_{n-1})$ may be singular at $x_1 = 0, x_2 = 0, \ldots, x_{n-1} = 0$, $q(t)$ may be singular at $t = 0$ and/or $t = 1$. Third, the nonlocal boundary conditions involving fractional derivatives of the unknown function are more general cases, which include two-point, three-point, multi-point, and some nonlocal problems as special cases.

The rest of the paper is organized as follows. In Section 2, we present some preliminaries and lemmas on fractional calculus theory, and then give the associated Green's function and develop some properties of the Green's function. In Section 3, we establish an existence result of a unique positive solution of BVP (1.1) under certain assumptions for the functions f and q. An example is given to illustrate the main result in Section 4.

2 Preliminaries and lemmas

Definition 2.1 ([4, 5]) The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $u : (0, +\infty) \to R$ is given by

$$I_{0^+}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)\, ds$$

provided the right-hand side is pointwise defined on $(0, +\infty)$.

Definition 2.2 ([4, 5]) The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $u : (0, +\infty) \to R$ is given by

$$D_{0^+}^{\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{u(s)}{(t-s)^{\alpha-n+1}}\, ds,$$

where n is the smallest integer not less than α, provided the right-hand side is pointwise defined on $(0, +\infty)$.

Lemma 2.1 ([4, 5])

(1) *If* $x \in L^1(0,1)$, $\rho > \sigma > 0$, *and* $n \in N$, *then*

$$I_{0^+}^{\rho} I_{0^+}^{\sigma} x(t) = I_{0^+}^{\rho+\sigma} x(t), \qquad D_{0^+}^{\sigma} I_{0^+}^{\rho} x(t) = I_{0^+}^{\rho-\sigma} x(t),$$
$$D_{0^+}^{\sigma} I_{0^+}^{\sigma} x(t) = x(t), \qquad \left(\frac{d}{dt}\right)^n \bigl(D_{0^+}^{\sigma} x(t)\bigr) = D_{0^+}^{n+\sigma} x(t).$$

(2) *If* $\nu > 0$, $\sigma > 0$, *then*

$$D_{0^+}^{\nu} t^{\sigma-1} = \frac{\Gamma(\sigma)}{\Gamma(\sigma-\nu)} t^{\sigma-\nu-1}.$$

Lemma 2.2 ([4]) *Let* $\alpha > 0$. *Then the following equality holds for* $u \in L^1(0,1)$ *and* $D_{0^+}^{\alpha} u \in L^1(0,1)$:

$$I_{0^+}^{\alpha} D_{0^+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n},$$

where $c_i \in R$, $i = 1, 2, 3, \ldots, n$, *here* $n-1 < \alpha \le n$.

Let

$$k_1(t,s) = \begin{cases} \frac{t^{\alpha-\mu_{n-2}-1}(1-s)^{\alpha-\mu-1} - (t-s)^{\alpha-\mu_{n-2}-1}}{\Gamma(\alpha-\mu_{n-2})}, & 0 \le s \le t \le 1, \\ \frac{t^{\alpha-\mu_{n-2}-1}(1-s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu_{n-2})}, & 0 \le t \le s \le 1, \end{cases}$$

$$k_2(t,s) = \begin{cases} \frac{(t(1-s))^{\alpha-\mu-1} - (t-s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu_{n-2})}, & 0 \le s \le t \le 1, \\ \frac{(t(1-s))^{\alpha-\mu-1}}{\Gamma(\alpha-\mu_{n-2})}, & 0 \le t \le s \le 1. \end{cases}$$

Obviously, for $t, s \in [0,1]$, we have

$$k_1(t,s) \le \frac{t^{\alpha-\mu_{n-2}-1}}{\Gamma(\alpha-\mu_{n-2})}, \qquad k_2(t,s) \le \frac{1}{\Gamma(\alpha-\mu_{n-2})}. \tag{2.1}$$

Proceeding as for the proof of Lemma 2.3 in [11], we have the following lemma.

Lemma 2.3 *If $h(t) \in L^1[0,1]$, then the boundary value problem*

$$\begin{cases} -D_{0^+}^{\alpha-\mu_{n-2}} w(t) = h(t), \\ w(0) = 0, \qquad D_{0^+}^{\mu-\mu_{n-2}} w(1) = \sum_{j=1}^{p-2} a_j D_{0^+}^{\mu-\mu_{n-2}} w(\xi_j), \end{cases} \tag{2.2}$$

has a unique solution

$$w(t) = \int_0^1 K(t,s) h(s)\, ds,$$

where

$$K(t,s) = k_1(t,s) + \frac{t^{\alpha-\mu_{n-2}-1}}{1 - \sum_{j=1}^{p-2} a_j \xi_j^{\alpha-\mu-1}} \sum_{j=1}^{p-2} a_j k_2(\xi_j, s) \tag{2.3}$$

is the Green's function of the boundary value problem (2.2).

