Zhou and Wang *Advances in Difference Equations* (2016) 2016:227
DOI 10.1186/s13662-016-0911-6

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

The Kamenev type oscillation criteria of nonlinear differential equations under impulse effects

Xiaoliang Zhou[1*] and Wu-Sheng Wang[2]

*Correspondence: zxlmath@163.com
[1]Department of Mathematics, Lingnan Normal University, Zhanjiang, Guangdong 524048, P.R. China
Full list of author information is available at the end of the article

Abstract

In this paper, a general class of second-order nonlinear damped differential equations under impulse effects is studied. By introducing an auxiliary function of two variables and using Riccati transformation, several Kamenev type interval oscillation criteria are established, which generalize and extend some known ones, such as those of Huang and Feng. Moreover, examples are given to illustrate the effectiveness and non-emptiness of our results.

MSC: 34C15; 34A37; 65L03

Keywords: interval oscillation; Riccati transformation; impulse; damping

1 Introduction and preliminaries

In 2003, Rogovchenko and Rogovchenko [1] first studied the oscillation of the nonlinear second-order differential equation with nonlinear damping of the form

$$\big(r(t)k_1\big(x(t),x'(t)\big)\big)' + p(t)k_2\big(x(t),x'(t)\big)x'(t) + q(t)h\big(x(t)\big) = 0. \tag{1.1}$$

Later, Tiryaki and Zafer [2], Zhao and Feng [3], Zhao *et al.* [4] and Huang and Meng [5] obtained several oscillation criteria for solutions of (1.1), which extended and improved the results in [1]. In this work, the conditions in terms of the coefficients involving integral averages over the whole half-line $[t_0,\infty)$ are used. However, oscillation is an interval property as pointed out early in [6, 7]. So, it is more reasonable to investigate solutions on an infinite set of bounded intervals. In 2005, Tiryaki and Zafer [8] investigated the interval oscillation of (1.1). Later, some interval oscillation criteria were also given by Shi [9] and Huang and Meng [10] for a forced equation of the form

$$\big(r(t)k_1\big(x(t),x'(t)\big)\big)' + p(t)k_2\big(x(t),x'(t)\big)x'(t) + q(t)h\big(x(t)\big) = f(t). \tag{1.2}$$

In recent years, interval oscillation of impulsive differential equations also aroused the interest of many researchers; see, for example, [11–16]. In 2009, under impulse effects,

Springer

© 2016 Zhou and Wang. This article is distributed under the terms of the Creative Commons Attribution 4.0 International License (http://creativecommons.org/licenses/by/4.0/), which permits unrestricted use, distribution, and reproduction in any medium, provided you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made.

Özbekler and Zafer [17] first considered the special case of equation (1.2) as follows:

$$\begin{cases} (r(t)\varphi_\alpha(x'))' + p(t)\varphi_\alpha(x') + q(t)\varphi_\beta(x) = f(t), & t \neq \theta_i, \\ \Delta(r(t)\varphi_\alpha(x')) + q_i\varphi_\beta(x) = f_i, & t = \theta_i, i \in \mathbb{N}. \end{cases} \tag{1.3}$$

They improved and extended the earlier results for the equations without impulse or damping of [18–21]. Recently, Li *et al.* [22] investigated equation (1.2) with impulses and obtained several interval oscillation theorems by introducing an auxiliary function of one variable and generalized and extended some known results in [9, 17, 23].

In this article, motivated mainly by [8–10, 17, 22], we continue to study the interval oscillation of (1.2) under impulse effects of the form

$$\begin{cases} (r(t)k_1(x(t),x'(t)))' + p(t)k_2(x(t),x'(t))x'(t) + q(t)g(x(t)) = f(t), & t \neq t_i, \\ \Delta(r(t)k_1(x(t),x'(t))) + q_i g(x(t)) = f_i, & t = t_i, i = 1,2,\ldots, \end{cases} \tag{1.4}$$

where $\Delta y(t) = y(t^+) - y(t^-)$, $y(t^\pm) = \lim_{t\to t^\pm} y(t)$, $\{t_i\}$ is an impulse moments sequence with $0 \leq t_0 < t_1 < t_2 < \cdots < t_i < t_{i+1} < \cdots$ and $\lim_{i\to\infty} t_i = \infty$.

By introducing an auxiliary function of two variables and using a Riccati transformation, we establish some Kamenev type (*cf.* Philos [24]) interval oscillation criteria. Our methods are different from those of Özbekler and Zafer [17] and Li *et al.* [22] because we use an auxiliary function of two variables and divide the considered interval into two parts to study oscillation of (1.4). The results obtained here are the generalization and extension of some known ones, such as those of Huang and Feng [10]. Moreover, we also give two examples to illustrate the effectiveness and non-emptiness of our results.

The organization of this paper is as follows. In the next section, we will give four theorems of Kamenev type interval oscillation criteria. Section 3 is devoted to the proofs of the theorems. Two examples will be illustrated in Section 4.

2 The main results

We first introduce a set $\mathrm{PLC}(I,\mathbb{R})$ and a function class $\mathscr{H}_D$ of bivariate functions $H(t,s)$ on a set D.

Let $I \subset \mathbb{R}$ be an interval, $\mathrm{PLC}(I,\mathbb{R}) := \{y : I \to \mathbb{R} | y \text{ is continuous on } I \setminus \{t_i\} \text{ and at each } t_i, y(t_i^+) \text{ and } y(t_i^-) \text{ exist, and the left continuity of } y \text{ is assumed, } i.e., y(t_i^-) = y(t_i), i \in \mathbb{N}\}$.

Let $D = \{(t,s) : t \geq s \geq t_0\}$, $\mathscr{H}_D$ is a class of functions $H(t,s) \in C(D,\mathbb{R})$ which satisfy the following assumptions:

(i) $H(t,t) = 0$, $H(t,s) > 0$, for $t > s \geq t_0$;

(ii) $\frac{\partial}{\partial t}H(t,s) = h_1(t,s)\sqrt{H(t,s)}$, $\frac{\partial}{\partial s}H(t,s) = -h_2(t,s)\sqrt{H(t,s)}$,

for some $h_1, h_2 \in L_{\mathrm{loc}}(D,\mathbb{R})$.

For any given t_0, we say a continuous function $x(t)$ defined on $[t_0,\infty)$ is a solution of (1.4) with initial value $x(t_0) = x_0$ and $x'(t_0) = x_0'$ if $x'(t), (r(t)k_1(x(t),x'(t)))' \in \mathrm{PLC}([t_0,\infty),\mathbb{R})$ and $x(t)$ satisfies (1.4) for $t \geq t_0$.

The following preparatory lemma will be useful to prove our theorems.

