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Existence of positive solutions for second-order impulsive differential equations with delay and three-point boundary value problem

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Abstract
We study the existence of three positive solutions to a type of three-point boundary value problems for second-order impulsive and delay differential equations, and we obtain the result that there exist at least three nonnegative symmetric positive solutions by means of a generalization of the Leggett-Williams fixed point theorem.

Keywords: symmetric positive solutions; fixed point theorem; impulsive differential equations

1 Introduction

Boundary value problems associated with second-order differential equations emerge in a variety of areas of applied mathematics and physics. In recent years many authors have paid attention to the research of boundary value problems for differential equations because of its potential applications. For example, the authors of [1–3] investigate the existence of nontrivial solution for the three-point boundary value problem, under certain growth conditions on the nonlinearity f, and several sufficient conditions for the existence of nontrivial solution are obtained by using Leray-Schauder nonlinear alternative. Reference [4] discusses the solvability of a three-point nonlinear boundary value problem for a second-order ordinary differential equation using the Leray-Schauder continuous theorem. Reference [5] studies the existence of three nonnegative solutions to a type of three-point boundary value problem for second-order impulsive differential equations, and one obtains the sufficient conditions for existence of three nonnegative solutions by means of the Leggett-Williams fixed point theorem.

On the other side, there is much current attention focusing on questions of symmetric positive solutions for second-order three-point boundary value problems.

In [6, 7], Avery imposed conditions on f to yield at least three symmetric positive solutions applying the Leggett-Williams fixed point theorem. Avery [6, 7] was concerned with

$$\begin{cases} u''(t) + f(u(t)) = 0, & t \in (0,1), \\ u(1) = 0, & u'(0) = 0, \end{cases} \tag{1}$$

and

$$\begin{cases} u''(t) + f(u(t)) = 0, & t \in (0,1), \\ u(1) = u(0) = 0, \end{cases} \tag{2}$$

applying the Leggett-Williams fixed point theorem.

Reference [8] is concerned with the existence of positive solutions to a second-order boundary value problem. By imposing growth conditions on f and using a generalization of the Leggett-Williams fixed point theorem, one proved the existence of at least three symmetric positive solutions.

Motivated by the work mentioned above, in this paper we consider the existence of at least three positive solutions to the following boundary value problem:

$$\begin{cases} u''(t) + f(u(t-\tau)) = 0, & t \in [a,b], t \neq t_k, \\ \Delta u'(t_k) = I_k(u(t_k)), & k = 1,2,\ldots,m, \\ u(a) = \mu u(\eta), \qquad u'(b) = 0, & \mu \in (0,1), \eta \in [a, \frac{a+b}{2}], \\ u(t) = 0, & a - \tau \leq t < a, \end{cases} \tag{3}$$

where $a < t_1 < t_2 < \cdots < t_k < \cdots < t_m < b$, $I_k \in C[P \times P, P]$, $\Delta(u(t_k))$ denotes the jump of $u(t)$ at t_k, *i.e.*

$$\Delta\bigl(u(t_k)\bigr) = u\bigl(t_k^+\bigr) - u\bigl(t_k^-\bigr), \tag{4}$$

where $u(t_k^+)$ and $u(t_k^-)$ represent the right-hand limit and left-hand limit of $u(t)$ at $t = t_k$, respectively. $\Delta u'(t_k)$ has a similar meaning for $u'(t)$.

Let $I' = I \setminus \{t_1, t_2, \ldots, t_k, \ldots, t_n\}$, denote $PC[I, C(I)] = \{u : u$ is a map from I into $C(I)$ such that it is continuous in I'; $u(t_k^+)$, $u(t_k^-)$ exist, and $u(t)$ is right continuous at $t = t_k\}$. $PC^1[I, C(I)] = \{u \in PC[I, C(I)] : u'(t)$ is continuous in I', and $u(t_k^+)$, $u(t_k^-)$ exist for $k = 1, 2, \ldots, m\}$, where $f : \mathbb{R} \to [0, +\infty)$ is continuous. A solution $u \in C^{(2)}[a,b]$ of (3) is both nonnegative and concave on $[a,b]$. We impose growth conditions on f to apply the Leggett-Williams fixed point theorem in finding three positive solutions of (3).

2 Preliminaries

In this section, we present some definitions and lemmas which are essential to prove following main results and we then state the generalization of the Leggett-Williams fixed point theorem.