Lemma 2.4 *The function $K(t,s)$ has the following properties*:

(1) $K(t,s) > 0$, *for* $t, s \in (0,1)$;

(2) $t^{\alpha-\mu_{n-2}-1}\mathcal{G}(s) \le K(t,s) \le B t^{\alpha-\mu_{n-2}-1}$, *for* $t,s \in (0,1)$, *where*

$$\mathcal{G}(s) = \frac{\sum_{j=1}^{p-2} a_j k_2(\xi_j, s)}{1 - \sum_{j=1}^{p-2} a_j \xi_j^{\alpha-\mu-1}}, \qquad B = \frac{1 + \sum_{j=1}^{p-2} a_j (1 - \xi_j^{\alpha-\mu-1})}{\Gamma(\alpha - \mu_{n-2})(1 - \sum_{j=1}^{p-2} a_j \xi_j^{\alpha-\mu-1})}.$$

Proof It is obvious that (1) holds. In the following, we will prove (2). First, from (2.3) we have

$$K(t,s) \ge \frac{t^{\alpha-\mu_{n-2}-1}}{1 - \sum_{j=1}^{p-2} a_j \xi_j^{\alpha-\mu-1}} \sum_{j=1}^{p-2} a_j k_2(\xi_j, s) = t^{\alpha-\mu_{n-2}-1}\mathcal{G}(s).$$

On the other hand, it follows from (2.1) that

$$\begin{aligned} K(t,s) &= k_1(t,s) + \frac{t^{\alpha-\mu_{n-2}-1}}{1 - \sum_{j=1}^{p-2} a_j \xi_j^{\alpha-\mu-1}} \sum_{j=1}^{p-2} a_j k_2(\xi_j, s) \\ &\le \frac{t^{\alpha-\mu_{n-2}-1}}{\Gamma(\alpha-\mu_{n-2})} + \frac{t^{\alpha-\mu_{n-2}-1} \sum_{j=1}^{p-2} a_j}{\Gamma(\alpha-\mu_{n-2})(1 - \sum_{j=1}^{p-2} a_j \xi_j^{\alpha-\mu-1})} \\ &= \left(1 + \frac{\sum_{j=1}^{p-2} a_j}{1 - \sum_{j=1}^{p-2} a_j \xi_j^{\alpha-\mu-1}}\right) \frac{t^{\alpha-\mu_{n-2}-1}}{\Gamma(\alpha-\mu_{n-2})}. \end{aligned}$$

This completes the proof. □

For the convenience of expression in the rest of the paper, we let $\mu_0 = 0$. Now let us consider the following modified problems of BVP (1.1):

$$\begin{cases} D_{0^+}^{\alpha-\mu_{n-2}}u(t) + q(t)f(t, I_{0^+}^{\mu_{n-2}-\mu_0}u(t), I_{0^+}^{\mu_{n-2}-\mu_1}u(t), \ldots, I_{0^+}^{\mu_{n-2}-\mu_{n-3}}u(t), u(t)) = 0, \\ u(0) = 0, \qquad D_{0^+}^{\mu-\mu_{n-2}}u(1) = \sum_{j=1}^{p-2} a_j D_{0^+}^{\mu-\mu_{n-2}}u(\xi_j). \end{cases} \tag{2.4}$$

By similar arguments to [11], we obtain the following lemma.

Lemma 2.5 *Let* $x(t) = I_{0^+}^{\mu_{n-2}-\mu_0}u(t)$, $u(t) \in C[0,1]$. *Then we can transform* (1.1) *into* (2.4). *Moreover, if* $u \in C([0,1],(0,+\infty))$ *is a positive solution of problem* (2.4), *then the function* $x(t) = I_{0^+}^{\mu_{n-2}-\mu_0}u(t)$ *is a positive solution of BVP* (1.1).

In order to obtain the uniqueness of a positive solution to BVP (1.1), we will consider the uniqueness of a positive solution to the following modified problem:

$$\begin{cases} D_{0^+}^{\alpha-\mu_{n-2}}u(t) + q(t)f(t, I_{0^+}^{\mu_{n-2}-\mu_0}u(t) + \frac{1}{m}, I_{0^+}^{\mu_{n-2}-\mu_1}u(t) + \frac{1}{m}, \ldots, u(t) + \frac{1}{m}) = 0, \\ u(0) = 0, \qquad D_{0^+}^{\mu-\mu_{n-2}}u(1) = \sum_{j=1}^{p-2} a_j D_{0^+}^{\mu-\mu_{n-2}}u(\xi_j), \end{cases} \tag{2.5}$$

where $t \in (0,1)$, $m \in \{2,3,\ldots\}$. Assume that $f : [0,1] \times (0,+\infty)^{n-1} \to [0,+\infty)$ is continuous, then u is a solution of system (2.5) if and only if u is a solution of the following nonlinear integral equation:

$$u(t) = \int_0^1 K(t,s)q(s)f\left(s, I_{0^+}^{\mu_{n-2}-\mu_0}u(s) + \frac{1}{m}, I_{0^+}^{\mu_{n-2}-\mu_1}u(s) + \frac{1}{m}, \ldots, u(s) + \frac{1}{m}\right) ds. \tag{2.6}$$

Let P be a normal cone of a Banach space E, and $e \in P$ with $\|e\| \le 1$, $e \ne \theta$ (θ is a zero element of E). Define $Q_e = \{x \in P |$there exist constants $m, M > 0$ such that $me \le x \le Me\}$. Now we give the following definition and theorem (see [15]).