Lemma 2.1 *Let $\eta_0, \eta_1, \eta_2 \in C([t_0,\infty),\mathbb{R})$ with $\eta_2 > 0$, and $w \in C^1([t_0,\infty),\mathbb{R})$, for $t \neq t_i$, together with $w(t_i^\pm)$ exist. If there exist $[a,b] \subset [t_0,\infty)$, $t_i \in (a,b)$, $w_i > 0$ for $i \in \mathbb{N}$, and $\alpha > 0$ such that*

$$w'(t) \leq -\eta_0(t) + \eta_1(t)w(t) - \eta_2(t)\left|w(t)\right|^{1+1/\alpha}, \quad t \in (a,b), t \neq t_i, \tag{2.1}$$

and

$$\Delta w(t) \le -w_i, \quad t = t_i \in (a,b), \tag{2.2}$$

then, for every $H \in \mathscr{H}_{(a,b)}$ *and* $c \in (a,b) \setminus \{t_i\}$,

$$\begin{aligned}
&\frac{1}{H^{\alpha+1}(c,a)} \int_a^c \bigl[\tilde{\eta}_2(s)\tilde{H}_1^{\alpha+1}(s,a) - \eta_0(s)H^{\alpha+1}(s,a)\bigr]\,\mathrm{d}s \\
&\quad + \frac{1}{H^{\alpha+1}(b,c)} \int_c^b \bigl[\tilde{\eta}_2(s)\tilde{H}_2^{\alpha+1}(b,s) - \eta_0(s)H^{\alpha+1}(b,s)\bigr]\,\mathrm{d}s \\
&\quad \ge \frac{1}{H^{\alpha+1}(c,a)} \sum_{a<t_i<c} H^{\alpha+1}(t_i,a)w_i + \frac{1}{H^{\alpha+1}(b,c)} \sum_{c<t_i<b} H^{\alpha+1}(b,t_i)w_i,
\end{aligned} \tag{2.3}$$

where

$$\begin{aligned}
&\tilde{\eta}_2(s) = \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}\eta_2^{-\alpha}(s), \\
&\tilde{H}_1(s,a) = \bigl|(\alpha+1)h_1(s,a)\sqrt{H(s,a)} + \eta_1(s)H(s,a)\bigr|, \\
&\tilde{H}_2(b,s) = \bigl|(\alpha+1)h_2(b,s)\sqrt{H(b,s)} - \eta_1(s)H(b,s)\bigr|.
\end{aligned}$$

Proof Multiplying (2.1) by $H^{\alpha+1}(t,s)$ and integrating it (with t replaced by s) over $[c,t)$ for $t \in [c,b)$, we have

$$\begin{aligned}
\int_c^t H^{\alpha+1}(t,s)w'(s)\,\mathrm{d}s \le{}& -\int_c^t \eta_0(s)H^{\alpha+1}(t,s)\,\mathrm{d}s + \int_c^t \eta_1(s)w(s)H^{\alpha+1}(t,s)\,\mathrm{d}s \\
& - \int_c^t \eta_2(s)\bigl|w(s)\bigr|^{1+1/\alpha}H^{\alpha+1}(t,s)\,\mathrm{d}s.
\end{aligned} \tag{2.4}$$

In view of (2.2) and the assumptions (i) and (ii) for $H(t,s)$, we get

$$\begin{aligned}
&\int_c^t \eta_0(s)H^{\alpha+1}(t,s)\,\mathrm{d}s + \sum_{c<t_i<t} H^{\alpha+1}(t,t_i)w_i - H^{\alpha+1}(t,c)w(c) \\
&\quad \le \int_c^t \Bigl[\bigl|(\alpha+1)h_2(t,s)\sqrt{H(t,s)} - \eta_1(s)H(t,s)\bigr|H^\alpha(t,s)\bigl|w(s)\bigr| \\
&\qquad - \eta_2(s)H^{\alpha+1}(t,s)\bigl|w(s)\bigr|^{1+1/\alpha}\Bigr]\,\mathrm{d}s.
\end{aligned} \tag{2.5}$$

Defining the integrand function in right-hand side of (2.5) by

$$\Phi(w) = \bigl|(\alpha+1)h_2(t,s)\sqrt{H(t,s)} - \eta_1(s)H(t,s)\bigr|H^\alpha(t,s)\bigl|w(s)\bigr| - \eta_2(s)H^{\alpha+1}(t,s)\bigl|w(s)\bigr|^{1+1/\alpha},$$

we easily see that

$$\max_{w\in\mathbb{R}} \Phi(w) = \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}\eta_2^{-\alpha}(s)\bigl|(\alpha+1)h_2(t,s)\sqrt{H(t,s)} - \eta_1(s)H(t,s)\bigr|^{\alpha+1}.$$

Therefore,

$$\int_c^t \eta_0(s)H^{\alpha+1}(t,s)\,\mathrm{d}s + \sum_{c<t_i<t} H^{\alpha+1}(t,t_i)w_i - H^{\alpha+1}(t,c)w(c)$$
$$\leq \int_c^t \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}\eta_2^{-\alpha}(s)\left|(\alpha+1)h_2(t,s)\sqrt{H(t,s)} - \eta_1(s)H(t,s)\right|^{\alpha+1}\mathrm{d}s. \tag{2.6}$$

Letting $t \to b^-$ in (2.6)

$$\int_c^b \eta_0(s)H^{\alpha+1}(b,s)\,\mathrm{d}s + \sum_{c<t_i<b} H^{\alpha+1}(b,t_i)w_i - H^{\alpha+1}(b,c)w(c)$$
$$\leq \int_c^b \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}\eta_2^{-\alpha}(s)\left|(\alpha+1)h_2(b,s)\sqrt{H(b,s)} - \eta_1(s)H(b,s)\right|^{\alpha+1}\mathrm{d}s. \tag{2.7}$$

Similarly, if (2.1) (with t replaced by s) is multiplied by $H^{\alpha+1}(s,t)$ and integrated over $(t,c]$ for $t \in (a,c]$, we have

$$\int_t^c H^{\alpha+1}(s,t)w'(s)\,\mathrm{d}s \leq -\int_t^c \eta_0(s)H^{\alpha+1}(s,t)\,\mathrm{d}s + \int_t^c \eta_1(s)w(s)H^{\alpha+1}(s,t)\,\mathrm{d}s$$
$$-\int_t^c \eta_2(s)\left|w(s)\right|^{1+1/\alpha}H^{\alpha+1}(s,t)\,\mathrm{d}s. \tag{2.8}$$

In view of (2.2) and using the assumptions (i) and (ii), we get

$$\int_t^c \eta_0(s)H^{\alpha+1}(s,t)\,\mathrm{d}s + \sum_{t<t_i<c} H^{\alpha+1}(t_i,t)w_i + H^{\alpha+1}(c,t)w(c)$$
$$\leq \int_t^c \Big[\left|(\alpha+1)h_1(s,t)\sqrt{H(s,t)} + \eta_1(s)H(s,t)\right|H^\alpha(s,t)\left|w(s)\right|$$
$$- \eta_2(s)H^{\alpha+1}(s,t)\left|w(s)\right|^{1+1/\alpha}\Big]\,\mathrm{d}s$$
$$\leq \int_t^c \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}\eta_2^{-\alpha}(s)\left|(\alpha+1)h_1(s,t)\sqrt{H(s,t)} + \eta_1(s)H(s,t)\right|^{\alpha+1}\mathrm{d}s. \tag{2.9}$$

Letting $t \to a^+$ in (2.9)

$$\int_a^c \eta_0(s)H^{\alpha+1}(s,a)\,\mathrm{d}s + \sum_{a<t_i<c} H^{\alpha+1}(t_i,a)w_i + H^{\alpha+1}(c,a)w(c)$$
$$\leq \int_a^c \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}\eta_2^{-\alpha}(s)\left|(\alpha+1)h_1(s,a)\sqrt{H(s,a)} + \eta_1(s)H(s,a)\right|^{\alpha+1}\mathrm{d}s. \tag{2.10}$$

