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Eigenvalue problem for fractional differential operator containing left and right fractional derivatives

Hua Jin and Wenbin Liu*

*Correspondence:
liuwenbin-xz@163.com
Department of Mathematics, China University of Mining and Technology, Xuzhou, 221116, P.R. China

Abstract
In this paper, we consider the eigenvalue problem for fractional differential operator containing left and right fractional derivatives with Dirichlet boundary value conditions. By critical point theory, we see that there exist an eigenvalue sequence which is increasing, tending to infinity, and an eigenfunction sequence which is a Hilbert basis of a fractional Sobolev space.

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1 Introduction

The study of the eigenvalues of a linear operator is a classical topic and many functional analytic tools of a general nature may be used to deal with it. The knowledge of eigenvalues of an ordinary or partial linear differential operator with some boundary value conditions plays an important role in the study of the existence of solutions of nonlinear perturbations of this operator. For the classical second-order ordinary differential equation of the form

$$-u'' = \lambda u, \tag{1}$$

it is well known that, under certain boundary value conditions, nontrivial solutions exist only for particular values λ^i, $i = 1, 2, \ldots$. The constant λ^i are called eigenvalues and the corresponding nontrivial solutions u^i are called eigenfunctions. By using critical point theory, the classical results as regards the eigenvalue problem (1) with Dirichlet boundary value conditions have been obtained on $W_0^{1,2}([0,T],\mathbb{R})$.

Recently, the subject of fractional calculus gained a considerable popularity and importance. During the last three decades or so, due to its demonstrated applications in numerous fields of science and engineering, such as viscoelasticity, neurons, electrochemistry, control, *etc.* [1–6], more attention was paid to the fractional differential equations. Many different analytic tools including the variational method are used to study the existence and multiplicity of solutions for fractional boundary value problems (BVPs for short); see [7–19]. Meanwhile, different from the integer order differential BVPs, most of BVPs for the fractional differential operators do not have a variational structure. For instance, there is no energy functional for the operator D^α ($\alpha \notin \mathbb{N}$). So the variational method cannot be applied here.

In recent years, by using critical point theory, some authors considered the existence of (weak) solutions for some fractional BVPs with variational structure, such as

$$\begin{cases} {}_tD_T^\alpha {}_0D_t^\alpha u = f(t,u), & t \in (0,T), \\ u(0) = u(T) = 0, \end{cases} \tag{2}$$

where $0 < \alpha \leq 1$, ${}_0D_t^\alpha$, and ${}_tD_T^\alpha$ are the left and right Riemann-Liouville fractional derivative operators of order α, respectively. The reader may refer to [12, 14, 15, 17, 18] and the references therein. The equations containing both left and right fractional differential operators have received renewed attention due to their applications, such as the physical phenomena exhibiting anomalous diffusion [15] and the height loss over time of the granular material contained in a silo [20]. Note that, when $\alpha = 1$, the fractional differential operator ${}_tD_T^\alpha {}_0D_t^\alpha$ reduces to the standard second-order differential operator $-D^2$.

Motivated by the above research, in this paper, we consider the following eigenvalue problem:

$$\begin{cases} {}_tD_T^\alpha {}_0D_t^\alpha u = \lambda u, & t \in (0,T), \\ u(0) = u(T) = 0. \end{cases} \tag{3}$$

Naturally, we want to use the fractional Sobolev space $E_0^{\alpha,p}$ in [15] to study our problem. Similar to the definition of $W_0^{1,2}([0,T],\mathbb{R})$, in [15], $E_0^{\alpha,p}$ is defined by the closure of $C_0^\infty([0,T],\mathbb{R})$ with respect to the norm

$$\|u\|_{\alpha,p} = \left(\int_0^T \left|u(t)\right|^p dt + \int_0^T \left|{}_0D_t^\alpha u(t)\right|^p dt \right)^{\frac{1}{p}},$$

where ${}_0D_t^\alpha u$ denotes the classical fractional derivative. We should point out that the space $E_0^{\alpha,p}$ they defined is not rigorous. In fact, taking a Cauchy sequence $\{u_n\} \subset C_0^\infty([0,T],\mathbb{R})$ with respect to the norm $\|\cdot\|_{\alpha,p}$, one has

$$u_n \to u_0, \qquad {}_0D_t^\alpha u_n \to v_0 \quad \text{in } L^p\bigl([0,T],\mathbb{R}\bigr).$$

Unfortunately ${}_0D_t^\alpha u_0$ may not exist. Even if ${}_0D_t^\alpha u_0$ exists, ${}_0D_t^\alpha u_0$ may not be equal to v_0. Moreover, for $u \in E_0^{\alpha,p}$, we think it is not clear that u satisfies $u(0) = u(T) = 0$. For studying the problem (3) by critical point theory, we first should look for an appropriate space. In [21, 22], based on the weak fractional derivative, they defined the fractional Sobolev space $W_{a+}^{\alpha,p}$, which is denoted by $E^{\alpha,p}$ in our paper. Meanwhile, it is not enough for our problem because of the Dirichlet boundary problems. Inspired by [15, 21, 22], we discuss the fractional Sobolev space still written as $E_0^{\alpha,p}$, and then we study the eigenvalue problem (3).