Definition 2.3 Let D be a subset of Banach space E. Let operator $A : D \times D \to E$. A is said to be mixed monotone if $A(x,y)$ is nondecreasing in x and nonincreasing in y, *i.e.*, $x_1 \le x_2$ ($x_1, x_2 \in D$) implies $A(x_1,y) \le A(x_2,y)$ for any $y \in D$, and $y_1 \le y_2$ ($y_1, y_2 \in D$) implies $A(x,y_1) \ge A(x,y_2)$ for any $x \in D$. The element $x^* \in D$ is called a fixed point of A if $A(x^*,x^*) = x^*$.

Lemma 2.6 *Suppose that* $A : Q_e \times Q_e \to Q_e$ *is a mixed monotone operator and there exists a constant* σ, $0 < \sigma < 1$, *such that*

$$A\big(cx, c^{-1}y\big) \ge c^\sigma A(x,y), \quad x, y \in Q_e, 0 < c < 1.$$

Then A *has a unique fixed point* $x^* \in Q_e$.

3 Main results

For convenience in the presentation, we now present some assumptions to be used in the rest of the paper.

($\mathbf{H}_1$) $f(t,x_1,\ldots,x_{n-1}) = \phi(t,x_1,\ldots,x_{n-1}) + \psi(t,x_1,\ldots,x_{n-1})$, where $\phi : [0,1] \times [0,+\infty)^{n-1} \to [0,+\infty)$ and $\psi : [0,1] \times (0,+\infty)^{n-1} \to [0,+\infty)$ are continuous, and for any fixed $t \in [0,1]$, $\phi(t,x_1,\ldots,x_{n-1})$ is nondecreasing and $\psi(t,x_1,\ldots,x_{n-1})$ is nonincreasing in $x_i > 0$ ($i = 1,2,\ldots,n-1$), respectively.

($\mathbf{H}_2$) There exists $\sigma \in (0,1)$ such that, for $x_i > 0$, $i = 1,2,\ldots,n-1$, and for any $t \in [0,1]$ and $c \in (0,1)$,

$$\begin{aligned} &\phi(t,cx_1,\ldots,cx_{n-1}) \geq c^\sigma \phi(t,x_1,\ldots,x_{n-1}), \\ &\psi\left(t,c^{-1}x_1,\ldots,c^{-1}x_{n-1}\right) \geq c^\sigma \psi(t,x_1,\ldots,x_{n-1}). \end{aligned}$$

($\mathbf{H}_3$) $\phi(t,1,1,\ldots,1) \not\equiv 0$, $\psi(t,1,1,\ldots,1) \not\equiv 0$,

$$0 < \int_0^1 q(s)\phi(s,1,1,\ldots,1)\,ds < +\infty, \qquad \int_0^1 s^{-\sigma(\alpha-1)}q(s)\psi(s,1,1,\ldots,1)\,ds < +\infty.$$

Remark 3.1 By ($\mathbf{H}_2$), for $c \geq 1$, we have

$$\begin{aligned} &\phi(t,cx_1,\ldots,cx_{n-1}) \leq c^\sigma \phi(t,x_1,\ldots,x_{n-1}), \\ &\psi\left(t,c^{-1}x_1,\ldots,c^{-1}x_{n-1}\right) \leq c^\sigma \psi(t,x_1,\ldots,x_{n-1}). \end{aligned}$$

Let $e(t) = t^{\alpha-\mu_{n-2}-1}$, $t \in [0,1]$, it is clear that $e \neq \theta$, $\|e\| = 1$. We here define a normal cone of $C[0,1]$ by

$$P = \left\{x \in C[0,1] : x(t) \geq 0, 0 \leq t \leq 1\right\},$$

and we also define

$$Q_e = \left\{x \in P : \frac{1}{M}e(t) \leq x(t) \leq Me(t), 0 \leq t \leq 1\right\},$$

where

$$\begin{aligned} M > \max\Bigg\{ &\left[2^\sigma B \int_0^1 q(s)\phi(s,1,1,\ldots,1)\,ds + B\eta^{-\sigma} \int_0^1 s^{-\sigma(\alpha-1)}q(s)\psi(s,1,1,\ldots,1)\,ds\right]^{\frac{1}{1-\sigma}}, 1, \\ &2\eta, \left[\eta^\sigma \int_0^1 q(s)\mathcal{G}(s)s^{\sigma(\alpha-1)}\phi(s,1,1,\ldots,1)\,ds \right. \\ &\left. + 2^{-\sigma} \int_0^1 q(s)\mathcal{G}(s)\psi(s,1,1,\ldots,1)\,ds\right]^{-\frac{1}{1-\sigma}}\Bigg\}, \\ 0 < \eta < \min&\left\{1, \frac{\Gamma(\alpha-\mu_{n-2})}{\Gamma(\alpha-\mu_0)}, \frac{\Gamma(\alpha-\mu_{n-2})}{\Gamma(\alpha-\mu_1)}, \ldots, \frac{\Gamma(\alpha-\mu_{n-2})}{\Gamma(\alpha-\mu_{n-3})}\right\}. \end{aligned}$$