Finally, dividing (2.7) and (2.10) by $H^{\alpha+1}(b,c)$ and $H^{\alpha+1}(c,a)$, respectively, and then adding them, we get (2.3). □

2.1 Interval oscillation when $g(x)$ is differentiable

In this section we need the following conditions:

(A_1) $r(t) \in C([t_0,\infty),(0,\infty))$ and is nondecreasing, $p(t), q(t), f(t) \in \mathrm{PLC}([t_0,\infty),\mathbb{R})$;
(A_2) $vk_1(u,v) \geq \mu|k_1(u,v)|^{1+1/\alpha}$ for some $\mu > 0$, $\alpha > 0$ and for all $(u,v) \in \mathbb{R}^2$;

(A_3) $vk_2(u,v)\varphi_{\frac{1}{\alpha}}(g(u)) \geq \tau|k_1(u,v)|^{1+1/\alpha}$ for some $\tau > 0$, $\alpha > 0$, and for all $(u,v) \in \mathbb{R}^2$, where $\varphi_*(g) = |g|^{*-1}g$;

(A_4) $g(u)$ is differentiable, $ug(u) > 0$ for $u \neq 0$ and $g'(u) \geq \varsigma|g(u)|^{1-1/\alpha}$ for some $\varsigma > 0$, $\alpha > 0$, and for all $u \in \mathbb{R} \setminus \{0\}$.

Theorem 2.1 *Assume that* (A_1)-(A_4) *hold. For any given* $T_0 \geq t_0$, *there exist intervals* $I_j = [a_j, b_j] \subset [T_0, \infty)$, $j = 1, 2$, *such that*

(A_5) $p(t) \geq 0$, $\forall t \in \{I_1 \cup I_2\}$;

(A_6) $q(t) \geq 0$, $\forall t \in \{I_1 \cup I_2\} \setminus \{t_i\}$, $q_i \geq 0$, $t_i \in \{I_1 \cup I_2\}$, $\forall i \in \mathbb{N}$;

(A_7) $f(t)\begin{cases} \leq 0, & t \in I_1 \setminus \{t_i\}, \\ \geq 0, & t \in I_2 \setminus \{t_i\}, \end{cases}$ $f_i\begin{cases} \leq 0, t_i \in I_1, \\ \geq 0, t_i \in I_2, \end{cases}$ $\forall i \in \mathbb{N}$.

If there exist $H_j \in \mathscr{H}_{(a_j,b_j)}$ *and* $c_j \in (a_j, b_j) \setminus \{t_i\}$, $j = 1, 2$, *such that*

$$\begin{aligned} &\frac{1}{H_j^{\alpha+1}(c_j,a_j)} \int_{a_j}^{c_j} \bigl[\tilde{p}(s)\tilde{H}_{j1}^{\alpha+1}(s,a_j) - q(s)H_j^{\alpha+1}(s,a_j)\bigr]\,\mathrm{d}s \\ &\quad + \frac{1}{H_j^{\alpha+1}(b_j,c_j)} \int_{c_j}^{b_j} \bigl[\tilde{p}(s)\tilde{H}_{j2}^{\alpha+1}(b_j,s) - q(s)H_j^{\alpha+1}(b_j,s)\bigr]\,\mathrm{d}s \\ &< \frac{1}{H_j^{\alpha+1}(c_j,a_j)} \sum_{a_j<t_i<c_j} H_j^{\alpha+1}(t_i,a_j)q_i + \frac{1}{H_j^{\alpha+1}(b_j,c_j)} \sum_{c_j<t_i<b_j} H_j^{\alpha+1}(b_j,t_i)q_i, \end{aligned} \tag{2.11}$$

where

$$\begin{aligned} &\tilde{p}(s) = \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} \frac{r^{\alpha+1}(s)}{(\tau p(s) + \mu\varsigma r(s))^\alpha}, \\ &\tilde{H}_{j1}(s,a_j) = \bigl|(\alpha+1)h_{j1}(s,a_j)\sqrt{H_j(s,a_j)}\bigr|, \\ &\tilde{H}_{j2}(b_j,s) = \bigl|(\alpha+1)h_{j2}(b_j,s)\sqrt{H_j(b_j,s)}\bigr|, \end{aligned} \tag{2.12}$$

then (1.4) *is oscillatory.*

Proof Suppose that there exists a nonoscillatory solution $x(t)$ of (1.4). Then there exists a $T_0 \geq t_0$ such that $x(t) \neq 0$ on $[T_0, \infty)$. Define

$$w(t) =: \frac{r(t)k_1(x(t),x'(t))}{g(x(t))}, \quad t \geq T_0. \tag{2.13}$$

Differentiating (2.13) and using (1.4) for $t \geq T_0$, we have

$$\begin{aligned} w'(t) = {}& -q(t) - \frac{p(t)k_2(x(t),x'(t))x'(t)}{g(x(t))} \\ & - \frac{r(t)k_1(x(t),x'(t))x'(t)g'(x(t))}{g^2(x(t))} + \frac{f(t)}{g(x(t))}, \quad t \neq t_i, \end{aligned} \tag{2.14}$$

and

$$\Delta w(t) = -q_i + \frac{f_i}{g(x(t))}, \quad t = t_i. \tag{2.15}$$

By using assumptions (A_1)-(A_6) and from (2.14), we obtain, for $t \in \{I_1 \cup I_2\}$,

$$w'(t) \leq -q(t) - \frac{\tau p(t) + \mu \varsigma r(t)}{r^{1+1/\alpha}(t)} \left|w(t)\right|^{1+1/\alpha} + \frac{f(t)}{g(x(t))}, \quad t \neq t_i. \tag{2.16}$$

If $x(t) > 0$, we choose the interval I_1 to consider. From (2.16) and (2.15) and in view of condition (A_7), we get

$$w'(t) \leq -q(t) - \frac{\tau p(t) + \mu \varsigma r(t)}{r^{1+1/\alpha}(t)} \left|w(t)\right|^{1+1/\alpha}, \quad t \in I_1, t \neq t_i, \tag{2.17}$$

and

$$\Delta w(t) \leq -q_i, \quad t \in I_1, t = t_i. \tag{2.18}$$

Applying Lemma 2.1 to (2.17) and (2.18), for every $H_1 \in \mathscr{H}_{(a_1,b_1)}$ and $c_1 \in (a_1, b_1) \setminus \{t_i\}$, we have

$$\begin{aligned} &\frac{1}{H_1^{\alpha+1}(c_1,a_1)} \int_{a_1}^{c_1} \left[\tilde{p}(s)\tilde{H}_{11}^{\alpha+1}(s,a_1) - q(s)H_1^{\alpha+1}(s,a_1)\right] \mathrm{d}s \\ &\quad + \frac{1}{H_1^{\alpha+1}(b_1,c_1)} \int_{c_1}^{b_1} \left[\tilde{p}(s)\tilde{H}_{12}^{\alpha+1}(b_1,s) - q(s)H_1^{\alpha+1}(b_1,s)\right] \mathrm{d}s \\ &\quad \geq \frac{1}{H_1^{\alpha+1}(c_1,a_1)} \sum_{a_1<t_i<c_1} H_1^{\alpha+1}(t_i,a_1)q_i + \frac{1}{H_1^{\alpha+1}(b_1,c_1)} \sum_{c_1<t_i<b_1} H_1^{\alpha+1}(b_1,t_i)q_i, \end{aligned}$$

where $\tilde{p}(s)$, $\tilde{H}_{11}(s,a_1)$ and $\tilde{H}_{12}(b_1,s)$ are defined by (2.12). This contradicts (2.11) for $j = 1$.