Definition 1 Let $\mathcal{E}$ be a real Banach space. A nonempty, closed, convex set $\mathcal{P} \subset \mathcal{E}$ is a cone if it satisfies the following two conditions:

(i) If $x \in \mathcal{P}$ and $\lambda \geq 0$, then $\lambda x \in \mathcal{P}$.

(ii) If $x \in \mathcal{P}$ and $-x \in \mathcal{P}$, then $x = 0$. Every cone $\mathcal{P} \subset \mathcal{E}$ induces an ordering in $\mathcal{E}$ given by $x \leq y$ if and only if $y - x \in \mathcal{P}$.

Definition 2 A map ϑ is said to be a nonnegative continuous concave functional on a cone $\mathcal{P}$ in a real Banach space $\mathcal{E}$ if $\vartheta : \mathcal{P} \to [0, +\infty)$ is continuous, and

$$\vartheta\bigl(tx + (1-t)y\bigr) \geq t\vartheta(x) + (1-t)\vartheta(y),$$

for all $x, y \in \mathcal{P}$ and $0 \le t \le 1$. Similarly, we say the map π is a nonnegative continuous convex functional on a cone $\mathcal{P}$ in a real Banach space $\mathcal{E}$ if $\pi : \mathcal{P} \to [0, +\infty)$ is continuous, and

$$\pi\big(tx + (1-t)y\big) \le t\pi(x) + (1-t)\pi(y),$$

for all $x, y \in \mathcal{P}$ and $0 \le t \le 1$.

Let ς, π, θ be nonnegative continuous convex functional on a cone $\mathcal{P}$, and ϑ, ψ be nonnegative continuous concave functional on a cone $\mathcal{P}$. Then for nonnegative real numbers f, v, w, d, and c, we define the following convex sets:

$$\begin{aligned}
&\mathbb{K}(\varsigma, c) = \big\{u \in \mathcal{P} : \varsigma(u) < c\big\},\\
&\mathbb{K}(\varsigma, \vartheta, v, c) = \big\{u \in \mathcal{P} : v \le \vartheta(u), \varsigma(u) < c\big\},\\
&\mathbb{L}(\varsigma, \pi, d, c) = \big\{u \in \mathcal{P} : \pi(u) \le d, \varsigma(u) < c\big\},\\
&\mathbb{K}(\varsigma, \theta, \vartheta, v, w, c) = \big\{u \in \mathcal{P} : v \le \vartheta(u), \theta(u) \le w, \varsigma(u) < c\big\},\\
&\mathbb{L}(\varsigma, \pi, \psi, h, d, c) = \big\{u \in \mathcal{P} : h \le \psi(u), \pi(u) \le d, \varsigma(u) < c\big\}.
\end{aligned}$$

Lemma 1 *Let $f \in L^1[a,b]$, the three-point boundary value problem* (3) *has a unique solution*

$$\begin{cases} u(t) = \int_a^b G(t,s) f(u(s-\tau))\,ds - \sum_{i=1}^m G(t,t_i) I_i(u(t_i)), & a \le t \le b,\\ u(t) = 0, \quad a - \tau \le t < a, \end{cases} \tag{5}$$

where the Green's function $G(t,s)$ is

$$G(t,s) = \frac{1}{1-\mu}\begin{cases} s - a, & a \le s < \eta \le b, a \le s < t \le b,\\ \mu s + (1-\mu)t - a, & a \le t \le s \le \eta \le b,\\ \mu\eta + (1-\mu)s - a, & a \le \eta \le s \le t \le b,\\ \mu\eta + (1-\mu)t - a, & a \le \eta \le s \le b, a \le t \le s \le b. \end{cases} \tag{6}$$

Proof Assume $u(t)$ is a solution of the three-point boundary value problem (3), then we have

$$u''(t) = -f\big(u(t-\tau)\big), \tag{7}$$

$$u'(t) = u'(a) - \int_a^t f\big(u(s-\tau)\big)\,ds, \tag{8}$$

$$u(t) = u(a) + u'(a)(t-a) - \int_a^t \int_a^\xi f\big(u(s-\tau)\big)\,ds\,d\xi, \tag{9}$$

we obtain

$$u(t) = u(a) + u'(a)(t-a) - \int_a^t (t-s) f\big(u(s-\tau)\big)\,ds, \tag{10}$$

$$u(t) = u(t_1) + u'(t_1)(t-t_1) - \int_{t_1}^t (t-s) f\big(u(s-\tau)\big)\,ds, \tag{11}$$

since

$$u(t_1^-) = u(a) + u'(a)(t_1 - a) - \int_a^{t_1} (t_1 - s)f\big(u(s-\tau)\big)\,ds, \tag{12}$$