The paper is divided into four parts, as follows. First of all, we present the preliminaries of the definitions of Riemann-Liouville fractional integral and derivative (Section 2). Second, based on the fractional Sobolev space $E^{\alpha,p}$, we construct the fractional Sobolev space $E_0^{\alpha,p}$ and discuss some properties of $E_0^{\alpha,p}$. Moreover, we obtain the compactness of the embedding from $E_0^{\alpha,p}$ into $C([0,T],\mathbb{R})$ under some conditions (Section 3). Third, we study the regularity of weak solutions of a fractional BVP (Section 4). Finally, we consider the eigenvalue problem (3) in $E_0^{\alpha,2}$ (Section 5).

2 Preliminaries

For the convenience of the reader, the definitions of fractional integral and fractional derivative and $E^{\alpha,p}$ are presented below [21–24].

Definition 2.1 (see [23]) For $\gamma > 0$, the left and right Riemann-Liouville fractional integrals of order γ of a function $u : [a,b] \to \mathbb{R}$ are given by

$$
\begin{aligned}
{}_aI_t^\gamma u(t) &= \frac{1}{\Gamma(\gamma)} \int_a^t (t-s)^{\gamma-1} u(s)\,ds, \\
{}_tI_b^\gamma u(t) &= \frac{1}{\Gamma(\gamma)} \int_t^b (s-t)^{\gamma-1} u(s)\,ds,
\end{aligned}
$$

provided that the right-hand side integrals are pointwise defined on $[a,b]$, where $\Gamma(\cdot)$ is the gamma function.

Definition 2.2 (see [23]) For $n-1 \le \gamma < n$ ($n \in \mathbb{N}$), the left and right Riemann-Liouville fractional derivatives of order γ of a function $u : [a,b] \to \mathbb{R}$ are given by

$$
\begin{aligned}
{}_aD_t^\gamma u(t) &= \frac{d^n}{dt^n} {}_aI_t^{n-\gamma} u(t), \\
{}_tD_b^\gamma u(t) &= (-1)^n \frac{d^n}{dt^n} {}_tI_b^{n-\gamma} u(t).
\end{aligned}
$$

Remark 2.3 When $\gamma = 1$, we can see from Definitions 2.1 and 2.2 that

$$
{}_aD_t^1 u(t) = u'(t), \qquad {}_tD_b^1 u(t) = -u'(t),
$$

where u' is the usual first-order derivative of u.

Definition 2.4 (see [22]) Let $0 < \alpha \le 1$, $u, v \in L^1([0,T],\mathbb{R})$, if

$$
\int_0^T \varphi(t) v(t)\,dt = \int_0^T u(t) \big({}_tD_T^\alpha \varphi\big)(t)\,dt, \quad \forall \varphi \in C_0^\infty\big([0,T],\mathbb{R}\big),
$$

then v is called the weak left fractional derivative of order α of u, and it is denoted by ${}_0\dot{D}_t^\alpha u$.

Definition 2.5 Let $0 < \alpha \le 1$, $u, v \in L^1([0,T],\mathbb{R})$, if

$$
\int_0^T \varphi(t) v(t)\,dt = \int_0^T u(t) \big({}_0D_t^\alpha \varphi\big)(t)\,dt, \quad \forall \varphi \in C_0^\infty\big([0,T],\mathbb{R}\big),
$$

then v is called the weak right fractional derivative of order α of u, and it is denoted by ${}_t\dot{D}_T^\alpha u$.

Remark 2.6 When $\alpha = 1$, we can conclude from Remark 2.3 that

$$_0\dot{D}_t^1 u(t) = \dot{u}(t), \qquad _t\dot{D}_T^1 u(t) = -\dot{u}(t),$$

where $\dot{u}$ is the usual first-order weak derivative of u.

Definition 2.7 (see [22]) For $0 < \alpha \leq 1$, $1 \leq p < \infty$, the space $_LE^{\alpha,p}$ is defined by

$$_LE^{\alpha,p} = \{u \in L^p([0,T],\mathbb{R}) | {}_0\dot{D}_t^\alpha u \in L^p([0,T],\mathbb{R})\}$$

with the norm

$$\|u\|_{_LE^{\alpha,p}} = \left(\|u\|_{L^p}^p + \|{}_0\dot{D}_t^\alpha u\|_{L^p}^p\right)^{\frac{1}{p}},$$

where $\|u\|_{L^p} = (\int_0^T |u(t)|^p \,dt)^{1/p}$ is the norm of $L^p([0,T],\mathbb{R})$.