If $x(t) < 0$, we choose the interval I_2 to consider. Similar to the above proof we can also obtain a contradiction to (2.11) for $j = 2$. The proof is complete. □

Remark 2.1 In many published papers on the subject of interval oscillation criteria, the authors have insisted on the use of the function $\rho(t)$ as a multiplier in the Riccati transformation (2.13). This function has no meaning, *i.e.*, it can be taken as $\rho(t) = 1$.

Remark 2.2 When the impulses in (1.4) disappear, *i.e.*, $q_i = f_i = 0$, for all $i \in \mathbb{N}$, (1.4) reduces to the equation (1.2) studied by Huang and Feng in [10]. Therefore Theorem 2.1 provides an extension of Theorem 2.1 of [10], to the impulsive differential equations.

Remark 2.3 If the inequality in condition (A_3) is strict, then the sign condition $p(t)$ can be replaced by

$$\tau p(t) + \mu \varsigma r(t) > 0$$

in the above theorem.

In the next theorem the sign condition $p(t) \geq 0$ will be sufficiently eliminated by replacing (A_3) with

(A_3') $\nu k_2(u,v) = \zeta k_1(u,v)$ for some $\zeta > 0$ and all $(u,v) \in \mathbb{R}^2$.

Theorem 2.2 *Assume that* (A_1), (A_2), (A_3'), *and* (A_4) *hold. For any given* $T_0 \geq t_0$, *there exist intervals* $I_j = [a_j, b_j] \subset [T_0, \infty)$, $j = 1, 2$ *such that* (A_6) *and* (A_7) *hold. If there exist* $H_j \in \mathscr{H}_{(a_j,b_j)}$ *and* $c_j \in (a_j, b_j) \setminus \{t_i\}$, $j = 1, 2$, *such that*

$$\begin{aligned}&\frac{1}{H_j^{\alpha+1}(c_j,a_j)}\int_{a_j}^{c_j}\bigl[\tilde{r}(s)\tilde{H}_{j1}^{\alpha+1}(s,a_j)-q(s)H_j^{\alpha+1}(s,a_j)\bigr]\,\mathrm{d}s\\&\quad+\frac{1}{H_j^{\alpha+1}(b_j,c_j)}\int_{c_j}^{b_j}\bigl[\tilde{r}(s)\tilde{H}_{j2}^{\alpha+1}(b_j,s)-q(s)H_j^{\alpha+1}(b_j,s)\bigr]\,\mathrm{d}s\\&<\frac{1}{H_j^{\alpha+1}(c_j,a_j)}\sum_{a_j<t_i<c_j}H_j^{\alpha+1}(t_i,a_j)q_i+\frac{1}{H_j^{\alpha+1}(b_j,c_j)}\sum_{c_j<t_i<b_j}H_j^{\alpha+1}(b_j,t_i)q_i,\end{aligned}\tag{2.19}$$

where

$$\begin{aligned}\tilde{r}(s)&=\frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}\frac{r(s)}{(\mu\varsigma r(s))^\alpha},\\\tilde{H}_{j1}(s,a_j)&=\left|(\alpha+1)h_{j1}(s,a_j)\sqrt{H_j(s,a_j)}-\frac{\zeta p(s)}{r(s)}H_j(s,a_j)\right|,\\\tilde{H}_{j2}(b_j,s)&=\left|(\alpha+1)h_{j2}(b_j,s)\sqrt{H_j(b_j,s)}+\frac{\zeta p(s)}{r(s)}H_j(b_j,s)\right|,\end{aligned}$$

then (1.4) *is oscillatory.*

Proof Suppose that there exists a nonoscillatory solution $x(t)$ of (1.4). Then there exists a $T_0 \geq t_0$ such that $x(t) \neq 0$ on $[T_0, \infty)$. Define

$$w(t) =: \frac{r(t)k_1(x(t),x'(t))}{g(x(t))}, \quad t \geq T_0.\tag{2.20}$$

Similar to the proof of Theorem 2.1, we can obtain (2.14) and (2.15). By using assumptions (A_1), (A_2), (A_3'), (A_4), and from (2.14), we obtain for $t \in \{I_1 \cup I_2\}$

$$w'(t) \leq -q(t) - \frac{\zeta p(t)}{r(t)}w(t) - \frac{\mu\varsigma}{r^{1/\alpha}(t)}\bigl|w(t)\bigr|^{1+1/\alpha} + \frac{f(t)}{g(x(t))}, \quad t \neq t_i.\tag{2.21}$$

If $x(t) > 0$, we choose the interval I_1 to consider. From (2.21) and (2.15) and in view of condition (A_7), we obtain

$$w'(t) \leq -q(t) - \frac{\zeta p(t)}{r(t)}w(t) - \frac{\mu\varsigma}{r^{1/\alpha}(t)}\bigl|w(t)\bigr|^{1+1/\alpha}, \quad t \neq t_i,\tag{2.22}$$

and

$$\Delta w(t) \leq -q_i, \quad t = t_i.\tag{2.23}$$

Using the same method as in the proof of Theorem 2.1, we can obtain a contradiction to the condition (2.19) for $j = 1$.

If $x(t) < 0$, we choose the interval I_2 to consider. Similar to the above proof we can also obtain a contradiction to (2.19) for $j = 2$.

Therefore the proof is complete. □

2.2 Interval oscillation when $g(x)$ is not differentiable

In this section we assume that the conditions (A_1), (A_2), (A_5)-(A_7) in Section 2.1 and the following conditions are satisfied:

(A_8) $uvk_2(u,v) \geq \xi |k_1(u,v)|^{1+1/\alpha}$ for some $\xi > 0$, $\alpha > 0$, and all $(u,v) \in \mathbb{R}^2$;

(A_9) $g(u)/\varphi_\alpha(u) \geq \vartheta |u|^{\beta-\alpha}$ for some $\vartheta > 0$, $\beta \geq \alpha > 0$, and all $u \in \mathbb{R} \setminus \{0\}$, where $\varphi_*(u) = |u|^{*-1}u$.