Remark 3.2 By Definition 2.1, for $t \in (0,1)$, $i = 0,1,2,\ldots,n-3$, we have

$$\begin{aligned} 0 < I^{\mu_{n-2}-\mu_i}e(t) &= \frac{1}{\Gamma(\mu_{n-2}-\mu_i)} \int_0^t (t-s)^{\mu_{n-2}-\mu_i-1}s^{\alpha-\mu_{n-2}-1}\,ds \\ &= \frac{\mathbf{B}(\mu_{n-2}-\mu_i, \alpha-\mu_{n-2})}{\Gamma(\mu_{n-2}-\mu_i)}t^{\alpha-\mu_i-1} = \frac{\Gamma(\alpha-\mu_{n-2})}{\Gamma(\alpha-\mu_i)}t^{\alpha-\mu_i-1} < 1. \end{aligned}$$

Theorem 3.1 *Assume that conditions* $(\mathbf{H}_1)$-$(\mathbf{H}_3)$ *hold. Then BVP* (1.1) *has a unique positive solution* $x(t)$, *which satisfies*

$$\frac{\Gamma(\alpha-\mu_{n-2})}{M\Gamma(\alpha)}t^{\alpha-1}\le x(t)\le\frac{M\Gamma(\alpha-\mu_{n-2})}{\Gamma(\alpha)}t^{\alpha-1},\quad t\in[0,1].$$

Proof We first consider the existence of a positive solution to problem (2.5). From the discussion in Section 2, we only need to consider the existence of a positive solution to the integral equation (2.6). For this purpose, we define the operator $T:Q_e\times Q_e\to P$ by

$$\begin{aligned}
&T(u,v)(t)\\
&\quad=\int_0^1K(t,s)q(s)\biggl[\phi\biggl(s,I_{0^+}^{\mu_{n-2}-\mu_0}u(s)+\frac{1}{m},I_{0^+}^{\mu_{n-2}-\mu_1}u(s)+\frac{1}{m},\dots,u(s)+\frac{1}{m}\biggr)\\
&\qquad+\psi\biggl(s,I_{0^+}^{\mu_{n-2}-\mu_0}v(s)+\frac{1}{m},I_{0^+}^{\mu_{n-2}-\mu_1}v(s)+\frac{1}{m},\dots,v(s)+\frac{1}{m}\biggr)\biggr]\,ds,\quad t\in[0,1].
\end{aligned}\tag{3.1}$$

Now we prove that $T:Q_e\times Q_e\to Q_e$. First, we will prove $T:Q_e\times Q_e\to P$ is well defined. For any $u\in Q_e$, $v\in Q_e$, we have $I^iu(t)+\frac{1}{m}>0$, $I^iv(t)+\frac{1}{m}>0$ ($i=\mu_{n-2}-\mu_0,\mu_{n-2}-\mu_1,\dots,\mu_{n-2}-\mu_{n-3}$), $u(t)+\frac{1}{m}>0$ and $v(t)+\frac{1}{m}>0$ for all $t\in[0,1]$. By $(\mathbf{H}_1)$, $(\mathbf{H}_2)$, and Remark 3.2, for $t\in[0,1]$ we have

$$\begin{aligned}
&\phi\biggl(t,I_{0^+}^{\mu_{n-2}-\mu_0}u(t)+\frac{1}{m},I_{0^+}^{\mu_{n-2}-\mu_1}u(t)+\frac{1}{m},\dots,u(t)+\frac{1}{m}\biggr)\\
&\quad\le\phi\biggl(t,I_{0^+}^{\mu_{n-2}-\mu_0}Me(t)+\frac{1}{m},I_{0^+}^{\mu_{n-2}-\mu_1}Me(t)+\frac{1}{m},\dots,Me(t)+\frac{1}{m}\biggr)\\
&\quad\le\phi\bigl(t,I_{0^+}^{\mu_{n-2}-\mu_0}Me(t)+1,I_{0^+}^{\mu_{n-2}-\mu_1}Me(t)+1,\dots,Me(t)+1\bigr)\\
&\quad\le\phi(t,M+1,M+1,\dots,M+1)\\
&\quad\le(M+1)^\sigma\phi(t,1,1,\dots,1)\\
&\quad\le2^\sigma M^\sigma\phi(t,1,1,\dots,1)
\end{aligned}\tag{3.2}$$

and

$$\begin{aligned}
&\psi\biggl(t,I_{0^+}^{\mu_{n-2}-\mu_0}v(t)+\frac{1}{m},I_{0^+}^{\mu_{n-2}-\mu_1}v(t)+\frac{1}{m},\dots,v(t)+\frac{1}{m}\biggr)\\
&\quad\le\psi\biggl(t,I_{0^+}^{\mu_{n-2}-\mu_0}\frac{1}{M}e(t)+\frac{1}{m},I_{0^+}^{\mu_{n-2}-\mu_1}\frac{1}{M}e(t)+\frac{1}{m},\dots,\frac{1}{M}e(t)+\frac{1}{m}\biggr)\\
&\quad=\psi\biggl(t,\frac{\Gamma(\alpha-\mu_{n-2})}{M\Gamma(\alpha-\mu_0)}t^{\alpha-\mu_0-1}+\frac{1}{m},\frac{\Gamma(\alpha-\mu_{n-2})}{M\Gamma(\alpha-\mu_1)}t^{\alpha-\mu_1-1}+\frac{1}{m},\dots,\frac{1}{M}t^{\alpha-\mu_{n-2}-1}+\frac{1}{m}\biggr)\\
&\quad\le\psi\biggl(t,\frac{\eta}{M}t^{\alpha-\mu_0-1}+\frac{1}{m},\frac{\eta}{M}t^{\alpha-\mu_1-1}+\frac{1}{m},\dots,\frac{\eta}{M}t^{\alpha-\mu_{n-2}-1}+\frac{1}{m}\biggr)\\
&\quad\le\psi\biggl(t,\frac{\eta}{M}t^{\alpha-1}+\frac{1}{m},\frac{\eta}{M}t^{\alpha-1}+\frac{1}{m},\dots,\frac{\eta}{M}t^{\alpha-1}+\frac{1}{m}\biggr)\\
&\quad\le\biggl(\eta M^{-1}t^{\alpha-1}+\frac{1}{m}\biggr)^{-\sigma}\psi(t,1,1,\dots,1)\\
&\quad\le\eta^{-\sigma}M^\sigma t^{-\sigma(\alpha-1)}\psi(t,1,1,\dots,1).
\end{aligned}\tag{3.3}$$