Definition 2.8 For $0 < \alpha \leq 1$, $1 \leq p < \infty$, the space $_RE^{\alpha,p}$ is defined by

$$_RE^{\alpha,p} = \{u \in L^p([0,T],\mathbb{R}) | {}_t\dot{D}_T^\alpha u \in L^p([0,T],\mathbb{R})\}$$

with the norm

$$\|u\|_{_RE^{\alpha,p}} = \left(\|u\|_{L^p}^p + \|{}_t\dot{D}_T^\alpha u\|_{L^p}^p\right)^{\frac{1}{p}}.$$

Remark 2.9 When $\alpha = 1$, it follows from Remark 2.6 that the spaces $_LE^{\alpha,p}$ and $_RE^{\alpha,p}$ are reduced to the usual Sobolev space $W^{1,p}([0,T],\mathbb{R})$.

For the purpose of our paper, we only discuss the space $_LE^{\alpha,p}$ and the norm $\|\cdot\|_{_LE^{\alpha,p}}$ are denoted by $E^{\alpha,p}$ and $\|\cdot\|_{E^{\alpha,p}}$, respectively.

Lemma 2.10 (see [22]) *Let $0 < \alpha < 1$, the space $E^{\alpha,p}$ is a Banach space for $1 \leq p < \infty$. Moreover, it is reflexive for $1 < p < \infty$ and separable for $1 \leq p < \infty$.*

3 Fractional Sobolev space $E_0^{\alpha,p}$

When one discusses the existence of weak solutions for fractional BVPs by critical point theory, a suitable function space is necessary. This section will propose a fractional Sobolev space and some properties of this space for $0 < \alpha \leq 1$.

Definition 3.1 For $0 < \alpha \leq 1$, $1 \leq p < \infty$, the fractional Sobolev space $E_0^{\alpha,p}$ is defined by the closure of $C_0^\infty([0,T],\mathbb{R})$ with respect to the norm of $E^{\alpha,p}$.

Remark 3.2 It follows from Lemma 2.10 that the space $E_0^{\alpha,p}$ is a separable Banach space and is reflexive for $1 < p < \infty$. Moreover, when $\alpha = 1$, the space $E_0^{\alpha,p}$ is reduced to the usual Sobolev space $W_0^{1,p}([0,T],\mathbb{R})$ because of Remark 2.9.

Now we give some lemmas that are useful in the proof of the properties of $E_0^{\alpha,p}$.

Lemma 3.3 *$_0I_t^\alpha : L^p([0,T],\mathbb{R}) \to E^{\alpha,p}$ is a bounded linear operator.*

Proof From the definition of fractional integral, we see that ${}_0I_t^\alpha$ is a linear operator. Let $u \in L^p([0,T],\mathbb{R})$, $v = {}_0I_t^\alpha u$, then Lemma 2.1 in [23] shows $v \in L^p([0,T],\mathbb{R})$. By the semigroup property of the fractional integral, we have ${}_0I_t^{1-\alpha} v = {}_0I_t^{1-\alpha}{}_0I_t^\alpha u = {}_0I_t^1 u$. So ${}_0D_t^\alpha v = u$ exists, which means ${}_0\dot{D}_t^\alpha v = {}_0D_t^\alpha v$. Thus we obtain

$$\left\|{}_0\dot{D}_t^\alpha v\right\|_{L^p} = \|u\|_{L^p} \tag{4}$$

and ${}_0I_t^\alpha u = v \in E^{\alpha,p}$. Together with Lemma 2.1 in [23] and (4), we have

$$\begin{aligned}\left\|{}_0I_t^\alpha u\right\|_{E^{\alpha,p}} &= \left(\left\|{}_0I_t^\alpha u\right\|_{L^p}^p + \left\|{}_0\dot{D}_t^\alpha {}_0I_t^\alpha u\right\|_{L^p}^p\right)^{\frac{1}{p}} \\ &\leq \left(\frac{T^{\alpha p}}{(\Gamma(\alpha+1))^p} + 1\right)^{\frac{1}{p}} \|u\|_{L^p},\end{aligned}$$

which means ${}_0I_t^\alpha : L^p([0,T],\mathbb{R}) \to E^{\alpha,p}$ is a bounded operator. □

Lemma 3.4 ${}_0\dot{D}_t^\alpha : E^{\alpha,p} \to L^p([0,T],\mathbb{R})$ *is a bounded linear operator.*