$$u'(t_1^-) = u'(a) - \int_a^{t_1} f\big(u(s-\tau)\big)\,ds, \tag{13}$$

we have

$$u(t_1) = u(a) + u'(a)(t_1 - a) - \int_a^{t_1} (t_1 - s)f\big(u(s-\tau)\big)\,ds, \tag{14}$$

$$u'(t_1) = u'(a) - \int_a^{t_1} f\big(u(s-\tau)\big)\,ds, \tag{15}$$

for $\forall t \in (t_1, t_2]$, we obtain

$$u(t) = u(a) + u'(a)(t-a) + I_1\big(u(t_1)\big)(t - t_1) - \int_a^{t} (t-s)f\big(u(s-\tau)\big), \tag{16}$$

for $\forall t \in (t_k, t_{k+1}]$, we have

$$\begin{aligned} u(t) &= u(a) + u'(a)(t-a) + \sum_{i=1}^{k} (t - t_i)I_i\big(u(t_i)\big) \\ &\quad - \int_a^{t} (t-s)f\big(u(s-\tau)\big)\,ds, \end{aligned} \tag{17}$$

with the three-point boundary value problem (3), then we obtain

$$\begin{aligned} u'(a) &= \int_a^{b} f\big(u(s-\tau)\big)\,ds - \sum_{i=1}^{m} I_i\big(u(t_i)\big), \\ u(a) &= \frac{\mu}{1-\mu}\Bigg[\int_a^{b} f\big(u(s-\tau)\big)\,ds - \sum_{i=1}^{m} I_i\big(u(t_i)\big)(\eta - a) \\ &\quad + \sum_{t_i<\eta} (\eta - t_i)I_i\big(u(t_i)\big) - \int_a^{\eta} (\eta - s)f\big(u(s-\tau)\big)\,ds\Bigg], \end{aligned} \tag{18}$$

this shows that

$$\begin{cases} u(t) = \int_a^b G(t,s)f(u(s-\tau))\,ds - \sum_{i=1}^m G(t,t_i)I_i(u(t_i)), & a \le t \le b, \\ u(t) = 0, \quad a - \tau \le t < a, \end{cases} \tag{19}$$

where

$$G(t,s) = \frac{1}{1-\mu}\begin{cases} s - a, & a \le s < \eta \le b, a \le s < t \le b, \\ \mu s + (1-\mu)t - a, & a \le t \le s \le \eta \le b, \\ \mu\eta + (1-\mu)s - a, & a \le \eta \le s \le t \le b, \\ \mu\eta + (1-\mu)t - a, & a \le \eta \le s \le b, a \le t \le s \le b. \end{cases} \tag{20}$$

This completes the proof. □

Lemma 2 *The function G satisfies*

$$\frac{\min\{t-a,s-a,\eta-a\}}{1-\mu} \le G(t,s) \le \frac{s-a}{1-\mu}.$$

Proof According to the Green's function $G(t,s)$, we have:
when $s \le \eta$, $s \le t$,

$$\frac{s-a}{1-\mu} \le G(t,s) = \frac{s-a}{1-\mu},$$

when $t \le s \le \eta$,

$$\frac{t-a}{1-\mu} \le G(t,s) = \frac{\mu s + (1-\mu)t - a}{1-\mu} \le \frac{s-a}{1-\mu},$$

when $\eta \le s \le t$,

$$\frac{\eta-a}{1-\mu} \le G(t,s) = \frac{\mu\eta + (1-\mu)s - a}{1-\mu} \le \frac{s-a}{1-\mu},$$

when $\eta < s$, $t < s$,

$$\frac{\eta-a}{1-\mu} \le G(t,s) = \frac{\mu\eta + (1-\mu)s - a}{1-\mu} \le \frac{s-a}{1-\mu},$$

then we obtain

$$\frac{\min\{t-a,s-a,\eta-a\}}{1-\mu} \le G(t,s) \le \frac{s-a}{1-\mu}.$$ □

Theorem 1 [6] *Let $\mathcal{P}$ be a cone in a real Banach space $\mathcal{E}$. Assume that there exist positive numbers c and M, nonnegative continuous concave functionals ϑ and ψ on $\mathcal{P}$, and nonnegative continuous convex functionals ς, π, and θ on $\mathcal{P}$ with*

$$\vartheta(u) \le \pi(u), \qquad \|u\| \le M\varsigma(u), \tag{21}$$

for all $u \in \overline{\mathcal{P}(\varsigma,c)}$. Suppose that $F : \overline{\mathcal{P}(\varsigma,c)} \to \overline{\mathcal{P}(\varsigma,c)}$ is a completely continuous operator and that there exist nonnegative numbers f, d, v, w, with $0 < d < v$ such that