Theorem 2.3 *Assume that* (A_1), (A_2), (A_8), *and* (A_9) *hold. For any given* $T_0 \geq t_0$, *there exist intervals* $I_j = [a_j, b_j] \subset [T_0, \infty)$, $j = 1, 2$, *such that* (A_5)-(A_7) *hold. If there exist* $H_j \in \mathscr{H}_{(a_j,b_j)}$ *and* $c_j \in (a_j, b_j) \setminus \{t_i\}$, $j = 1, 2$, *such that*

$$\begin{aligned} &\frac{1}{H_j^{\alpha+1}(c_j,a_j)} \int_{a_j}^{c_j} \bigl[\tilde{p}(s)\tilde{H}_{j1}^{\alpha+1}(s,a_j) - \tilde{q}(s)H_j^{\alpha+1}(s,a_j)\bigr]\,\mathrm{d}s \\ &\quad + \frac{1}{H_j^{\alpha+1}(b_j,c_j)} \int_{c_j}^{b_j} \bigl[\tilde{p}(s)\tilde{H}_{j2}^{\alpha+1}(b_j,s) - \tilde{q}(s)H_j^{\alpha+1}(b_j,s)\bigr]\,\mathrm{d}s \\ &< \frac{1}{H_j^{\alpha+1}(c_j,a_j)} \sum_{a_j<t_i<c_j} H_j^{\alpha+1}(t_i,a_j)\tilde{q}_i + \frac{1}{H_j^{\alpha+1}(b_j,c_j)} \sum_{c_j<t_i<b_j} H_j^{\alpha+1}(b_j,t_i)\tilde{q}_i, \end{aligned} \tag{2.24}$$

where

$$\begin{aligned} \tilde{p}(s) &= \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} \frac{r^{\alpha+1}(s)}{(\xi p(s) + \alpha\mu r(s))^\alpha}, \\ \tilde{H}_{j1}(s,a_j) &= \bigl|(\alpha+1)h_{j1}(s,a_j)\sqrt{H_j(s,a_j)}\bigr|, \\ \tilde{H}_{j2}(b_j,s) &= \bigl|(\alpha+1)h_{j2}(b_j,s)\sqrt{H_j(b_j,s)}\bigr|, \end{aligned} \tag{2.25}$$

and

$$\begin{aligned} \tilde{q}(s) &= \beta\alpha^{-\alpha/\beta}(\beta-\alpha)^{\alpha/\beta-1}\bigl(\vartheta q(s)\bigr)^{\alpha/\beta}\bigl|f(s)\bigr|^{1-\alpha/\beta}, \\ \tilde{q}_i &= \beta\alpha^{-\alpha/\beta}(\beta-\alpha)^{\alpha/\beta-1}(\vartheta q_i)^{\alpha/\beta}|f_i|^{1-\alpha/\beta}, \end{aligned} \tag{2.26}$$

then (1.4) *is oscillatory.*

Proof Suppose that there exists a nonoscillatory solution $x(t)$ of (1.4). Then there exists a $T_0 \geq t_0$ such that $x(t) \neq 0$ on $[T_0, \infty)$. Define

$$w(t) =: \frac{r(t)k_1(x(t),x'(t))}{\varphi_\alpha(x(t))}, \quad t \geq T_0. \tag{2.27}$$

Differentiating (2.27) and using (1.4) for $t \geq T_0$, we have, for $t \in \{I_1 \cup I_2\}$ and $t \neq t_i$,

$$\begin{aligned} w'(t) = &-\frac{p(t)k_2(x(t),x'(t))x'(t)}{\varphi_\alpha(x(t))} - q(t)\frac{g(x(t))}{\varphi_\alpha(x(t))} \\ &- \frac{\alpha r(t)k_1(x(t),x'(t))x'(t)}{x(t)\varphi_\alpha(x(t))} + \frac{f(t)}{\varphi_\alpha(x(t))}, \end{aligned} \tag{2.28}$$

and, for $t = t_i$,

$$\Delta w(t) = -q_i \frac{g(x(t))}{\varphi_\alpha(x(t))} + \frac{f_i}{\varphi_\alpha(x(t))}. \tag{2.29}$$

If $x(t) > 0$, we choose the interval I_1 to consider. By using assumptions (A_1), (A_2), (A_5)-(A_7), (A_9), (A_{10}), and from (2.28), we obtain, for $t \in I_1$,

$$w'(t) \leq -\left(\vartheta q(t)\left|x(t)\right|^{\beta-\alpha} + \left|f(t)\right|\left|x(t)\right|^{-\alpha}\right) - \frac{\xi p(t) + \alpha\mu r(t)}{(r(t))^{1+1/\alpha}}\left|w(t)\right|^{1+1/\alpha}, \quad t \neq t_i. \tag{2.30}$$

Defining the functions

$$Q(x) := \vartheta q|x|^{\beta-\alpha} + |f||x|^{-\alpha}$$

and

$$\tilde{q}(t) := \beta\alpha^{-\alpha/\beta}(\beta-\alpha)^{\alpha/\beta-1}\left(\vartheta q(t)\right)^{\alpha/\beta}\left|f(t)\right|^{1-\alpha/\beta},$$

we easily see that, when $\beta > \alpha$,

$$Q(x) \geq \min_{x\in\mathbb{R}\setminus\{0\}} Q(x) = \tilde{q}(t) \tag{2.31}$$

and when $\beta = \alpha$,

$$Q(x) = \vartheta q(t) = \tilde{q}(t). \tag{2.32}$$

Applying (2.31) and (2.32) into (2.30), we have

$$w'(t) \leq -\tilde{q}(t) - \frac{\xi p(t) + \alpha\mu r(t)}{(r(t))^{1+1/\alpha}}\left|w(t)\right|^{1+1/\alpha}, \quad t \neq t_i. \tag{2.33}$$

On the other hand, from (2.29) we have, for $t \in I_1$,

$$\Delta w(t) \leq -\left(\vartheta q_i\left|x(t)\right|^{\beta-\alpha} + |f_i|\left|x(t)\right|^{-\alpha}\right) \leq -\tilde{q}_i, \quad t = t_i, \tag{2.34}$$

where $\tilde{q}_i = \beta\alpha^{-\alpha/\beta}(\beta-\alpha)^{\alpha/\beta-1}(\vartheta q_i)^{\alpha/\beta}|f_i|^{1-\alpha/\beta}$.

Using Lemma 2.1 to (2.33) and (2.44), for $H_1 \in \mathscr{H}_{(a_1,b_1)}$ and $c_1 \in (a_1,b_1)\setminus\{t_i\}$, we have

$$\begin{aligned}
&\frac{1}{H_1^{\alpha+1}(c_1,a_1)}\int_{a_1}^{c_1}\left[\tilde{p}(s)\tilde{H}_{11}^{\alpha+1}(s,a_1) - \tilde{q}(s)H_1^{\alpha+1}(s,a_1)\right]\mathrm{d}s \\
&\quad + \frac{1}{H_1^{\alpha+1}(b_1,c_1)}\int_{c_1}^{b_1}\left[\tilde{p}(s)\tilde{H}_{12}^{\alpha+1}(b_1,s) - \tilde{q}(s)H_1^{\alpha+1}(b_1,s)\right]\mathrm{d}s \\
&\geq \frac{1}{H_1^{\alpha+1}(c_1,a_1)}\sum_{a_1<t_i<c_1} H_1^{\alpha+1}(t_i,a_1)\tilde{q}_i + \frac{1}{H_1^{\alpha+1}(b_1,c_1)}\sum_{c_1<t_i<b_1} H_1^{\alpha+1}(b_1,t_i)\tilde{q}_i,
\end{aligned}$$

where $\tilde{p}(s)$, $\tilde{H}_{11}(s,a_1)$ and $\tilde{H}_{12}(b_1,s)$ are defined by (2.25). This contradicts (2.24) for $j = 1$.