Proof From the definition of left weak fractional derivative, we can see that the operator ${}_0\dot{D}_t^\alpha$ is linear. Moreover, it is easy to see

$$\left\|{}_0\dot{D}_t^\alpha u\right\|_{L^p} \leq \left(\|u\|_{L^p}^p + \left\|{}_0\dot{D}_t^\alpha u\right\|_{L^p}^p\right)^{\frac{1}{p}} = \|u\|_{E^{\alpha,p}},$$

which means ${}_0\dot{D}_t^\alpha : E^{\alpha,p} \to L^p([0,T],\mathbb{R})$ is a bounded operator. □

Lemma 3.5 *Let* $u \in E_0^{\alpha,p}$, *then* $u = {}_0I_t^\alpha {}_0\dot{D}_t^\alpha u$ *a.e. on* $(0,T)$.

Proof From the definition of space $E_0^{\alpha,p}$, there exists a sequence $\{\varphi_n\} \subset C_0^\infty([0,T],\mathbb{R})$ such that

$$\|\varphi_n - u\|_{E^{\alpha,p}} \to 0 \quad \text{as } n \to \infty.$$

For $\varphi_n \in C_0^\infty([0,T],\mathbb{R})$, one has ${}_0\dot{D}_t^\alpha \varphi_n = {}_0D_t^\alpha \varphi_n$, which together with Lemma 2.5 in [23] yields

$${}_0I_t^\alpha {}_0\dot{D}_t^\alpha \varphi_n = {}_0I_t^\alpha {}_0D_t^\alpha \varphi_n = \varphi_n - \frac{{}_0I_t^{1-\alpha}\varphi_n(0)}{\Gamma(\alpha)} t^{\alpha-1}.$$

Since ${}_0I_t^{1-\alpha}\varphi_n(0) = 0$, we have

$${}_0I_t^\alpha {}_0\dot{D}_t^\alpha \varphi_n = \varphi_n.$$

So, based on Lemmas 3.3 and 3.4, we obtain

$$\begin{aligned}\left\|{}_0I_t^\alpha {}_0\dot{D}_t^\alpha u - u\right\|_{E^{\alpha,p}} &\leq \left\|{}_0I_t^\alpha {}_0\dot{D}_t^\alpha u - {}_0I_t^\alpha {}_0\dot{D}_t^\alpha \varphi_n\right\|_{E^{\alpha,p}} + \|\varphi_n - u\|_{E^{\alpha,p}} \\ &\leq c\|u - \varphi_n\|_{E^{\alpha,p}} + \|\varphi_n - u\|_{E^{\alpha,p}} \\ &= (c+1)\|\varphi_n - u\|_{E^{\alpha,p}} \to 0 \quad \text{as } n \to \infty,\end{aligned}$$

where $c > 0$ is a constant. □

Corollary 3.6 (Fractional Poincaré-Friedrichs inequality) *Let $u \in E_0^{\alpha,p}$, one has*

$$\|u\|_{L^p} \leq \frac{T^\alpha}{\Gamma(\alpha+1)} \left\| {}_0\dot{D}_t^\alpha u \right\|_{L^p}.$$

Proof Combining Lemma 3.5 with Lemma 2.1 in [23], we have

$$\begin{aligned} \|u\|_{L^p} &= \left\| {}_0I_t^\alpha {}_0\dot{D}_t^\alpha u \right\|_{L^p} \\ &\leq \frac{T^\alpha}{\Gamma(\alpha+1)} \left\| {}_0\dot{D}_t^\alpha u \right\|_{L^p}. \end{aligned}$$ □

Remark 3.7 According to Corollary 3.6, we know that the norm $\|\cdot\|_{E^{\alpha,p}}$ of the space $E_0^{\alpha,p}$ is equivalent to the norm $\|{}_0\dot{D}_t^\alpha \cdot\|_{L^p}$. Hence we can consider $E_0^{\alpha,p}$ with the norm $\|{}_0\dot{D}_t^\alpha \cdot\|_{L^p}$ in the following analysis.

Theorem 3.8 *Let $\alpha > 1/p$ and $u \in E_0^{\alpha,p}$, then there exists a function $\tilde{u} \in C([0,T],\mathbb{R})$ such that*

$$u = \tilde{u} \quad \text{a.e. on } (0,T). \tag{5}$$

Proof Let $u \in E_0^{\alpha,p}$ and $\tilde{u} = {}_0I_t^\alpha {}_0\dot{D}_t^\alpha u$. From Lemma 3.5, (5) is obtained. Next we prove that $\tilde{u}$ is uniformly continuous on $[0,T]$.