(B1) *$u \in \mathbb{K}(\varsigma,\theta,\vartheta,v,w,c) : \vartheta(u) > v \neq \emptyset$ and $\vartheta(Fu) > v$ for $u \in \mathbb{K}(\varsigma,\theta,\vartheta,v,w,c)$;*
(B2) *$u \in \mathbb{L}(\varsigma,\pi,\psi,f,d,c) : \pi(u) < d \neq \emptyset$ and $\pi(Fu) < d$ for $u \in \mathbb{L}(\varsigma,\pi,\psi,f,d,c)$;*
(B3) *$\vartheta(Fu) > v$ for $u \in \mathbb{K}(\varsigma,\vartheta,v,c)$ with $\theta(Fu) > w$;*
(B4) *$\pi(Fu) < d$ for $u \in \mathbb{L}(\varsigma,\pi,d,c)$ with $\psi(Fu) < f$.*

Then F has at least three fixed points $u_1, u_2, u_3 \in \overline{\mathcal{P}(\varsigma,c)}$ such that

$$\pi(u_1) < d, \qquad v < \vartheta(u_2) \quad \text{and} \quad d < \pi(u_3) \quad \text{with } \vartheta(u_3) < v.$$

3 Main results

In this section, we utilize the growth conditions on f in order to apply the generalization of the Leggett-Williams fixed point theorem in a study of the existence of at least three symmetric positive solutions of (3). Now we give some properties of the Green's function $G(t,s)$ which include the following.

For any $b \ge t > \eta \ge a$, we have

$$\int_a^b G(t,s)\,ds = \frac{1}{2(1-\mu)}\bigl[a^2 - t^2 - \mu\eta^2 + \mu t^2 + 2(\mu\eta b - \mu t b - ab + tb)\bigr],$$

and we have

$$\begin{aligned}
&\int_a^{\frac{1}{r}} G\left(\frac{a+b}{2},s\right) ds = \frac{1}{2(1-\mu)}\left(\frac{1+a^2r^2-2ar}{r^2}\right), \quad r > \frac{1}{\eta} > \frac{2}{a+b},\\
&\int_{a+b-\frac{1}{r}}^{b} G\left(\frac{a+b}{2},s\right) ds\\
&\quad = \frac{1}{2(1-\mu)}\left[\frac{(1-ar)(2(\mu\eta - a) + (1-\mu)(a+b))}{r}\right], \quad r > \frac{1}{\eta} > \frac{2}{a+b},\\
&\int_{\frac{1}{r}}^{\frac{a+b}{2}} G\left(\frac{a+b}{2},s\right) ds = \frac{1}{2(1-\mu)}\left[\frac{\eta^2r^2 - 1 + 2ar + \mu\eta(a+b-2\eta)r^2}{r^2}\right.\\
&\qquad\qquad \left. + \frac{(1-\mu)(a+b-2\eta^2) - 2a(a+b)}{2}\right], \quad r > \frac{1}{\eta} > \frac{2}{a+b},\\
&\int_{\frac{1}{r}}^{\eta} G\left(\frac{a+b}{2},s\right) ds = \frac{1}{2(1-\mu)}\frac{\eta^2r^2 - 2a\eta r^2 + 2ar - 1}{r^2}, \quad r > \frac{1}{\eta} > \frac{2}{a+b},\\
&\int_{\eta}^{\frac{a+b}{2}} G\left(\frac{a+b}{2},s\right) ds\\
&\quad = \frac{1}{2(1-\mu)}\left[\mu\eta(a+b-2\eta) + (1-\mu)\frac{a^2+b^2+2ab-4\eta^2}{4} - a(a+b-2\eta)\right],\\
&\int_{\frac{a+b}{2}}^{a+b-\frac{1}{r}} G\left(\frac{a+b}{2},s\right) ds\\
&\quad = \frac{1}{2(1-\mu)}\frac{(2+ar+br)[(1-\mu)(a+b) - a + 2\mu\eta]}{2r}, \quad r > \frac{1}{\eta} > \frac{2}{a+b},\\
&\int_{\iota_1}^{\iota_2} G(t,s)\,ds + \int_{a+b-\iota_2}^{a+b-\iota_1} G(t,s)\,ds\\
&\quad = \frac{1}{2(1-\mu)}\bigl[\iota_2^2 - \iota_1^2 + (\iota_2 - \iota_1)\bigl(2(1-\mu)\iota_1 + 2\mu\eta - 4a\bigr)\bigr], \qquad a < \iota_1 < \iota_2 < \eta < \frac{a+b}{2},\\
&\min_{r\in[a,b]} \frac{G(\iota_1,r)}{G(\iota_2,r)} = 1, \quad a < \iota_1 < \iota_2 < \eta < \frac{a+b}{2},\\
&\max_{r\in[a,b]} \frac{G(\frac{a+b}{2},r)}{G(t,r)} = 1, \quad a < \eta < t \le \frac{a+b}{2}.
\end{aligned}$$