If $x(t) < 0$, we choose the interval I_2 to consider. Similar to the above proof we can also obtain a contradiction to (2.24) for $j = 2$. The proof is complete. □

In the next theorem we will consider the special case of (1.4) of the following form:

$$\begin{cases} (r(t)k_1(x(t),x'(t)))' + p(t)k_1(x(t),x'(t)) + q(t)g(x(t)) = f(t), & t \neq t_i, \\ \Delta(r(t)k_1(x(t),x'(t))) + q_i g(x(t)) = f_i, & t = t_i, i = 1,2,\ldots. \end{cases} \tag{2.35}$$

In this theorem the sign condition for $p(t)$ will be eliminated.

Theorem 2.4 *Assume that* (A_1), (A_2), (A_9) *hold. For any given* $T_0 \geq t_0$, *there exist intervals* $I_j = [a_j, b_j] \subset [T_0, \infty)$, $j = 1,2$, *such that* (A_6) *and* (A_7) *hold. If there exist* $H_j \in \mathscr{H}_{(a_j,b_j)}$ *and* $c_j \in (a_j, b_j) \setminus \{t_i\}$, $j = 1,2$ *such that*

$$\begin{aligned} &\frac{1}{H_j^{\alpha+1}(c_j,a_j)} \int_{a_j}^{c_j} \bigl[\tilde{r}(s)\tilde{H}_{j1}^{\alpha+1}(s,a_j) - \tilde{q}(s)H_j^{\alpha+1}(s,a_j)\bigr]\,ds \\ &\quad + \frac{1}{H_j^{\alpha+1}(b_j,c_j)} \int_{c_j}^{b_j} \bigl[\tilde{r}(s)\tilde{H}_{j2}^{\alpha+1}(b_j,s) - \tilde{q}(s)H_j^{\alpha+1}(b_j,s)\bigr]\,ds \\ &< \frac{1}{H_j^{\alpha+1}(c_j,a_j)} \sum_{a_j<t_i<c_j} H_j^{\alpha+1}(t_i,a_j)\tilde{q}_i + \frac{1}{H_j^{\alpha+1}(b_j,c_j)} \sum_{c_j<t_i<b_j} H_j^{\alpha+1}(b_j,t_i)\tilde{q}_i, \end{aligned} \tag{2.36}$$

where

$$\begin{aligned} \tilde{r}(s) &= \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} \frac{r(s)}{(\alpha\mu)^\alpha}, \\ \tilde{H}_{j1}(s,a_j) &= \left|(\alpha+1)h_{j1}(s,a_j)\sqrt{H_j(s,a_j)} - \frac{p(s)}{r(s)}H_j(s,a_j)\right|, \\ \tilde{H}_{j2}(b_j,s) &= \left|(\alpha+1)h_{j2}(b_j,s)\sqrt{H_j(b_j,s)} + \frac{p(s)}{r(s)}H_j(b_j,s)\right|, \end{aligned} \tag{2.37}$$

and

$$\begin{aligned} \tilde{q}(s) &= \beta\alpha^{-\alpha/\beta}(\beta-\alpha)^{\alpha/\beta-1}\bigl(\vartheta q(s)\bigr)^{\alpha/\beta}\bigl|f(s)\bigr|^{1-\alpha/\beta}, \\ \tilde{q}_i &= \beta\alpha^{-\alpha/\beta}(\beta-\alpha)^{\alpha/\beta-1}(\vartheta q_i)^{\alpha/\beta}|f_i|^{1-\alpha/\beta}, \end{aligned}$$

then (1.4) *is oscillatory.*

Proof Suppose that there exists a nonoscillatory solution $x(t)$ of (1.4). Then there exists a $T_0 \geq t_0$ such that $x(t) \neq 0$ on $[T_0, \infty)$. Similar to the proof of Theorem 2.3 we have

$$\begin{aligned} w'(t) = {}& -q(t)\frac{g(x(t))}{\varphi_\alpha(x(t))} - \frac{p(t)k_1(x(t),x'(t))}{\varphi_\alpha(x(t))} \\ & - \frac{\alpha r(t)k_1(x(t),x'(t))x'(t)}{x(t)\varphi_\alpha(x(t))} + \frac{f(t)}{\varphi_\alpha(x(t))}, \quad t \neq t_i, \end{aligned} \tag{2.38}$$

and

$$\Delta w(t) = -q_i\frac{g(x(t))}{\varphi_\alpha(x(t))} + \frac{f_i}{\varphi_\alpha(x(t))}, \quad t = t_i, \tag{2.39}$$

where $w(t)$ is defined as of (2.27). If $x(t) > 0$, we choose the interval I_1 to consider. By using assumptions (A_1), (A_2), (A_6), (A_7), and (A_9) and from (2.38), we obtain, for $t \in I_1$,

$$\begin{aligned} w'(t) \le & -\left(\vartheta q(t)\left|x(t)\right|^{\beta-\alpha} + \left|f(t)\right|\left|x(t)\right|^{-\alpha}\right) \\ & - \frac{p(t)}{r(t)}w(t) - \frac{\alpha\mu}{r^{1/\alpha}(t)}\left|w(t)\right|^{1+1/\alpha}, \quad t \neq t_i, \end{aligned} \tag{2.40}$$

and

$$\Delta w(t) \le -\left(\vartheta q_i\left|x(t)\right|^{\beta-\alpha} + |f_i|\left|x(t)\right|^{-\alpha}\right), \quad t = t_i. \tag{2.41}$$

Then we have

$$w'(t) \le -\tilde{q}(t) - \frac{p(t)}{r(t)}w(t) - \frac{\alpha\mu}{r^{1/\alpha}(t)}\left|w(t)\right|^{1+1/\alpha}, \quad t \neq t_i, \tag{2.42}$$

and

$$\Delta w(t) \le -\tilde{q}_i, \quad t = t_i, \tag{2.43}$$

where

$$\begin{aligned} &\tilde{q}(t) = \beta\alpha^{-\alpha/\beta}(\beta-\alpha)^{\alpha/\beta-1}\left(\vartheta q(t)\right)^{\alpha/\beta}\left|f(t)\right|^{1-\alpha/\beta}, \\ &\tilde{q}_i = \beta\alpha^{-\alpha/\beta}(\beta-\alpha)^{\alpha/\beta-1}(\vartheta q_i)^{\alpha/\beta}|f_i|^{1-\alpha/\beta}. \end{aligned} \tag{2.44}$$

Using Lemma 2.1 in (2.42) and (2.43) we obtain a contradiction to (2.36) for $t \in I_1$.

If $x(t) < 0$, we choose the interval I_2 to consider. Similar to the above proof we can also obtain a contradiction to (2.36) for $j = 2$. The proof is complete. □

3 Examples

In this section, we give two examples to illustrate the effectiveness and non-emptiness of our results. We will see that if there is no impulse then no oscillation conclusion can be drawn.