By the Hölder inequality and $\alpha > 1/p$, we have

$$\begin{aligned} \left|\tilde{u}(t)\right| &= \left| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} {}_0\dot{D}_t^\alpha u(s)\, ds \right| \\ &\leq \frac{1}{\Gamma(\alpha)} \left(\int_0^t (t-s)^{q(\alpha-1)}\, ds \right)^{\frac{1}{q}} \left\| {}_0\dot{D}_t^\alpha u \right\|_{L^p} \\ &\leq \frac{T^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)(\alpha q - q + 1)^{\frac{1}{q}}} \left\| {}_0\dot{D}_t^\alpha u \right\|_{L^p}, \quad \forall t \in [0,T], \end{aligned} \tag{6}$$

which means $\tilde{u}$ is uniformly bounded.

In addition, from Theorem 3.6 in [24] and $\alpha > 1/p$, one has

$$\begin{aligned} \left|\tilde{u}(t_1) - \tilde{u}(t_2)\right| &= \left| {}_0I_t^\alpha {}_0\dot{D}_t^\alpha u(t_1) - {}_0I_t^\alpha {}_0\dot{D}_t^\alpha u(t_2) \right| \\ &\leq c|t_1 - t_2|^{\alpha-\frac{1}{p}} \left\| {}_0\dot{D}_t^\alpha u \right\|_{L^p}, \quad \forall t_1, t_2 \in [0,T], \end{aligned} \tag{7}$$

where $c > 0$ is a constant. So $\tilde{u}$ is uniformly continuous on $[0,T]$. □

Remark 3.9 For $u \in E_0^{\alpha,p}$, it follows from Theorem 3.8 that

$$u \in {}_0I_t^\alpha \big(L^p\big([0,T],\mathbb{R}\big)\big) = \big\{ u | u = {}_0I_t^\alpha v, v \in L^p\big([0,T],\mathbb{R}\big)\big\}.$$

More precisely, $\tilde{u} \in {}_0I_t^\alpha(L^p([0,T],\mathbb{R}))$ and $\tilde{u}$ is uniformly continuous on $[0,T]$. Hence we do not distinguish u and $\tilde{u}$, that is, $E_0^{\alpha,p} \subset C([0,T],\mathbb{R})$ and ${}_0\dot{D}_t^\alpha u = {}_0D_t^\alpha u$. Moreover, by (6), one has

$$\|u\|_\infty \le C_0\|u\|_{E^{\alpha,p}}, \tag{8}$$

where $\|u\|_\infty = \max_{t\in[0,T]}|u(t)|$ is the norm of $C([0,T],\mathbb{R})$ and

$$C_0 = \frac{T^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)(\alpha q - q + 1)^{\frac{1}{q}}} > 0, \quad q = \frac{p}{p-1} > 1$$

are two constants.

Theorem 3.10 *Let $\alpha > 1/p$, then the imbedding of $E_0^{\alpha,p}$ in $C([0,T],\mathbb{R})$ is compact.*

Proof Let $\{u_n\} \subset E_0^{\alpha,p}$ is a bounded sequence, from (7) and (8), we know $\{u_n\} \subset C([0,T],\mathbb{R})$ is uniformly bounded and equicontinuous. Thus, by the Arzelà-Ascoli theorem, $\{u_n\}$ is relatively compact in $C([0,T],\mathbb{R})$. □

Theorem 3.11 *Let $u \in E_0^{\alpha,p}$ with $\alpha > 1/p$, then*

$$u(0) = u(T) = 0.$$

Proof For $u \in E_0^{\alpha,p}$, from the definition of $E_0^{\alpha,p}$, there exists a sequence $\{u_n\} \subset C_0^\infty([0,T],\mathbb{R})$ such that

$$u_n \to u \quad \text{in } E_0^{\alpha,p}.$$

Thus, by (8), we get

$$\max_{t\in[0,T]}\bigl|u_n(t) - u(t)\bigr| \le C_0\|u_n - u\|_{E^{\alpha,p}} \to 0.$$

So we have

$$u(0) = u_n(0) = 0, \qquad u(T) = u_n(T) = 0.$$ □

4 Regularity of weak solutions

Consider the following BVP:

$$\begin{cases} {}_tD_T^\alpha {}_0D_t^\alpha u = f, & t \in (0,T), \\ u(0) = u(T) = 0, \end{cases} \tag{9}$$

where $1/2 < \alpha \le 1$ and f is a given function in $C([0,T],\mathbb{R})$ or $L^2([0,T],\mathbb{R})$. First of all, we give the definitions of weak solutions and classical solutions of BVP (9).