Since $\eta M^{-1}t^{\alpha-1}+\frac{1}{m}\in(0,1)$, we also have

$$\begin{aligned}
&\phi\left(t, I_{0^+}^{\mu_{n-2}-\mu_0}u(t)+\frac{1}{m}, I_{0^+}^{\mu_{n-2}-\mu_1}u(t)+\frac{1}{m},\dots,u(t)+\frac{1}{m}\right)\\
&\quad\geq \phi\left(t, I_{0^+}^{\mu_{n-2}-\mu_0}\frac{1}{M}e(t)+\frac{1}{m}, I_{0^+}^{\mu_{n-2}-\mu_1}\frac{1}{M}e(t)+\frac{1}{m},\dots,\frac{1}{M}e(t)+\frac{1}{m}\right)\\
&\quad= \phi\left(t, \frac{1}{M}I_{0^+}^{\mu_{n-2}-\mu_0}e(t)+\frac{1}{m}, \frac{1}{M}I_{0^+}^{\mu_{n-2}-\mu_1}e(t)+\frac{1}{m},\dots,\frac{1}{M}e(t)+\frac{1}{m}\right)\\
&\quad\geq \phi\left(t, \frac{\eta}{M}t^{\alpha-1}+\frac{1}{m}, \frac{\eta}{M}t^{\alpha-1}+\frac{1}{m},\dots,\frac{\eta}{M}t^{\alpha-1}+\frac{1}{m}\right)\\
&\quad\geq \left(\eta M^{-1}t^{\alpha-1}+\frac{1}{m}\right)^{\sigma}\phi(t,1,1,\dots,1)\\
&\quad\geq \eta^{\sigma}M^{-\sigma}t^{\sigma(\alpha-1)}\phi(t,1,1,\dots,1)
\end{aligned}\tag{3.4}$$

and

$$\begin{aligned}
&\psi\left(t, I_{0^+}^{\mu_{n-2}-\mu_0}v(t)+\frac{1}{m}, I_{0^+}^{\mu_{n-2}-\mu_1}v(t)+\frac{1}{m},\dots,v(t)+\frac{1}{m}\right)\\
&\quad\geq \psi\left(t, I_{0^+}^{\mu_{n-2}-\mu_0}Me(t)+\frac{1}{m}, I_{0^+}^{\mu_{n-2}-\mu_1}Me(t)+\frac{1}{m},\dots,Me(t)+\frac{1}{m}\right)\\
&\quad\geq \psi\left(t, I_{0^+}^{\mu_{n-1}-\mu_0}Me(t)+1, I_{0^+}^{\mu_{n-1}-\mu_1}Me(t)+1,\dots,M+1\right)\\
&\quad\geq \psi(t,M+1,M+1,\dots,M+1)\\
&\quad\geq (M+1)^{-\sigma}\psi(t,1,1,\dots,1)\\
&\quad\geq 2^{-\sigma}M^{-\sigma}\psi(t,1,1,\dots,1).
\end{aligned}\tag{3.5}$$

It follows from (3.2), (3.3), ($\mathbf{H}_3$), and Lemma 2.4 that

$$\begin{aligned}
&\int_0^1 K(t,s)q(s)\phi\left(s, I_{0^+}^{\mu_{n-2}-\mu_0}u(s)+\frac{1}{m}, I_{0^+}^{\mu_{n-2}-\mu_1}u(s)+\frac{1}{m},\dots,u(s)+\frac{1}{m}\right)ds\\
&\quad\leq Bt^{\alpha-\mu_{n-2}-1}\int_0^1 q(s)\phi\left(s, I_{0^+}^{\mu_{n-2}-\mu_0}u(s)+\frac{1}{m}, I_{0^+}^{\mu_{n-2}-\mu_1}u(s)+\frac{1}{m},\dots,u(s)+\frac{1}{m}\right)ds\\
&\quad\leq B2^{\sigma}M^{\sigma}t^{\alpha-\mu_{n-2}-1}\int_0^1 q(s)\phi(s,1,1,\dots,1)\,ds
\end{aligned}\tag{3.6}$$