Example 3.1 Consider second-order nonlinear impulsive differential equations

$$\begin{cases} (k_1(x(t),x'(t)))' + (\cos t)k_2(x(t),x'(t))x'(t) + (\cos t)g(x(t)) = \sin(3t), & t \neq t_i, \\ \Delta k_1(x(t),x'(t)) + \gamma(\cos t)g(x(t)) = \lambda\sin(3t), & t = t_i, \end{cases} \tag{3.1}$$

where γ, λ are positive real numbers, and

$$k_1(u,v) = \frac{u^{2/3}\varphi_{1/3}(v)}{(1+u^2)^{1/3}}, \qquad k_2(u,v) = \frac{u^{5/3}\varphi_{1/3}(v)}{(1+u^2)^{4/3}}, \qquad g(u) = \varphi_{1/3}(u).$$

Then it is easy to see $\mu = \tau = 1$, $\varsigma = \alpha = 1/3$. For any given $T_0 \ge 0$ we may choose $n \in \mathbb{N}$ sufficiently large so that $n\pi \ge T_0$. If we let $I_1 = [a_1, b_1] = [2n\pi - \pi/3, 2n\pi]$, $I_2 = [a_2, b_2] = [2n\pi, 2n\pi + \pi/3]$, $c_1 = 2n\pi - 5\pi/24$, $c_2 = 2n\pi + 5\pi/24$, impulsive points $t_{i,j} =$

$2n\pi + (-1)^j i\pi/12$ ($i = 1,2,3$, $j = 1,2$), then the conditions (A_1)-(A_6) hold, $t_{i,1} \in [a_1, b_1]$ and $t_{i,2} \in [a_2, b_2]$ for $i = 1,2,3$. Letting $\rho(t) = 1$ and $H_j(t,s) = (t-s)^2$ for $j = 1,2$, we have

$$\begin{aligned}
&\frac{1}{H_1^{\alpha+1}(c_1,a_1)}\int_{a_1}^{c_1}\left[\tilde{p}(s)\tilde{H}_{11}^{\alpha+1}(s,a_1) - \rho(s)q(s)H_1^{\alpha+1}(s,a_1)\right]\mathrm{d}s \\
&\quad= \left(\frac{8}{\pi}\right)^{8/3}\int_{2n\pi-\pi/3}^{2n\pi-5\pi/24}\left[\frac{2^{4/3}(s-2n\pi+\pi/3)^{4/3}}{(3\cos s+1)^{1/3}} - (s-2n\pi+\pi/3)^{8/3}\cos s\right]\mathrm{d}s \\
&\quad= \left(\frac{8}{\pi}\right)^{8/3}\int_0^{\pi/8}\left[\frac{(2s)^{4/3}}{(3\cos(s-\pi/3)+1)^{1/3}} - s^{8/3}\cos(s-\pi/3)\right]\mathrm{d}s \approx 0.929, \\
&\frac{1}{H_1^{\alpha+1}(b_1,c_1)}\int_{c_1}^{b_1}\left[\tilde{p}(s)\tilde{H}_{12}^{\alpha+1}(b_1,s) - \rho(s)q(s)H_1^{\alpha+1}(b_1,s)\right]\mathrm{d}s \\
&\quad= \left(\frac{24}{5\pi}\right)^{8/3}\int_0^{5\pi/24}\left[\frac{(2s)^{4/3}}{(3\cos s+1)^{1/3}} - s^{8/3}\cos s\right]\mathrm{d}s \approx 0.653, \\
&\frac{1}{H_1^{\alpha+1}(c_1,a_1)}\sum_{a_1<t_{i,1}<c_1} H_1^{\alpha+1}(t_{i,1},a_1)\rho(t_{i,1})q_{i,1} = \left(\frac{8}{\pi}\right)^{8/3}(t_{i,1}-a_1)^{8/3}\gamma\cos t_{i,1} \\
&\quad= \gamma\left(\frac{8}{\pi}\right)^{8/3}\left(\frac{\pi}{12}\right)^{8/3}\cos\left(-\frac{\pi}{4}\right) \approx 0.240\gamma, \\
&\frac{1}{H_1^{\alpha+1}(b_1,c_1)}\sum_{c_1<t_{i,1}<b_1} H_1^{\alpha+1}(t_{i,1},a_1)\rho(t_{i,1})q_{i,1} \\
&\quad= \gamma\left(\frac{24}{5\pi}\right)^{8/3}\left[\left(\frac{\pi}{12}\right)^{8/3}\cos\left(\frac{\pi}{12}\right) + \left(\frac{2\pi}{12}\right)^{8/3}\cos\left(\frac{2\pi}{12}\right)\right] \approx 0.562\gamma.
\end{aligned}$$

From (2.11) for $j = 1$, we have

$$\gamma > 1.905. \tag{3.2}$$

For $j = 2$, by similarly calculating we also obtain the above condition (3.2). It follows from Theorem 2.1 that (3.1) is oscillatory if $\gamma > 1.905$.

Example 3.2 Consider the second-order nonlinear impulsive differential equations

$$\begin{cases} ((2+\sin t)k_1(x(t),x'(t)))' + (2-\sin t)k_2(x(t),x'(t))x'(t) + |\tan t|g(x(t)) \\ \quad = \sin t, \quad t \neq t_i, \\ \Delta((2+\sin t)k_1(x(t),x'(t))) + \gamma(2-\sin t)g(x(t)) = \lambda\sin t, \quad t = t_i, \end{cases} \tag{3.3}$$

where γ, λ are positive real numbers, and

$$k_1(u,v) = \frac{u^2 v}{1+u^2}, \qquad k_2(u,v) = \frac{u^3 v(1+u^2+v^2)}{(1+u^2)^2}, \qquad g(u) = 10^{-3}\varphi_2(u).$$

If we let $I_1 = [(2n-1)\pi, 2n\pi]$, $I_2 = [2n\pi, (2n+1)\pi]$, $t_i = (2n + (-1)^i/2)\pi$ (for $i = 1,2$), and $H(t,s) = t - s$, it is easy to see $\mu = \xi = 1$, $\vartheta = 10^{-2}$, $\alpha = 1$, $\beta = 2$, and the conditions (A_1), (A_2), (A_5)-(A_9) hold. For any given $T_0 \geq 0$ we may choose $n \in \mathbb{N}$ sufficiently large so that