Denote $\mathcal{E} = C[a,b]$ endowed with the maximum norm, $\|u\| = \max_{t\in[a,b]} |u(t)|$. Then for $a < \iota_3 \le \frac{a+b}{2}$, we define the cone $\mathcal{P} \subset \mathcal{E}$ by

$$\mathcal{P} = \Bigl\{u \in \mathcal{E} : u \text{ is concave, symmetric, nonnegative valued, } \min_{t\in[\iota_3, a+b-\iota_3]} u(t) \ge 2\iota_3\|u\|\Bigr\}.$$

Define the nonnegative, continuous concave functionals ϑ, ψ and nonnegative, continuous convex functionals π, θ, ς on the cone $\mathcal{P}$ by

$$\vartheta(u) = \min_{t\in[\iota_1,\iota_2]\cup[a+b-\iota_2,a+b-\iota_1]} u(t) = u(\iota_1),$$
$$\pi(u) = \max_{t\in[\frac{1}{r},a+b-\frac{1}{r}]} u(t) = u\left(\frac{a+b}{2}\right),$$
$$\varsigma(u) = \max_{t\in[a,\iota_3]\cup[a+b-\iota_3,b]} u(t) = u(\iota_3),$$
$$\theta(u) = \max_{t\in[\iota_1,\iota_2]\cup[a+b-\iota_2,a+b-\iota_1]} u(t) = u(\iota_2),$$
$$\psi(u) = \min_{t\in[\frac{1}{r},a+b-\frac{1}{r}]} u(t) = u\left(\frac{1}{r}\right),$$

where ι_1, ι_2, and r are nonnegative numbers such that

$$a < \iota_1 < \iota_2 < \eta \le \frac{a+b}{2} \quad \text{and} \quad \frac{1}{r} \le \iota_2.$$

We see that, for all $u \in \mathcal{P}$,

$$\vartheta(u) = u(\iota_1) \le u\left(\frac{a+b}{2}\right) = \pi(u), \tag{22}$$
$$\|u\| = u\left(\frac{a+b}{2}\right) \le \frac{a+b}{2\iota_3}u(\iota_3) = \frac{a+b}{2\iota_3}\varsigma(u), \tag{23}$$

and also that $u \in \mathcal{P}$ is a solution of (3) if and only if

$$u(t) = \int_a^b G(t,s)f\big(u(s-\tau)\big)\,ds - \sum_{i=1}^m G(t,t_i)I_i\big(u(t_i)\big), \quad \text{for } t \in [a,b]. \tag{24}$$

We will show our main result of the paper.

In this paper, we assume that (A1)-(A3) hold:

(A1) $f : [a,b] \times [0,+\infty] \to [0,+\infty]$ is continuous;

(A2) $I_i : [0,+\infty] \to R$ is continuous;

(A3) there exist l_i, L_i, such that $l_i \le I_i(u(t)) \le L_i$ for any $t \in [a,b]$.

For convenience, let

$$\Theta = a^2 - \iota_3^2 - \mu\eta^2 + \mu\iota_3^2 + 2\mu\eta b - 2\mu b\iota_3 - 2ab + 2b\iota_3,$$
$$\Lambda = \iota_2^2 - \iota_1^2 + (\iota_2 - \iota_1)\big(2(1-\mu)\iota_1 + 2\mu\eta - 4a\big),$$
$$\Psi = \eta^2 - \frac{1}{r^2} - 2a\left(\eta - \frac{1}{r}\right) + \mu\eta(a+b-2\eta) + (1-\mu)\left[\left(\frac{a+b}{2}\right)^2 - \eta^2\right] - a(a+b-2\eta),$$
$$\Delta = \frac{1}{1-\mu}\left[\frac{1}{r^2} + a^2 - \frac{2a}{r}\right],$$
$$\Xi = 2\left(c - \frac{m\mu\min\{t-a, s-a, \eta-a\}}{1-\mu}\sum_{i=1}^m l_i\right)(1-\mu).$$