and

$$\begin{aligned}
&\int_0^1 K(t,s)q(s)\psi\left(s, I_{0^+}^{\mu_{n-2}-\mu_0}v(s)+\frac{1}{m}, I_{0^+}^{\mu_{n-2}-\mu_1}v(s)+\frac{1}{m},\dots,v(s)+\frac{1}{m}\right)ds\\
&\quad\leq Bt^{\alpha-\mu_{n-2}-1}\int_0^1 q(s)\psi\left(s, I_{0^+}^{\mu_{n-2}-\mu_0}v(s)+\frac{1}{m}, I_{0^+}^{\mu_{n-2}-\mu_1}v(s)+\frac{1}{m},\dots,v(s)+\frac{1}{m}\right)ds\\
&\quad\leq B\eta^{-\sigma}M^{\sigma}t^{\alpha-\mu_{n-2}-1}\int_0^1 s^{-\sigma(\alpha-1)}q(s)\psi(s,1,1,\dots,1)\,ds,
\end{aligned}\tag{3.7}$$

which imply that

$$
\begin{aligned}
T(u,v)(t) &\leq B2^{\sigma} M^{\sigma} t^{\alpha-\mu_{n-2}-1} \int_0^1 q(s)\phi(s,1,1,\ldots,1)\,ds \\
&\quad + B\eta^{-\sigma} M^{\sigma} t^{\alpha-\mu_{n-2}-1} \int_0^1 s^{-\sigma(\alpha-1)} q(s)\psi(s,1,1,\ldots,1)\,ds \\
&< +\infty.
\end{aligned}
$$

Hence, $T : Q_e \times Q_e \to P$ is well defined.

By (3.6) and (3.7), we see that

$$
T(u,v)(t) \leq Mt^{\alpha-\mu_{n-2}-1} = Me(t), \quad t \in [0,1]. \tag{3.8}
$$

From (3.4), (3.5), ($\mathbf{H}_3$), and Lemma 2.4, it follows that

$$
\begin{aligned}
&\int_0^1 q(s)K(t,s)\phi\left(s, I_{0^+}^{\mu_{n-2}-\mu_0} u(s) + \frac{1}{m}, I_{0^+}^{\mu_{n-2}-\mu_1} u(s) + \frac{1}{m}, \ldots, u(s) + \frac{1}{m}\right) ds \\
&\quad \geq t^{\alpha-\mu_{n-2}-1} \int_0^1 q(s)\mathcal{G}(s)\phi\left(s, I_{0^+}^{\mu_{n-2}-\mu_0} u(s) + \frac{1}{m}, I_{0^+}^{\mu_{n-2}-\mu_1} u(s) + \frac{1}{m}, \ldots, u(s) + \frac{1}{m}\right) ds \\
&\quad \geq t^{\alpha-\mu_{n-2}-1} \eta^{\sigma} M^{-\sigma} \int_0^1 s^{\sigma(\alpha-1)} q(s)\mathcal{G}(s)\phi(s,1,1,\ldots,1)\,ds
\end{aligned} \tag{3.9}
$$

and

$$
\begin{aligned}
&\int_0^1 q(s)K(t,s)\psi\left(s, I_{0^+}^{\mu_{n-2}-\mu_0} v(s) + \frac{1}{m}, I_{0^+}^{\mu_{n-2}-\mu_1} v(s) + \frac{1}{m}, \ldots, v(s) + \frac{1}{m}\right) ds \\
&\quad \geq t^{\alpha-\mu_{n-2}-1} \int_0^1 q(s)\mathcal{G}(s)\psi\left(s, I_{0^+}^{\mu_{n-2}-\mu_0} v(s) + \frac{1}{m}, I_{0^+}^{\mu_{n-2}-\mu_1} v(s) + \frac{1}{m}, \ldots, v(s) + \frac{1}{m}\right) ds \\
&\quad \geq t^{\alpha-\mu_{n-2}-1} 2^{-\sigma} M^{-\sigma} \int_0^1 q(s)\mathcal{G}(s)\psi(s,1,1,\ldots,1)\,ds,
\end{aligned} \tag{3.10}
$$

which imply that

$$
T(u,v)(t) \geq \frac{1}{M} t^{\alpha-\mu_{n-2}-1} = \frac{1}{M} e(t), \quad t \in [0,1]. \tag{3.11}
$$

Hence, by (3.8) and (3.11), $T(u,v) \in Q_e$. Therefore, $T : Q_e \times Q_e \to Q_e$.

Next, we shall show that $T : Q_e \times Q_e \to Q_e$ is a mixed monotone operator. In fact, if $u_1 \leq u_2$ ($u_1, u_2 \in Q_e$), from the monotonicity of $I^{(i)}$ ($i > 0$) and ϕ, we obtain

$$
\begin{aligned}
&\int_0^1 K(t,s)q(s)\phi\left(s, I_{0^+}^{\mu_{n-2}-\mu_0} u_1(s) + \frac{1}{m}, I_{0^+}^{\mu_{n-2}-\mu_1} u_1(s) + \frac{1}{m}, \ldots, u_1(s) + \frac{1}{m}\right) ds \\
&\quad \leq \int_0^1 K(t,s)q(s)\phi\left(s, I_{0^+}^{\mu_{n-2}-\mu_0} u_2(s) + \frac{1}{m}, I_{0^+}^{\mu_{n-2}-\mu_1} u_2(s) + \frac{1}{m}, \ldots, u_2(s) + \frac{1}{m}\right) ds,
\end{aligned}
$$