$(2n-1)\pi \geq T_0$. Letting $\rho(t)=1$, $c_1=(2n-1/4)\pi$, we have

$$\begin{aligned}
&\frac{1}{H_1^{\alpha+1}(c_1,a_1)}\int_{a_1}^{c_1}\bigl[\tilde{p}(s)\tilde{H}_{11}^{\alpha+1}(s,a_1)-\rho(s)\tilde{q}(s)H_1^{\alpha+1}(s,a_1)\bigr]\,\mathrm{d}s\\
&\quad+\frac{1}{H_1^{\alpha+1}(b_1,c_1)}\int_{c_1}^{b_1}\bigl[\tilde{p}(s)\tilde{H}_{12}^{\alpha+1}(b_1,s)-\rho(s)\tilde{q}(s)H_1^{\alpha+1}(b_1,s)\bigr]\,\mathrm{d}s\\
&\quad=\left(\frac{4}{3\pi}\right)^2\int_0^{3\pi/4}\left[\frac{(2-\sin s)^2}{4}-\frac{2\sin s}{10\sqrt{\cos s}}s^2\right]\mathrm{d}s\\
&\qquad+\left(\frac{4}{\pi}\right)^2\int_0^{\pi/4}\left[\frac{(2-\sin s)^2}{4}-\frac{2\sin s}{10\sqrt{\cos s}}s^2\right]\mathrm{d}s\approx 0.927,\\
&\frac{1}{H_2^{\alpha+1}(c_2,a_2)}\int_{a_2}^{c_2}\bigl[\tilde{p}(s)\tilde{H}_{21}^{\alpha+1}(s,a_1)-\rho(s)\tilde{q}(s)H_2^{\alpha+1}(s,a_2)\bigr]\,\mathrm{d}s\\
&\quad+\frac{1}{H_2^{\alpha+1}(b_2,c_2)}\int_{c_2}^{b_2}\bigl[\tilde{p}(s)\tilde{H}_{22}^{\alpha+1}(b_2,s)-\rho(s)\tilde{q}(s)H_2^{\alpha+1}(b_2,s)\bigr]\,\mathrm{d}s\\
&\quad=\left(\frac{4}{\pi}\right)^2\int_0^{\pi/4}\left[\frac{(2+\sin s)^2}{4}-\frac{2\sin s}{10\sqrt{\cos s}}s^2\right]\mathrm{d}s\\
&\qquad+\left(\frac{4}{3\pi}\right)^2\int_0^{3\pi/4}\left[\frac{(2+\sin s)^2}{4}-\frac{2\sin s}{10\sqrt{\cos s}}s^2\right]\mathrm{d}s\approx 2.492,\\
&\frac{1}{H_1^{\alpha+1}(c_1,a_1)}\sum_{a_1<t_i<c_1}H_1^{\alpha+1}(t_i,a_1)\rho(t_i)\tilde{q}_i\\
&\quad=\left(\frac{4}{3\pi}\right)^2\left(\frac{\pi}{2}\right)^2\frac{2}{10}\sqrt{3\gamma}\sqrt{\lambda}\approx 0.154\sqrt{\gamma\lambda},
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{H_2^{\alpha+1}(b_2,c_2)}\sum_{c_2<t_i<b_2}H_2^{\alpha+1}(b_2,t_i)\rho(t_i)\tilde{q}_i\\
&\quad=\left(\frac{4}{3\pi}\right)^2\left(\frac{\pi}{2}\right)^2\frac{2}{10}\sqrt{\gamma}\sqrt{\lambda}\approx 0.889\sqrt{\gamma\lambda}.
\end{aligned}$$

It follows from Theorem 2.3 that (3.3) is oscillatory if $\gamma\lambda>36.228$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Each of the authors, XZ and WSW contributed to each part of this study equally and read and approved the final version of the manuscript.

Author details
[1]Department of Mathematics, Lingnan Normal University, Zhanjiang, Guangdong 524048, P.R. China. [2]Department of Mathematics, Hechi University, Yizhou, Guangxi 546300, P.R. China.

Acknowledgements
The authors thank the main editor and anonymous referees for their valuable comments and suggestions, leading to improvement of this paper. This work has been supported by the NNSF of China (Grant 11561019), the Innovation and Developing School Project (Grant 2014KZDXM065) and the Science Innovation Project (Grant 2013KJCX0125) of Department of Education of Guangdong Province, the Training Program Project for OYTCU in Guangdong Province (Grant Yq2014116), and the NSFP of Lingnan Normal University (Grant ZL1303).

Received: 11 April 2016 Accepted: 26 June 2016 Published online: 31 August 2016

References

1. Rogovchenko, SP, Rogovchenko, YV: Oscillation theorems for differential equations with a nonlinear damping term. J. Math. Anal. Appl. **279**, 121-134 (2003)
2. Tiryaki, A, Zafer, A: Oscillation of second-order nonlinear differential equations with nonlinear damping. Math. Comput. Model. **39**, 197-208 (2004)
3. Zhao, X, Meng, F: Oscillation of second-order nonlinear ODE with damping. Appl. Math. Comput. **182**, 1861-1871 (2006)
4. Zhao, A, Wang, Y, Yan, J: Oscillation criteria for second-order nonlinear differential equations with nonlinear damping. Comput. Math. Appl. **56**, 542-555 (2008)
5. Huang, Y, Meng, F: Oscillation of second-order nonlinear ODE with damping. Appl. Math. Comput. **199**, 644-652 (2008)
6. El-Sayed, MA: An oscillation criteria for forced second order linear differential equations. Proc. Am. Math. Soc. **118**, 813-817 (1993)
7. Kong, Q: Interval criteria for oscillation of second-order linear ordinary differential equations. J. Math. Anal. Appl. **229**, 258-270 (1999)
8. Tiryaki, A, Zafer, A: Interval oscillation of a general class of second-order nonlinear differential equations with nonlinear damping. Nonlinear Anal. **60**, 49-63 (2005)
9. Shi, W: Interval oscillation criteria for a forced second-order differential equation with nonlinear damping. Math. Comput. Model. **43**, 170-177 (2006)
10. Huang, Y, Meng, F: Oscillation criteria for forced second-order nonlinear differential equations with damping. J. Comput. Appl. Math. **224**, 339-345 (2009)
11. Özbekler, A, Zafer, A: Forced oscillation of super-half-linear impulsive differential equations. Comput. Math. Appl. **54**, 785-792 (2007)
12. Liu, X, Xu, Z: Oscillation criteria for a forced mixed type Emden-Fowler equation with impulses. Appl. Math. Comput. **215**, 283-291 (2009)
13. Liu, X, Xu, Z: Oscillation of a forced super-linear second order differential equation with impulses. Comput. Math. Appl. **53**, 1740-1749 (2007)
14. Özbekler, A, Zafer, A: Oscillation of solutions of second order mixed nonlinear differential equations under impulsive perturbations. Comput. Math. Appl. **61**, 933-940 (2011)
15. Guo, Z, Zhou, X, Ge, W: Interval oscillation criteria for second-order forced impulsive differential equations with mixed nonlinearities. J. Math. Anal. Appl. **381**, 187-201 (2011)
16. Özbekler, A: Oscillation criteria for second-order nonlinear impulsive differential equations. In: Further Progress in Analysis, pp. 545-554. World Scientific, Hackensack (2009)
17. Özbekler, A, Zafer, A: Interval criteria for the forced oscillation of super-half-linear differential equations under impulse effects. Math. Comput. Model. **50**, 59-65 (2009)
18. Li, WT: Interval oscillation theorems for second-order quasi-linear nonhomogeneous differential equations with damping. Appl. Math. Comput. **147**, 753-763 (2004)
19. Zheng, Z, Meng, F: Oscillation criteria for forced second-order quasi-linear differential equations. Math. Comput. Model. **45**, 215-220 (2007)
20. Nasr, AH: Sufficient conditions for the oscillation of forced super-linear second order differential equations with oscillatory potential. Proc. Am. Math. Soc. **126**, 123-125 (1998)
21. Wong, JSW: Oscillation criteria for a forced second-order linear differential equation. J. Math. Anal. Appl. **231**, 235-240 (1999)
22. Li, X, Zhou, X, Wang, WS: Interval oscillation of a general class of second-order nonlinear impulsive differential equations with nonlinear damping. Appl. Math. Comput. **219**, 8803-8810 (2013)
23. Xing, L, Zheng, Z: New oscillation criteria for forced second order half-linear differential equations with damping. Appl. Math. Comput. **198**, 481-486 (2008)
24. Philos, CG: Oscillation theorems for linear differential equations of second order. Arch. Math. **53**, 482-492 (1989)

Submit your manuscript to a SpringerOpen journal and benefit from:

- ► Convenient online submission
- ► Rigorous peer review
- ► Immediate publication on acceptance
- ► Open access: articles freely available online
- ► High visibility within the field
- ► Retaining the copyright to your article

Submit your next manuscript at ► springeropen.com