Theorem 2 *Assume that there exist nonnegative numbers* v, w, *and* c, *such that* $0 < v < w \leq \frac{c\iota_1}{\iota_2}$, *and suppose that* f *satisfies the following growth conditions*:

$$\text{(C1)}\quad f(\omega) \leq 2\left(v - \frac{m\mu \min\{t-a, s-a, \eta-a\}}{1-\mu}\sum_{i=1}^{m} l_i - \frac{\Xi}{\Theta}\Delta\right)(1-\mu)\frac{1}{\Psi}, \quad \frac{2v}{r} < \omega < v;$$

$$\text{(C2)}\quad f(\omega) > 2\left(\omega + \frac{mwb\iota_2}{(1-\mu)\iota_1}\sum_{i=1}^{m} L_i\right)(1-\mu)\frac{1}{\Lambda}, \quad w < \omega < \frac{w\iota_2}{\iota_1};$$

$$\text{(C3)}\quad f(\omega) \leq 2\left(c - \frac{2m\mu v}{(1-\mu)r}\sum_{i=1}^{m} l_i\right)(1-\mu)\frac{1}{\Theta}, \quad a < \omega < \frac{c(a+b)}{2\iota_3}.$$

Then the boundary value problem (3) *has three symmetric positive solutions* u_1, u_2, u_3 *satisfying*

$$\max_{t\in[a,\iota_3]\cup[a+b-\iota_3,b]} u_i(t) \leq c, \quad i = 1,2,3,$$

$$\min_{t\in[\iota_1,\iota_2]\cup[a+b-\iota_2,a+b-\iota_1]} \iota_1(t) > w, \qquad \max_{t\in[\frac{1}{r},a+b-\frac{1}{r}]} \iota_2(t) < v,$$

$$\min_{t\in[\iota_1,\iota_2]\cup[a+b-\iota_2,a+b-\iota_1]} \iota_3(t) < w, \quad \text{with} \max_{t\in[\frac{1}{r},a+b-\frac{1}{r}]} \iota_3(t) > v.$$

Proof Let us define the completely continuous operator F by

$$(Fu)(t) = \begin{cases} \int_a^b G(t,s)f(u(s-\tau))\,ds - \sum_{i=1}^m G(t,t_i)I_i(u(t_i)), & a \leq t \leq b, \\ 0, & a-\tau \leq t < a. \end{cases}$$

We will seek fixed points of F in the cone. We note that, if $u \in \mathcal{P}$, and from some properties of $G(t,s)$, then $Fu(t) \geq 0$, and $(Fu)''(t) = -f(u(s-\tau)) \leq 0$, $a < t < b$, $Fu(\iota_3) \geq \frac{a+b}{2}\iota_3 Fu(\frac{a+b}{2})$, and $Fu(t) = Fu(b+a-t)$, $a < t < \frac{a+b}{2}$, and this implies that $Fu \in \mathcal{P}$, and so $F : \mathcal{P} \to \mathcal{P}$.

Now, for all $u \in \mathcal{P}$, from (22), we get $\vartheta(u) \leq \pi(u)$, and from (23), we also get $\|u\| \leq \frac{a+b}{2\iota_3}\varsigma(u)$.

If, $u \in \overline{\mathcal{P}(\varsigma,c)}$, then $\|u\| \leq \frac{a+b}{2\iota_3}\varsigma(u) < \frac{a+b}{2\iota_3}c$ and from assumption (C3), we get

$$\begin{aligned} \varsigma(Fu) &= \max_{t\in[a,\iota_3]\cup[a+b-\iota_3,b]}\left[\int_a^b G(t,s)f\big(u(s-\tau)\big)\,ds - \sum_{i=1}^m G(t,t_i)I_i\big(u(t_i)\big)\right] \\ &= \int_a^b G(\iota_3,s)f\big(u(s-\tau)\big)\,ds - \min\sum_{i=1}^m G(t,t_i)I_i\big(u(t_i)\big) \\ &\leq c. \end{aligned}$$

Thus, $F : \overline{\mathcal{P}(\varsigma,c)} \to \overline{\mathcal{P}(\varsigma,c)}$, and it is immediate that

$$\left\{u \in \mathbb{K}\left(\varsigma,\theta,\vartheta,w,\frac{w\iota_2}{\iota_1},c\right) : \vartheta > w\right\} \neq \emptyset \quad \text{and}$$

$$\left\{u \in \mathbb{L}\left(\varsigma,\pi,\psi,\frac{2v}{(a+b)c},v,c\right) : \pi(u) < v\right\} \neq \emptyset.$$

We will show the remaining conditions of Theorem 1.