which implies that

$$
T(u_1,v)(t) \leq T(u_2,v)(t), \quad v \in Q_e, t \in [0,1].
$$

That is, $T(u,v)$ is nondecreasing in u for any $v \in Q_e$. Similarly, if $v_1 \geq v_2$ ($v_1, v_2 \in Q_e$), from the monotonicity of $I^{(i)}$ ($i > 0$) and ψ, we deduce that

$$\begin{aligned}
&\int_0^1 K(t,s)q(s)\psi\left(s, I_{0^+}^{\mu_{n-2}-\mu_0} v_1(s) + \frac{1}{m}, I_{0^+}^{\mu_{n-2}-\mu_1} v_1(s) + \frac{1}{m}, \ldots, v_1(s) + \frac{1}{m}\right) ds \\
&\quad \leq \int_0^1 K(t,s)q(s)\psi\left(s, I_{0^+}^{\mu_{n-2}-\mu_0} v_2(s) + \frac{1}{m}, I_{0^+}^{\mu_{n-2}-\mu_1} v_2(s) + \frac{1}{m}, \ldots, v_2(s) + \frac{1}{m}\right) ds,
\end{aligned}$$

which implies that

$$T(u, v_1)(t) \leq T(u, v_2)(t), \quad u \in Q_e, t \in [0,1].$$

That is, $T(u,v)$ is nonincreasing in v for any $u \in Q_e$. Hence, $T : Q_e \times Q_e \to Q_e$ is a mixed monotone operator.

Finally, we prove that $T(cu, c^{-1}v)(t) \geq c^\sigma T(u,v)(t)$ for $c \in (0,1)$, $t \in [0,1]$. In fact, for $u, v \in Q_e$, $c \in (0,1)$, from $(\mathbf{H}_2)$, we have

$$\begin{aligned}
&\int_0^1 K(t,s)q(s)\phi\left(s, I_{0^+}^{\mu_{n-2}-\mu_0} cu(s) + \frac{1}{m}, I_{0^+}^{\mu_{n-2}-\mu_1} cu(s) + \frac{1}{m}, \ldots, cu(s) + \frac{1}{m}\right) ds \\
&\quad \geq \int_0^1 K(t,s)q(s)\phi\left(s, I_{0^+}^{\mu_{n-2}-\mu_0} cu(s) + \frac{c}{m}, I_{0^+}^{\mu_{n-2}-\mu_1} cu(s) + \frac{c}{m}, \ldots, cu(s) + \frac{c}{m}\right) ds \\
&\quad = \int_0^1 K(t,s)q(s) \\
&\qquad \times \phi\left(s, c\left(I_{0^+}^{\mu_{n-2}-\mu_0} u(s) + \frac{1}{m}\right), c\left(I_{0^+}^{\mu_{n-2}-\mu_1} u(s) + \frac{1}{m}\right), \ldots, c\left(u(s) + \frac{1}{m}\right)\right) ds \\
&\quad \geq c^\sigma \int_0^1 K(t,s)q(s)\phi\left(s, I_{0^+}^{\mu_{n-2}-\mu_0} u(s) + \frac{1}{m}, I_{0^+}^{\mu_{n-2}-\mu_1} u(s) + \frac{1}{m}, \ldots, u(s) + \frac{1}{m}\right) ds
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1 K(t,s)q(s)\psi\left(s, I_{0^+}^{\mu_{n-2}-\mu_0} c^{-1}v(s) + \frac{1}{m}, I_{0^+}^{\mu_{n-2}-\mu_1} c^{-1}v(s) + \frac{1}{m}, \ldots, c^{-1}v(s) + \frac{1}{m}\right) ds \\
&\quad \geq \int_0^1 K(t,s)q(s) \\
&\qquad \times \psi\left(s, I_{0^+}^{\mu_{n-2}-\mu_0} c^{-1}v(s) + \frac{1}{cm}, I_{0^+}^{\mu_{n-2}-\mu_1} c^{-1}v(s) + \frac{1}{cm}, \ldots, c^{-1}v(s) + \frac{1}{cm}\right) ds \\
&\quad = \int_0^1 K(t,s)q(s) \\
&\qquad \times \psi\left(s, c^{-1}\left(I_{0^+}^{\mu_{n-2}-\mu_0} v(s) + \frac{1}{m}\right), c^{-1}\left(I_{0^+}^{\mu_{n-2}-\mu_1} v(s) + \frac{1}{m}\right), \ldots, c^{-1}\left(v(s) + \frac{1}{m}\right)\right) ds \\
&\quad \geq c^\sigma \int_0^1 K(t,s)q(s)\psi\left(s, I_{0^+}^{\mu_{n-2}-\mu_0} v(s) + \frac{1}{m}, I_{0^+}^{\mu_{n-2}-\mu_1} v(s) + \frac{1}{m}, \ldots, v(s) + \frac{1}{m}\right) ds,
\end{aligned}$$

which imply that

$$T\left(cu, c^{-1}v\right)(t) \geq c^\sigma T(u,v)(t), \quad t \in [0,1].$$

Thus, Lemma 2.6 ensures that there exists a unique positive solution $u_m^* \in Q_e$ such that $T(u_m^*, u_m^*) = u_m^*$. Consequently, problem (2.5) has a unique positive solution for every $m \in \{2, 3, \ldots\}$.