Definition 4.1 A weak solution of BVP (9) is a function $u \in E_0^{\alpha,2}$ satisfying

$$\int_0^T \bigl({}_0\dot{D}_t^\alpha u\bigr)\bigl({}_0D_t^\alpha \varphi\bigr) = \int_0^T f\varphi, \quad \forall \varphi \in C_0^\infty\bigl([0,T],\mathbb{R}\bigr). \tag{10}$$

In the Sobolev space $E_0^{\alpha,2}$, we define the energy functional Φ by

$$\Phi(u) = \int_0^T \frac{1}{2}\left|{}_0\dot{D}_t^\alpha u\right|^2 - \int_0^T fu.$$

It is easy to verify

$$\left\langle \Phi'(u), v\right\rangle = \int_0^T \left({}_0\dot{D}_t^\alpha u\right)\left({}_0\dot{D}_t^\alpha v\right) - \int_0^T fv, \quad \forall v \in E_0^{\alpha,2}.$$

Hence the weak solutions of BVP (9) are the critical points of C^1 functional Φ.

Definition 4.2 A classical solution of BVP (9) is a function $u \in E_0^{\alpha,2}$ satisfying BVP (9) in the usual sense, that is:
(1) ${}_0\dot{D}_t^\alpha u = {}_0D_t^\alpha u$,
(2) ${}_tI_T^{1-\alpha}{}_0D_t^\alpha u$ is derivable for every $t \in (0, T)$.

Lemma 4.3 ([25]) *Let $f \in L^1_{\mathrm{loc}}(I)$ be such that*

$$\int_I f\phi' = 0, \quad \forall \phi \in C_c^1(I).$$

Then there exists a constant C such that $f = C$ a.e. on I, where I is an open interval.

Theorem 4.4 *If $f \in L^2([0, T], \mathbb{R})$, BVP* (9) *has a unique weak solution.*

Proof Define the linear operator $\mathcal{F} : E_0^{\alpha,2} \to \mathbb{R}$ by

$$\mathcal{F}(v) = \int_0^T fv, \quad \forall v \in E_0^{\alpha,2}.$$

Obviously we have

$$\mathcal{F}(v) = \int_0^T fv \le \|f\|_{L^2}\|v\|_{L^2} \le c\|f\|_{L^2}\|v\|_{E_0^{\alpha,2}}.$$

So the operator $\mathcal{F}$ is well defined in $E_0^{\alpha,2}$ and continuous. Thus, it follows from the Lax-Milgram theorem that there exists a unique $u \in E_0^{\alpha,2}$ such that

$$\int_0^T \left({}_0\dot{D}_t^\alpha u\right)\left({}_0D_t^\alpha v\right) = \int_0^T fv. \qquad \square$$

Now we give the regularity of weak solutions of BVP (9).

Theorem 4.5 *If $f \in C([0, T], \mathbb{R})$, the unique weak solution of BVP* (9) *is the classical solution.*

Proof Let u is a weak solution of BVP (9), from Remark 3.9, we can get ${}_0\dot{D}_t^\alpha u = {}_0D_t^\alpha u$.

Next we prove ${}_tI_T^{1-\alpha}{}_0D_t^\alpha u$ is derivable for every $t \in (0, T)$. Let $F = {}_tI_T^\alpha f$, then $F \in L^2([0, T], \mathbb{R})$ and

$${}_tD_T^\alpha F = -\left({}_tI_T^{1-\alpha}{}_tI_T^\alpha f\right)' = -({}_tI_T f)' = f.$$

Thus, from the definition of right weak fractional derivative, we have

$$\int_0^T F\big({}_0D_t^\alpha\varphi\big) = \int_0^T \big({}_tD_T^\alpha F\big)\varphi = \int_0^T f\varphi, \quad \forall\varphi\in C_0^\infty\big([0,T],\mathbb{R}\big),$$

which together with (10) yields

$$\begin{aligned}
&\int_0^T \big({}_0D_t^\alpha u\big)\big({}_0D_t^\alpha\varphi\big) = \int_0^T F\big({}_0D_t^\alpha\varphi\big)\\
&\quad\Rightarrow\quad \int_0^T \big({}_0D_t^\alpha u - F\big)\big({}_0D_t^\alpha\varphi\big) = 0\\
&\quad\Rightarrow\quad \int_0^T \big({}_0D_t^\alpha u - F\big)\big({}_0I_t^{1-\alpha}\varphi'\big) = 0\\
&\quad\Rightarrow\quad \int_0^T \big({}_tI_T^{1-\alpha}\big({}_0D_t^\alpha u - F\big)\big)\varphi' = 0.
\end{aligned}$$

Hence, from Lemma 4.3, we have

$${}_tI_T^{1-\alpha}\big({}_0D_t^\alpha u - F\big) = c \quad \text{a.e. on } (0,T).$$

That is,

$$\begin{aligned}
{}_tI_T^{1-\alpha}\,{}_0D_t^\alpha u &= {}_tI_T^{1-\alpha}F + c\\
&= {}_tI_T^{1-\alpha}\,{}_tI_T^{\alpha}f + c\\
&= {}_tI_Tf + c.
\end{aligned}\tag{11}$$

The right side of (11) indicates that if $f\in C([0,T],\mathbb{R})$, then ${}_tI_Tf + c\in C^1([0,T],\mathbb{R})$. So ${}_tI_T^{1-\alpha}\,{}_0D_t^\alpha u$ is derivable for every $t\in(0,T)$.