(1) If $u \in \mathbb{L}(\varsigma,\pi,\nu,c)$ with $\psi(Fu) < \frac{2\nu}{(a+b)c}$, then $\pi(Fu) < \nu$. We have

$$\begin{aligned}
\pi(Fu) &= \max_{t\in[\frac{1}{r},a+b-\frac{1}{r}]}\left[\int_a^b G(t,s)f\big(u(s-\tau)\big)\,ds - \sum_{i=1}^m G(t,t_i)I_i\big(u(t_i)\big)\right] \\
&= \int_a^b G\left(\frac{a+b}{2},s\right)f\big(u(s-\tau)\big)\,ds - \min\sum_{i=1}^m G(t,t_i)I_i\big(u(t_i)\big) \\
&= \int_a^b \frac{G(\frac{a+b}{2},s)}{G(\frac{1}{r},s)}G\left(\frac{1}{r},s\right)f\big(u(s-\tau)\big)\,ds - \min\sum_{i=1}^m G(t,t_i)I_i\big(u(t_i)\big) \\
&\leq \int_a^b G\left(\frac{1}{r},s\right)f\big(u(s-\tau)\big)\,ds - \min\sum_{i=1}^m G(t,t_i)I_i\big(u(t_i)\big) = \psi(Fu) < \nu.
\end{aligned}$$

(2) If $u \in \mathbb{L}(\vartheta,\pi,\psi,\frac{2\nu}{c},\nu,c)$, then $\pi(Fu) < \nu$. We have

$$\begin{aligned}
\pi(Fu) &= \max_{t\in[\frac{1}{r},a+b-\frac{1}{r}]}\left[\int_a^b G(t,s)f\big(u(s-\tau)\big)\,ds - \sum_{i=1}^m G(t,t_i)I_i\big(u(t_i)\big)\right] \\
&= \int_a^b G\left(\frac{a+b}{2},s\right)f\big(u(s-\tau)\big)\,ds - \min\sum_{i=1}^m G(t,t_i)I_i\big(u(t_i)\big) \\
&= 2\int_a^{\frac{1}{r}} G\left(\frac{a+b}{2},s\right)f\big(u(s-\tau)\big)\,ds + 2\int_{\frac{1}{r}}^{\frac{a+b}{2}} G\left(\frac{a+b}{2},s\right)f\big(u(s-\tau)\big)\,ds \\
&\quad - \min\sum_{i=1}^m G(t,t_i)I_i\big(u(t_i)\big) \leq \nu.
\end{aligned}$$

(3) If $u \in \mathbb{L}(\varsigma,\vartheta,w,c)$ with $\theta(Fu) > \frac{w\iota_2}{\iota_1}$, then $\vartheta(Fu) > w$. We have

$$\begin{aligned}
\vartheta(Fu) &= \min_{t\in[\iota_1,\iota_2]\cup[a+b-\iota_2,a+b-\iota_1]}\left[\int_a^b G(t,s)f\big(u(s-\tau)\big)\,ds - \sum_{i=1}^m G(t,t_i)I_i\big(u(t_i)\big)\right] \\
&= \int_a^b G(\iota_1,s)f\big(u(s-\tau)\big)\,ds - \max\sum_{i=1}^m G(t,t_i)I_i\big(u(t_i)\big) \\
&= \int_a^b \frac{G(\iota_1,s)}{G(\iota_2,s)}G(\iota_2,s)f\big(u(s-\tau)\big)\,ds - \max\sum_{i=1}^m G(t,t_i)I_i\big(u(t_i)\big) \\
&\geq \int_a^b G(\iota_2,s)f\big(u(s-\tau)\big)\,ds - \max\sum_{i=1}^m G(t,t_i)I_i\big(u(t_i)\big) = \theta(Fu) > w.
\end{aligned}$$