Since $u_m^* \in C([0,1],[0,+\infty))$, it implies that $\phi(s, I^{\mu_{n-2}-\mu_0} u_m^* + \frac{1}{m}, \ldots, u_m^* + \frac{1}{m})$ and $\psi(s, I^{\mu_{n-2}-\mu_0} u_m^* + \frac{1}{m}, I^{\mu_{n-2}-\mu_1} u_m^* + \frac{1}{m}, \ldots, u_m^* + \frac{1}{m})$ are continuous. Also, u_m^* has uniform lower and upper bounds from $u_m^* \in Q_e$. Hence, in order to pass the solution u_m^* of (2.5) to that of (2.4), we need the fact that $\{u_m^*\}_{m\geq 2}$ is an equicontinuous family on $[0,1]$. In fact, by the Arzela-Ascoli theorem and the Lebesgue dominated convergence theorem, we can complete the proof. Since this process is easy and standard, here we omit the details. Let $u^* = \lim_{m\to+\infty} u_m^*$, then, by Lemma 2.5, $x(t) = I^{\mu_{n-2}-\mu_0} u^*(t)$ is the unique positive solution of BVP (1.1), and

$$\frac{\Gamma(\alpha-\mu_{n-2})}{M\Gamma(\alpha)} t^{\alpha-1} \leq x(t) \leq \frac{M\Gamma(\alpha-\mu_{n-2})}{\Gamma(\alpha)} t^{\alpha-1}, \quad t \in [0,1]. \qquad \square$$

4 An example

Example 4.1 Consider the following singular boundary value problem:

$$\begin{cases} D^{\frac{5}{2}} x(t) + t^{-\frac{1}{4}} [t^2 x^{\frac{1}{6}} + t^{\frac{1}{2}} x^{-\frac{1}{6}} + 2t(D^{\frac{9}{8}} x(t))^{\frac{1}{8}} + (D^{\frac{9}{8}} x(t))^{-\frac{1}{8}}] = 0, & 0 < t < 1, \\ x(0) = D^{\frac{9}{8}} x(0) = 0, \qquad D^{\frac{11}{8}} x(1) = \frac{\sqrt{2}}{2} D^{\frac{11}{8}} x(\frac{1}{2}), \end{cases} \tag{4.1}$$

where

$$q(t) = t^{-\frac{1}{4}}, \qquad f(t, x_1, x_2) = t^2 x_1^{\frac{1}{6}} + t^{\frac{1}{2}} x_1^{-\frac{1}{6}} + 2t x_2^{\frac{1}{8}} + x_2^{-\frac{1}{8}}.$$

Then the singular BVP (4.1) has a unique positive solution.

Proof Let $\alpha = \frac{5}{2}$, $\mu_1 = \frac{9}{8}$, $\mu = \frac{11}{8}$, $p = 3$, then $\sum_{j=1}^{p-2} a_j \xi_j^{\alpha-\mu-1} = \frac{\sqrt{2}}{2}(\frac{1}{2})^{\frac{1}{8}} \approx 0.0027 < 1$. Let $f(t, x_1, x_2) = \phi(t, x_1, x_2) + \psi(t, x_1, x_2)$, where

$$\phi(t, x_1, x_2) = t^2 x_1^{\frac{1}{6}} + 2t x_2^{\frac{1}{8}}, \qquad \psi(t, x_1, x_2) = t^{\frac{1}{2}} x_1^{-\frac{1}{6}} + x_2^{-\frac{1}{8}}.$$

Then, for any $(t, x_1, x_2) \in [0,1] \times (0,\infty)^2$ and $0 < c < 1$,

$$\phi(t, cx_1, cx_2) = c^{\frac{1}{6}} t^2 x_1^{\frac{1}{6}} + 2t c^{\frac{1}{8}} x_2^{\frac{1}{8}} \geq c^{\frac{1}{6}} \phi(t, x_1, x_2)$$

and

$$\psi\left(t, c^{-1} x_1, c^{-1} x_2\right) = c^{\frac{1}{6}} t^{\frac{1}{2}} x_1^{-\frac{1}{6}} + c^{\frac{1}{8}} x_2^{-\frac{1}{8}} \geq c^{\frac{1}{6}} \psi(t, x_1, x_2).$$

Noting $\sigma = \frac{1}{6}$, $\phi(t,1,1) = t^2 + 2t$, $\psi(t,1,1) = t^{\frac{1}{2}} + 1$, we have

$$\int_0^1 q(s)\phi(s,1,1)\,ds = \int_0^1 s^{-\frac{1}{4}}\left(s^2 + 2s\right) ds = \frac{116}{77}$$

and

$$\int_0^1 s^{-\sigma(\alpha-1)} q(s)\psi(s,1,1)\,ds = \int_0^1 s^{-\frac{1}{2}}\left(s^{\frac{1}{2}} + 1\right) ds = 3.$$

Hence, the assumptions $(\mathbf{H}_1)$-$(\mathbf{H}_3)$ of Theorem 3.1 hold. Then Theorem 3.1 implies that BVP (4.1) has a unique positive solution. □

Competing interests
The author declares that there are no competing interests.

Author's contributions
The author declares that he carried out all the work in this manuscript and read and approved the final manuscript.

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