Taking derivatives on both sides of (11), one has

$$-{}_tD_T^\alpha\,{}_0D_t^\alpha u = ({}_tI_Tf + c)' = -f,$$

which means u is a classical solution of BVP (9). □

5 Spectral structure of fractional Dirichlet BVP

In this section, we study the spectral structure of a class of fractional Dirichlet BVPs with variational structure. For proving our theorem, we first present a lemma.

Lemma 5.1 ([25]) *Let H be a separable Hilbert space and $\mathcal{T}$ be a compact self-adjoint operator. Then there exists a Hilbert basis composed of eigenvectors of $\mathcal{T}$.*

Definition 5.2 Let $\mathcal{L} = {}_tD_T^\alpha\,{}_0D_t^\alpha$ with $\operatorname{dom}\mathcal{L}\subset E_0^{\alpha,2}$, if there exists a function $u\neq 0$ such that $\mathcal{L}u=\lambda u$, one says λ is the eigenvalue of $\mathcal{L}$ and u is the associated eigenfunction.

Theorem 5.3 *For the following BVP*:

$$\begin{cases} {}_tD_T^\alpha\,{}_0D_t^\alpha u = \lambda u, & t\in(0,T),\\ u(0)=u(T)=0, \end{cases}\tag{12}$$

where $1/2 < \alpha \leq 1$, *there exist an eigenvalue sequence* $\{\lambda_n\}$ *and an eigenfunction sequence* $\{\phi_n\} \subset E_0^{\alpha,2}$. *Furthermore,* $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots$, $\lambda_n \to \infty$ *as* $n \to \infty$ *and* $\{\phi_n\}$ *is a Hilbert basis of* $E_0^{\alpha,2}$.

Proof For $f \in L^2([0,T],\mathbb{R})$, by Theorem 4.4, we see that there exists a unique $u \in E_0^{\alpha,2}$ such that

$$\int_0^T \big({}_0D_t^\alpha u\big)\big({}_0D_t^\alpha \varphi\big) = \int_0^T f\varphi, \quad \forall \varphi \in C_0^\infty\big([0,T],\mathbb{R}\big). \tag{13}$$

So we can define $\mathcal{T} = \mathcal{L}^{-1} : f \to u$ as an operator from L^2 into $E_0^{\alpha,2}$.

First of all, we show that $\mathcal{T}$ is compact. From the definition of $E_0^{\alpha,2}$ and (13), we have

$$\int_0^T \big({}_0D_t^\alpha u\big)\big({}_0D_t^\alpha u\big) = \int_0^T fu.$$

Thus $\|u\|_{E^{\alpha,2}}^2 \leq \|f\|_{L^2}\|u\|_{L^2}$. By Corollary 3.6, one has $\|u\|_{L^2} \leq c\|u\|_{E^{\alpha,2}}$. So we get $\|u\|_{E^{\alpha,2}} \leq c\|f\|_{L^2}$. This can be written as

$$\|\mathcal{T}f\|_{E^{\alpha,2}} \leq c\|f\|_{L^2}, \quad \forall f \in L^2\big([0,T],\mathbb{R}\big).$$

Moreover, from Theorem 3.10, the injection map from $E_0^{\alpha,2}$ into $L^2([0,T],\mathbb{R})$ is compact. Hence, we deduce that $\mathcal{T}$ is a compact operator from $L^2([0,T],\mathbb{R})$ into $L^2([0,T],\mathbb{R})$.

Second, we show that $\mathcal{T}$ is self-adjoint. Let $\mathcal{T}f = u$, $\mathcal{T}g = v$, by (13), we have

$$\int_0^T \big({}_0D_t^\alpha u\big)\big({}_0D_t^\alpha v\big) = \int_0^T fv, \qquad \int_0^T \big({}_0D_t^\alpha v\big)\big({}_0D_t^\alpha u\big) = \int_0^T gu.$$

So we get $\int_0^T fv = \int_0^T gu$, which means $\int_0^T f(\mathcal{T}g) = \int_0^T (\mathcal{T}f)g$.