(4) If $u \in \mathbb{L}(\varsigma,\theta,\vartheta,w,\frac{w\iota_2}{\iota_1},c)$, then $\vartheta(Fu) > w$. We have

$$\begin{aligned}
\vartheta(Fu) &= \min_{t\in[\iota_1,\iota_2]\cup[a+b-\iota_2,a+b-\iota_1]}\left[\int_a^b G(t,s)f\big(u(s-\tau)\big)\,ds - \max\sum_{i=1}^m G(t,t_i)I_i\big(u(t_i)\big)\right] \\
&= \int_a^b G(\iota_1,s)f\big(u(s-\tau)\big)\,ds - \max\sum_{i=1}^m G(t,t_i)I_i\big(u(t_i)\big)
\end{aligned}$$

$$
\begin{aligned}
&> \int_{\iota_1}^{\iota_2} G(\iota_1, s) f\big(u(s-\tau)\big)\, ds + \int_{a+b-\iota_2}^{a+b-\iota_1} G(\iota_1, s) f\big(u(s-\tau)\big)\, ds \\
&\quad - \max \sum_{i=1}^{m} G(t, t_i) I_i\big(u(t_i)\big) \geq w.
\end{aligned}
$$

Since all the conditions of the generalized Leggett-Williams fixed point theorem are satisfied, (3) has three positive solutions $u_1, u_2, u_3 \in \overline{\mathcal{P}(\varsigma, c)}$ such that

$$\vartheta(u_1) > w, \qquad \pi(u_2) < v, \qquad \vartheta(u_3) < w \quad \text{with } \pi(u_3) > v.$$

This completes the proof. □

Remark 1 For the case $t \geq \eta$, the method in Theorem 2 is similar, it is unnecessary to go into details here.

Remark 2 For the case $u(t) \neq 0$, $a - \tau \leq t < a$, it is clear to see, and it is unnecessary to go into details here.

4 Example

In this section, we present a simple example to explain our results.

Example 1 Consider the following second-order impulsive differential equations with delay and three-point boundary value problem,

$$
\begin{cases}
u''(t) + u(t-\tau) = 0, \quad t \in [0,1], t \neq \frac{1}{2}, \\
\Delta u'(\frac{1}{2}) = \frac{1}{3} u(\frac{1}{2}), \\
u(0) = \frac{1}{4} u(\frac{1}{3}), \qquad u'(1) = 0,
\end{cases}
\tag{25}
$$

where $f(u(t-\tau)) = u(t-\tau)$, $[a,b] = [0,1]$, $t_k = \frac{1}{2}$, $\mu = \frac{1}{4}$, $\eta = \frac{1}{3}$, then $u(t)$, $u(t-\tau)$ satisfy the assumptions (C1)-(C3).

We choose $v = \frac{1}{3}$, $w = \frac{1}{2}$, $c = 2$, so we obtain

$$
\begin{aligned}
&\Theta = -\frac{3}{4}\iota_3^2 + \frac{3}{2}\iota_3 + \frac{5}{36}, \\
&\Lambda = \iota_2^2 - \frac{5}{2}\iota_1^2 + \frac{3}{2}\iota_1\iota_2 + \frac{1}{6}(\iota_2 - \iota_1), \\
&\Psi = \frac{25}{96} - \frac{1}{r^2}, \\
&\Delta = \frac{4}{3r^2}, \\
&\Xi = 3.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&\text{(C1)} \quad f(\omega) \leq 2\left(\frac{1}{3} - \frac{\Xi}{\Theta}\Delta\right)\frac{1}{\Psi}, \quad \frac{2}{3r} < \omega < \frac{1}{3}; \\
&\text{(C2)} \quad f(\omega) > 2\left(\frac{1}{2} + \frac{2\iota_2}{9\iota_1}\right)\frac{1}{4\Lambda}, \quad \frac{1}{2} < \omega < \frac{\iota_2}{2\iota_1}; \\
&\text{(C3)} \quad f(\omega) \leq \frac{3}{\Theta}, \quad 0 < \omega < \frac{1}{\iota_3}.
\end{aligned}
$$

Let $\omega = u(t), f = u(t)$, obviously, we can find

$$\frac{1}{3} \leq 2\left(\frac{1}{3} - \frac{\Xi}{\Theta}\Delta\right)\frac{1}{\Psi};$$
$$\frac{1}{2} > 2\left(\frac{1}{2} + \frac{2\iota_2}{9\iota_1}\right)\frac{1}{4\Lambda};$$
$$\frac{1}{\iota_3} \leq \frac{3}{\Theta}, \quad 0 < \omega < \frac{1}{\iota_3}.$$

By Theorem 2, we know the BVP (25) has at least three symmetric positive solutions.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
WX wrote the first draft of a paper and did revising and editing; WD was in charge of the choice of the topic, the method, revising, and editing. Both authors read and approved the final manuscript.

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