Finally, we show that $\mathcal{T}$ is a positive operator. It is obvious that

$$\int_0^T (\mathcal{T}f)f = \int_0^T uf = \int_0^T \big({}_0D_t^\alpha u\big)\big({}_0D_t^\alpha u\big) \geq 0.$$

In addition, $\int_0^T (\mathcal{T}f)f = 0$ implies $f = 0$.

Therefore, applying Lemma 5.1, we know that $L^2([0,T],\mathbb{R})$ admits a Hilbert basis $\{\phi_n\}$ consisting of eigenvectors of $\mathcal{T}$ with corresponding eigenvalues μ_n. Also we have $\mu_n > 0$ ($\forall n$), $\mu_n \to 0$ ($n \to \infty$). Thus we have $\phi_n \in E_0^{\alpha,2}$ and

$$\int_0^T \big({}_0D_t^\alpha \phi_n\big)\big({}_0D_t^\alpha \varphi\big) = \frac{1}{\mu_n}\int_0^T \phi_n\varphi, \quad \forall \varphi \in C_0^\infty\big([0,T],\mathbb{R}\big).$$

In other words, the ϕ_n are the weak solutions of (12) with $\lambda_n = 1/\mu_n$.

Since $\phi_n \in E_0^{\alpha,2}$, from Theorem 3.8, the ϕ_n are continuous. Thus, by Theorem 4.5, we see that the ϕ_n satisfy the following BVP:

$$\begin{cases} {}_tD_T^\alpha {}_0D_t^\alpha \phi_n = \lambda_n\phi_n, & t \in (0,T), \\ \phi_n(0) = \phi_n(T) = 0, \end{cases} \tag{14}$$

where $\lambda_n = 1/\mu_n \to \infty$ ($n \to \infty$).

Now we show that $\{\frac{\phi_n}{\sqrt{\lambda_n}}\}$ is a Hilbert basis of $E_0^{\alpha,2}$. It is easy to see

$$\begin{aligned}(\phi_n,\phi_m)_{E_0^{\alpha,2}} &= \int_0^T ({}_0D_t^\alpha\phi_n)({}_0D_t^\alpha\phi_m) \\ &= \lambda_n(\phi_n,\phi_m)_{L^2} = 0, \quad n \neq m,\end{aligned}$$

which means $\{\frac{\phi_n}{\sqrt{\lambda_n}}\}$ is orthonormal in $E_0^{\alpha,2}$. It remains to be proved that if $f \in E_0^{\alpha,2}$, satisfying $(f,\phi_n)_{E_0^{\alpha,2}} = 0$, then $f = 0$. From (14), we have

$$\int_0^T ({}_0D_t^\alpha\phi_n)({}_0D_t^\alpha f) = \lambda_n \int_0^T \phi_n f.$$

That is, $(f,\phi_n)_{E_0^{\alpha,2}} = \lambda_n(f,\phi_n)_{L^2}$. Because $\{\phi_n\}$ is a Hilbert basis of $L^2([0,T],\mathbb{R})$ and $(f,\phi_n)_{L^2} = 0$, we get $f = 0$. □

Corollary 5.4 *For* $1/2 < \alpha \le 1$, *the first eigenvalue* λ_1 *of operator* ${}_tD_T^\alpha {}_0D_t^\alpha$ *with Dirichlet boundary value conditions satisfies*

$$\lambda_1 \ge \left(\frac{\Gamma(\alpha+1)}{T^\alpha}\right)^2.$$

Proof From Theorem 5.3, we obtain

$$\lambda_1 = \inf_{u\in E_0^{\alpha,2}} \frac{(\mathcal{L}u,u)_{L^2}}{(u,u)_{L^2}} = \inf_{u\in E_0^{\alpha,2}} \frac{\|{}_0D_t^\alpha u\|_{L^2}^2}{\|u\|_{L^2}^2},$$

which together with Corollary 3.6 yields $\lambda_1 \ge (\frac{\Gamma(\alpha+1)}{T^\alpha})^2$. □

Remark 5.5 When $\alpha = 1$, the first eigenvalue of operator $-D^2$ in $W_0^{1,2}([0,T],\mathbb{R})$ is $\tilde{\lambda}_1 = (\pi/T)^2$, which satisfies $\tilde{\lambda}_1 \ge (1/T)^2$.

Remark 5.6 When $0 < \alpha \le 1/2$, the continuity of $u \in E_0^{\alpha,2}$ (Theorem 3.8) and the compactness of injection from $E_0^{\alpha,2}$ into $C([0,T],\mathbb{R})$ (Theorem 3.10) may not be obtained. If we want to get the conclusions of Sections 3-5, some new methods and techniques need to be explored.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors contributed equally in this article. They read and approved the final manuscript.

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